Applied Mathematics E-Notes, 12(2012), 44-52 © ISSN 1607-2510
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A Multipoint Iterative Method For Semistable Solutions*

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Received 28 June 2011

Abstract

This paper deals with variational inclusions of the form : $0 \in \varphi(z) + F(z)$ where φ is a single-valued function admitting a second order Fréchet derivative and F is a set-valued map from $\mathbb{R}^q$ to the closed subsets of $\mathbb{R}^q$. In order to approximate a solution $\bar{z}$ of the previous inclusion, we use an iterative scheme based on a multipoint method. We obtain, thanks to some semistability properties of $\bar{z}$, local superquadratic or cubic convergent sequences.

1 Introduction

This paper is devoted to the study of a multipoint iterative method for approximating a solution of the variational inclusion

$$0 \in \varphi(z) + F(z), \tag{1}$$

where φ is a single-valued function and F is a set-valued map.

Variational inclusions are an abstract model of a wide variety of variational problems including linear and non-linear complementarity problems, systems of non-linear equations, variational inequalities. In the last decade, several iterative methods for solving the inclusion (1) have been introduced. These methods consist in generating an iterative sequence (z_n) obtained by subsequently solving implicit subproblems of the form $0 \in A(z_n, z_{n+1}) + F(z_{n+1})$ where A denotes some approximation of the mapping φ.

Dontchev [7, 8] associates to (1) a Newton-type method based on a partial linearization. Inspired and motivated by his works, various authors proved the convergence of some methods based on different techniques such as one or second order Taylor's expansion, interpolation formula and a multipoint formula given in [23]. For more details on these methods, the reader could refer to [6, 12, 13, 14, 15, 17, 20]. Let us point out that all these methods have been studied when a metric regularity property is satisfied for the set-valued map $(\varphi + F)^{-1}$ or one of its approximation. For more details on this property, the reader could refer to [1, 2, 9, 10, 11, 18, 19, 22].

*Mathematics Subject Classifications: 49J53, 47H04, 65K10.

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In this paper, the function φ is defined from $\mathbb{R}^q$ to $\mathbb{R}^q$ and F is a set-valued map from $\mathbb{R}^q$ to the closed subsets of $\mathbb{R}^q$. Our aim is to study inclusion (1) using an assumption which is directly connected to a solution : the semistability concept. This concept has been introduced by Bonnans [3] for variational inequalities. A solution $\bar{z}$ of a variational inclusion is said to be semistable if, given a small perturbation on the left-hand side, a solution z of the perturbed variational inclusion that is sufficiently close to $\bar{z}$ is such that the distance of z to $\bar{z}$ is of the order of the magnitude of the perturbation.

Studies using such property has recently been made in [4, 5] for methods based on the second order Fréchet derivative. Following these works, we consider the relation

$$0 \in \varphi(z_k) + \sum_{i=1}^{N} a_i M_k^i (z_{k+1} - z_k) + F(z_{k+1}), \tag{2}$$

where M_k^i is a $q \times q$ matrix satisfying some conditions and we prove the convergence of this procedure under the semistability property. We can observe that if $M_k^i = \varphi'\big(z_k + \beta_i(z_{k+1} - z_k)\big)$, we get the method proposed in [6] and if $N = 2$, $a_1 = a_2 = \frac{1}{2}$, $\beta_1 = 0$ and $\beta_2 = 1$, we obtain the method introduced in [4].

The rest of the paper is organized as follows. In section 2, we collect a number of basic definitions regarding semistability of solutions and regularity for set-valued maps that we will need afterwards. Then, in section 3, we study the behaviour of the method (2) and we give a classical problem which could be treated with our method.

In this paper, for simplicity reasons, all the norms are denoted by $\|.\|$.

2 Background Materials

Here, we recall the concept of semistability introduced by Bonnans in [3].

DEFINITION 1. A solution $\bar{z}$ of (1) is said to be *semistable* if $c_1 > 0$ and $c_2 > 0$ exist such that, for all $(z, \delta) \in \mathbb{R}^q \times \mathbb{R}^q$, solution of $\delta \in \varphi(z) + F(z)$, and $\|z - \bar{z}\| \leq c_1$, then $\|z - \bar{z}\| \leq c_2 \|\delta\|$.

