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Meromorphic Solutions Of Conjugacy Equations*

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Abstract

This paper characterizes the relation between a conjugacy equation $\varphi \circ f = g \circ \varphi$ and a permutable functional equation $\phi \circ f = f \circ \phi$ where $f : X \to X$, $g : Y \to Y$ are given self-maps, and φ, ϕ are unknown maps. When f and g are Möbius transformations, we prove that there exists a bijective meromorphic solution of a conjugacy equation if and only if f and g have the same normal form. Moreover, every bijective meromorphic solution is expressed by a permutable meromorphic function with their normal form.

1 Introduction

Let X and Y be topological spaces, and let $f : X \to X$ and $g : Y \to Y$ be continuous maps. We say that $f : X \to X$ is *topologically conjugate* (or simply *conjugate*) to $g : Y \to Y$ if there exists a homeomorphism $\varphi : X \to Y$ satisfying the conjugacy equation (cf. [1,2])

$$\varphi \circ f = g \circ \varphi, \tag{1}$$

where $\circ$ denotes the composition of maps. For instance,

$$\varphi(z+1) = \frac{\varphi(z)}{\varphi(z)+1}, \tag{2}$$

once arose in mathematical competitions or applied mathematics. Taking $f(z) = z+1$ and $g(z) = z/(z+1)$, Eq.(2) becomes a conjugacy equation.

In particular, when $g = f$ and φ is replaced with ϕ, the conjugacy equation (1) becomes

$$\phi \circ f = f \circ \phi, \tag{3}$$

which is called a *permutable functional equation*. f is said to be *permutable* with ϕ if the relation (3) holds. Permutable functions and close form solutions of functional equations have been extensively studied by many authors (see [3-9]). The monograph [2] collects many results including analytic solutions on a neighborhood of the origin of

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Eq.(3). In 1996, Coonce [10] studied some families of permutable functions of several variables. Later, Singh and Wang [11] investigated Julia sets of permutable holomorphic functions. In [4], Ciepliński discussed commuting functions on the circle. Zheng et al. [12] considered permutable entire functions satisfying algebraic differential equations. Recently, for conjugacy equations, we constructed all non-monotonic solutions and continuously differentiable solutions of conjugacy equations in [13]. Using a conjugacy equation, all meromorphic iterative roots of Möbius transformations were calculated in [14].

This paper characterizes the relation between Eq.(1) and Eq.(3) for two given $f: X \to X$ and $g: Y \to Y$. When f and g are Möbius transformations, we prove that there exists a bijective meromorphic solution of a conjugacy equation if and only if f and g have the same normal form. Moreover, every bijective meromorphic solution is expressed by a permutable meromorphic function with their normal form. Some examples are illustrated to apply these results.

2 Preliminaries

The following lemma states a relation between permutable functional equation and conjugacy equation.

LEMMA 1. Let $\varphi_0 : X \to Y$ be a particular solution of Eq.(1). Then every solution of (1) is given by

$$\varphi = \varphi_0 \circ \phi,$$

where $\phi : X \to X$ is a solution of Eq.(3).

PROOF. Since φ_0 is a solution of Eq.(1), we have $g \circ \varphi_0 = \varphi_0 \circ f$. For any solution $\phi : X \to X$ of Eq.(3), let $\varphi = \varphi_0 \circ \phi$, then

$$\varphi \circ f = \varphi_0 \circ \phi \circ f = \varphi_0 \circ f \circ \phi = g \circ \varphi_0 \circ \phi = g \circ \varphi$$

This completes the proof.

A Möbius transformation on the complex plane is given by

$$\ell(z) = \frac{az + b}{cz + d}$$

where a, c, b, d are any complex numbers satisfying $ad - bc \neq 0$. In case $c \neq 0$, this definition is extended to the whole Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ by defining $\ell(-d/c) = \infty$ and $\ell(\infty) = a/c$, if $c = 0$ we define $\ell(\infty) = \infty$. This turns ℓ into a bijective meromorphic function from $\widehat{\mathbb{C}}$ to itself.