Note that a sufficient condition for semistability is the strong regularity of Robinson [21]. Recently, Izmailov and Solodov in [16] used this concept in order to study the convergence of Inexact Josephy-Newton method for solving generalized equations. We will also need the following Hölder-type property.

DEFINITION 2. Let $\varphi : \mathbb{R}^q \longrightarrow \mathbb{R}^q$ be a function. One says that φ satisfies a Hölder-type condition on a neighborhood Ω of $\bar{z}$ if

$$\exists K > 0, \alpha \in]0,1], \ \|\varphi(x) - \varphi(y)\| \leq K \|x - y\|^{\alpha}, \forall\, x, y \in \Omega.$$

Note that when $\alpha = 1$, we have the Lipschitz condition for φ.

3 Convergence Analysis

Our purpose in this section is to provide an iterative procedure for solving (1) and to show how the semistability property can be an efficient tool to estimate the rate of convergence of the method (2).

First of all, we consider the approximation ψ of φ defined by

$$\psi(u,v) = \varphi(u) + \sum_{i=1}^{N} a_i \varphi'\big(u + \beta_i(v-u)\big)(v-u) \tag{3}$$

and then we introduce the algorithm:

- given any starting point z_0 in some neighborhood of $\bar{z}$ which is a solution of (1),
- for $k = 0, 1, \ldots$, while z_k does not satisfy (1), choose $\Xi_k(.)$ an approximation of $\psi(z_k, \cdot)$ defined by $\Xi_k(z) = \varphi(z_k) + \sum_{i=1}^{N} a_i M_k^i (z - z_k)$,
- compute a solution z_{k+1} of

$$0 \in \Xi_k(z) + F(z). \tag{4}$$

The main result of this study reads as follows.

THEOREM 1. Let $\bar{z}$ be a semistable solution of (1) and let (z_k) be a sequence computed by (2) which converges towards $\bar{z}$. We suppose that φ'' is a locally Lipschitz function. Then :

(i) if $\psi(z_k, z_{k+1}) - \Xi_k(z_{k+1}) = o\big(\|z_{k+1} - z_k\|^2\big)$ then (z_k) converges superquadratically.

(ii) if $\psi(z_k, z_{k+1}) - \Xi_k(z_{k+1}) = O\big(\|z_{k+1} - z_k\|^3\big)$ then (z_k) converges cubically.

To prove Theorem 1, we will need the following lemma:

LEMMA 1. Let $\varphi : \mathbb{R}^q \to \mathbb{R}^q$ be a function admitting a second order Fréchet derivative which is L-Lipschitz on Ω and let ψ be defined by (3). One has the following inequality

$$\|\varphi(v) - \psi(u,v)\| \le \left(\frac{L}{6} + \frac{L}{2}\sum_{i=1}^{N} |a_i|\, \beta_i^2\right) \|v-u\|^3 . \tag{5}$$

PROOF. We have

$$\begin{aligned}
\|\varphi(v) - \psi(u,v)\| &= \left\|\varphi(v) - \varphi(u) - \sum_{i=1}^{N} a_i \varphi'\big(u + \beta_i(v-u)\big)(v-u)\right\| \\
&= \Bigg\|\varphi(v) - \varphi(u) - \varphi'(u)(v-u) - \frac{1}{2}\varphi''(u)(v-u)^2 + \varphi'(u)(v-u) \\
&\quad + \frac{1}{2}\varphi''(u)(v-u)^2 - \sum_{i=1}^{N} a_i \varphi'\big(u + \beta_i(v-u)\big)(v-u)\Bigg\|.
\end{aligned}$$

Let

$$A_1 = \left\|\varphi(v) - \varphi(u) - \varphi'(u)(v-u) - \frac{1}{2}\varphi''(u)(v-u)^2\right\|$$

and

$$A_2 = \left\| \varphi'(u)(v-u) + \frac{1}{2}\varphi''(u)(v-u)^2 - \sum_{i=1}^{N} a_i \varphi'\big(u+\beta_i(v-u)\big)(v-u) \right\|.$$

Since φ'' is L-Lipschitz on Ω, we obtain $A_1 \le \frac{L}{6}\|v-u\|^3$. Moreover,

$$\begin{aligned}
A_2 &= \left\| \varphi'(u)(v-u) + \frac{1}{2}\varphi''(u)(v-u)^2 - \sum_{i=1}^{N} a_i \varphi'\big(u+\beta_i(v-u)\big)(v-u) \right\| \\
&= \left\| \sum_{i=1}^{N} a_i \Big(\big(\varphi'(u) - \varphi'\big(u+\beta_i(v-u)\big)\big)(v-u) \Big) + \frac{1}{2}\varphi''(u)(v-u)^2 \right\| \\
&= \left\| \sum_{i=1}^{N} a_i \Big(-\beta_i \int_0^1 \varphi''\big(u+\beta_i t(v-u)\big)(v-u)(v-u)dt \Big) + \frac{1}{2}\varphi''(u)(v-u)^2 \right\| \\
&= \left\| \sum_{i=1}^{N} -a_i\beta_i \int_0^1 \varphi''\big(u+\beta_i t(v-u)\big)(v-u)^2 dt + \sum_{i=1}^{N} a_i\beta_i \int_0^1 \varphi''(u)(v-u)^2 dt \right\| \\
&\le \sum_{i=1}^{N} |a_i|\,\beta_i^2 L\,\|v-u\|^3 \int_0^1 t dt \;\le\; \frac{L}{2}\sum_{i=1}^{N} |a_i|\,\beta_i^2\,\|v-u\|^3.
\end{aligned}$$

And then

$$\|\varphi(v) - \psi(u,v)\| \le \left(\frac{L}{6} + \frac{L}{2}\sum_{i=1}^{N} |a_i|\,\beta_i^2 \right) \|v-u\|^3.$$

PROOF of THEOREM 1. We write (4) as

$$r_k \in \varphi(z_{k+1}) + F(z_{k+1}), \tag{6}$$

with

$$r_k := \Theta_k + \Phi(z_{k+1}, z_k),$$

where $\Theta_k = \psi(z_k, z_{k+1}) - \Xi_k(z_{k+1})$ and $\Phi(z_{k+1}, z_k) = \varphi(z_{k+1}) - \psi(z_k, z_{k+1})$. Using (5), we get $\|\Phi(z_{k+1}, z_k)\| = o(\|z_{k+1} - z_k\|^2)$. Then $\|r_k\| = o(\|z_{k+1} - z_k\|^2)$.

Since $\bar{z}$ is semistable, we get $\|z_{k+1} - \bar{z}\| = O(\|r_k\|)$. Then

$$\|z_{k+1} - \bar{z}\| = o(\|z_{k+1} - \bar{z}\|^2 + 2\|z_{k+1} - \bar{z}\|\,\|z_k - \bar{z}\| + \|z_k - \bar{z}\|^2),$$

i.e.,

$$\begin{aligned}
0 &= \lim_{k\to\infty} \frac{\|z_{k+1} - \bar{z}\|}{\|z_{k+1} - \bar{z}\|^2 + 2\|z_{k+1} - \bar{z}\|\,\|z_k - \bar{z}\| + \|z_k - \bar{z}\|^2} \\
&= \lim_{k\to\infty} \frac{1}{\|z_{k+1} - \bar{z}\| + 2\|z_k - \bar{z}\| + \frac{\|z_k - \bar{z}\|^2}{\|z_{k+1} - \bar{z}\|}}.
\end{aligned}$$

Since $z_k \to \bar{z}$, the latter relation implies that $\lim_{k\to\infty} \frac{\|z_k-\bar{z}\|^2}{\|z_{k+1}-\bar{z}\|} = +\infty$, i.e.,

$$\|z_{k+1} - \bar{z}\| = o(\|z_k - \bar{z}\|^2)$$

which proves (i).

Similarly for (ii), with (5), we get

$$\|r_k\| = O(\|z_{k+1} - z_k\|^3).$$

Thanks to the semistability of $\bar{z}$, we obtain

$$\|z_{k+1} - \bar{z}\| = O(\|z_{k+1} - z_k\|^3).$$

Since (z_k) converges superquadratically then $\frac{\|z_{k+1}-z_k\|}{\|z_k-\bar{z}\|} \to 1$, and then

$$\|z_{k+1} - \bar{z}\| = O(\|z_k - \bar{z}\|^3),$$

which completes the proof.

Let us extend this study to the case where the second order Fréchet derivative φ'' of the function φ satisfies a Hölder-type condition.

LEMMA 2. Let $\varphi : \mathbb{R}^q \to \mathbb{R}^q$ be a function admitting a second order Fréchet derivative and let ψ be defined by (3). If φ'' satisfies a Hölder-type condition with constants α and L on Ω, one has, for all $u, v \in \Omega$, the following inequality

$$\|\varphi(v) - \psi(u,v)\| \le \left(\frac{L}{(1+\alpha)(2+\alpha)} + \frac{L}{1+\alpha} \sum_{i=1}^{N} |a_i|\, \beta_i^{1+\alpha} \right) \|v-u\|^{2+\alpha}. \tag{7}$$

PROOF. The proof is similar to the proof of Lemma 1. We write

$$\|\varphi(v) - \psi(u,v)\| \le A_1 + A_2$$

with

$$A_1 = \left\| \varphi(v) - \varphi(u) - \varphi'(u)(v-u) - \frac{1}{2}\varphi''(u)(v-u)^2 \right\|$$

and

$$A_2 = \left\| \varphi'(u)(v-u) + \frac{1}{2}\varphi''(u)(v-u)^2 - \sum_{i=1}^{N} a_i \varphi'\big(u + \beta_i(v-u)\big)(v-u) \right\|.$$

Since φ'' satisfies a Hölder-type condition with constants α and L on Ω, we obtain

$$A_1 \le \frac{L}{(1+\alpha)(2+\alpha)} \|v-u\|^{2+\alpha}.$$

Moreover,

$$A_2 \le \sum_{i=1}^{N} |a_i|\, \beta_i^{1+\alpha} L \|v-u\|^{2+\alpha} \int_0^1 t^\alpha dt \le \frac{L}{1+\alpha} \sum_{i=1}^{N} |a_i|\, \beta_i^{1+\alpha} \|v-u\|^{2+\alpha}.$$

Then

$$\|\varphi(v)-\psi(u,v)\| \leq \left(\frac{L}{(1+\alpha)(2+\alpha)} + \frac{L}{1+\alpha}\sum_{i=1}^{N} |a_i|\, \beta_i^{1+\alpha}\right) \|v-u\|^{2+\alpha}.$$

THEOREM 2. Let $\bar{z}$ be a semistable solution of (1) and let (z_k) be a sequence computed by (2) which converges towards $\bar{z}$. If φ'' satisfies a Hölder-type condition with constants α and L then :

(i) if $\psi(z_k, z_{k+1}) - \Xi_k(z_{k+1}) = o\big(\|z_{k+1}-z_k\|^{1+\alpha}\big)$ then (z_k) converges superlinearly;

(ii) if $\psi(z_k, z_{k+1}) - \Xi_k(z_{k+1}) = O\big(\|z_{k+1}-z_k\|^{2+\alpha}\big)$ then (z_k) converges superquadratically.

PROOF. The proof is similar to that of Theorem 1, but using Lemma 2 instead of Lemma 1.

As an illustration of our results let us consider the standard following nonlinear programming problem :

$$\text{minimize } f(x) \tag{8}$$

subject to

$$\begin{cases} g_i(x) \leq 0, & \forall i \in I \\ g_j(x) = 0, & \forall i \in J \end{cases}$$

where I, J are a partition of $\{1, ..., p\}$; $f : \mathbb{R}^n \to \mathbb{R}$ is twice continuously differentiable on $\mathbb{R}^n$ while the functions $g_i : \mathbb{R} \to \mathbb{R}$, $i = 1, ..., p$ are differentiable on $\mathbb{R}^n$ and are twice differentiable in a neighborhood of a solution x^* of (8).

To problem (8) is associated the first order optimality condition (in which $\lambda \in \mathbb{R}^p$):

$$\begin{cases} \nabla f(x) + \nabla g(x) * \lambda = 0 \\ g_i(x) \leq 0, \forall i \in I, \;\; g_j(x) = 0, \forall j \in J, \quad \lambda_I \geq 0, \quad \lambda_i g_i(x) = 0, \quad \forall i \in I. \end{cases} \tag{9}$$

As observed in [3], we may embed (9) into (2) in putting $z = (x, \lambda)$,

$$\varphi(x,\lambda) = \begin{pmatrix} \nabla f(x) + \nabla g(x) * \lambda \\ -g(x) \end{pmatrix}$$

and $F(x, \lambda) = N(x, \lambda) = \{0\} \times N_\Lambda(\lambda)$ where $\Lambda = \{\lambda \in \mathbb{R}^p, \lambda_I \geq 0, \forall i = 1, ..., p\}$ and $N_\Lambda(\lambda)$ denotes the normal cone to Λ at the point λ i.e.

$$N_\Lambda(\lambda) = \begin{cases} \emptyset \quad \text{if} \quad \lambda \notin \Lambda \\ \{\mu \in \mathbb{R}^p;\, \mu_J = 0;\; \mu_I \leq 0;\; \mu_i = 0 \quad \text{if} \quad \lambda_i > 0, \forall i \in I\} \end{cases}.$$

The corresponding variational inequality can be written in the following way:

$$\begin{cases} \nabla f(x) + \nabla g(x) * \lambda = 0 \\ g(x) \in N_\Lambda(\lambda) \end{cases}. \tag{10}$$

Thanks to a result given in Bonnans [3] and using a similar reasoning, we obtain that the semistability of (10) is equivalent to :

$$\left\{\begin{array}{l} (y,\mu)=0 \quad \text{is the unique solution of } H(\bar{x},\bar{\lambda})y+\nabla g(\bar{x})*\mu=0, \\ y\in C,\ \mu_{I^0}\geq 0; \quad \mu_i=0\ \ if\ \ g_i(\bar{x})<0,\ \ \forall i\in I; \\ \mu_i\nabla g_i(\bar{x})y=0, \quad \forall i\in I^0 \end{array}\right. \tag{11}$$

where

$$H(x,\lambda)=\nabla^2 f(x)+\sum_{i=1}^{p}\lambda_i\nabla^2 g_i(x),$$

$$\bar{I}=\{i\in I;\ g_i(\bar{x})=0\},$$

$$I^0=\{i\in\bar{I};\ \bar{\lambda}_i=0\},$$

$$I^+=\{i\in\bar{I};\ \bar{\lambda}_i>0\}$$

and

$$C=\{y\in\mathbb{R}^n;\ \nabla g_I(\bar{x})y\leq 0,\ \nabla g_J(\bar{x})y=0;\ \nabla g_{I^+}(\bar{x})y=0\}.$$

For more details, the reader can refer to [3].

Now, let us apply the method (2) presented in this paper to (10). The subproblem to be solved at the step k is :

$$\left\{\begin{array}{l} \nabla f(x_k)+H(x_k,\lambda_k)(x_{k+1}-x_k)+\nabla g(x_k)*\lambda_{k+1}=0 \\ g(x^k)+\displaystyle\sum_{i=1}^{p}a_i\nabla g(x_k+\beta_i(x_{k+1}-x_k))(x_{k+1}-x_k)\in N_\Lambda(\lambda_{k+1}). \end{array}\right.$$

As the evaluation of $\nabla g(x_k)$ is already necessary in order to evaluate $\varphi(x_k,\lambda_k)$, the only part of the Jacobian that perhaps needs to be approximated is $H(x_k,\lambda_k)$. We then obtain the following algorithm:

- given any starting point $(x_0,\lambda_0)\in\mathbb{R}^n\times\mathbb{R}^p$
- if (x_k,λ_k) is not solution of (10), choose M_k, an $n\times n$ matrix, compute the (x_{k+1},λ_{k+1}) solution of

$$\left\{\begin{array}{l} \nabla f(x_k)+M_k(x_{k+1}-x_k)+\nabla g(x_k)*\lambda_{k+1}=0 \\ g(x_k)+\displaystyle\sum_{i=1}^{p}a_i\nabla g(x_k+\beta_i(x_{k+1}-x_k))(x_{k+1}-x_k)\in N_\Lambda(\lambda_{k+1}). \end{array}\right.$$

When $M_k=H(x_k,\lambda_k)$, by applying Theorem 1 and the relation (11),we obtain the convergence of the sequence computed by (2) with $M_k^i=\nabla g(x_k+\beta_i(x_{k+1}-x_k))$.

Acknowledgment. We would like to thank the referee for valuable suggestions that enabled us to improve the presentation of this paper.

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