The set of all Möbius transformations forms a group under composition called the Möbius group. It is the automorphism group of the Riemann sphere, denoted by $\mathrm{Aut}(\widehat{\mathbb{C}})$.

Let $GL_2(\mathbb{C})$ denote the group of all non-singular 2×2 matrices in the field $\mathbb{C}$. Define $h : GL_2(\mathbb{C}) \to \mathrm{Aut}(\widehat{\mathbb{C}})$ by

$$h\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \frac{az + b}{cz + d}. \tag{4}$$

The map h is surjective, but not injective because $h(\mu A) = h(A)$ for all nonzero $\mu \in \mathbb{C}$. Define an equivalence in $GL_2(\mathbb{C})$ with $A \sim B$ if and only if $A = \mu B$ and consider the corresponding quotient space $\tilde{GL}_2(\mathbb{C}){:=}GL_2(\mathbb{C})/\sim$. Then the induced map

$$\tilde{h} : \tilde{GL}_2(\mathbb{C}) \to \mathrm{Aut}(\widehat{\mathbb{C}}) \tag{5}$$

is bijective.

The following is a well-known fact, which states that the composition of two Möbius transformations corresponds to the multiplication of their corresponding matrices.

LEMMA 2. Suppose that A_1, A_2 are the corresponding matrices of $\ell_1, \ell_2 \in \mathrm{Aut}(\widehat{\mathbb{C}})$, respectively. Then

$$\ell_1 \circ \ell_2 = h(A_1) \circ h(A_2) = h(A_1 A_2).$$

In what follows, we consider Eq.(1) and Eq.(3) where $f, g \in \mathrm{Aut}(\widehat{\mathbb{C}})$.

Using the induced mapping $\tilde{h}$ on the quotient space $\tilde{GL}_2(\mathbb{C})$, the following lemma gives a relation between solutions of the permutable functional equation (3) and a matrix equation

$$XA = AX, \quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}. \tag{6}$$

LEMMA 3. Let A be a corresponding matrix of $f \in \mathrm{Aut}(\widehat{\mathbb{C}})$. Then every bijective meromorphic solution of Eq.(3) is given by

$$\phi(z) = \tilde{h}(X),$$

where X is a solution of Eq.(6).

PROOF. It is known from the proof of [1, Theorem 11.1.1] that if ϕ is a bijective meromorphic function, then $\phi \in \mathrm{Aut}(\widehat{\mathbb{C}})$. So assume X is a corresponding matrix of ϕ. So we see that

$$\tilde{h}(X) \circ \tilde{h}(A) = \tilde{h}(A) \circ \tilde{h}(X)$$

It follows from Lemma 2 that

$$\tilde{h}(XA) = \tilde{h}(AX).$$

Since $\tilde{h}$ is bijective, the matrix equation $XA = AX$ on the quotient space $\tilde{GL}_2(\mathbb{C})$ is equivalent to Eq.(3). Thus every bijective meromorphic solution of Eq.(3) is given by

$$\phi(z) = \tilde{h}(X),$$

The proof is complete.

If $A \in GL_2(\mathbb{C})$, there exists a nonzero constant $\mu \in \mathbb{C}$ such that μA can be transformed into one of the three Jordan canonical forms

$$J_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ J_2 = \begin{bmatrix} \lambda & 0 \\ 0 & 1 \end{bmatrix},\ J_3 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \tag{7}$$

where $\lambda \in \mathbb{C}$ is a constant and $\lambda \neq 0, 1$. By Lemmas 2 and 3, it suffices to discuss the case that A can be transformed into one of the above three Jordan canonical forms.

For each $j = 1, 2, 3$ we let $\mathcal{A}_j$ denote the collection of matrices A which are similar to J_j.

LEMMA 4. Let Q be an invertible matrix such that $Q^{-1}AQ$ is of a Jordan canonical form. Then Eq.(3) has (i) all Möbius transformations as bijective meromorphic solutions when $A \in \mathcal{A}_1$; (ii) infinitely many bijective meromorphic solutions

$$\phi(z) = \tilde{h}\left(Q\begin{bmatrix} c_1 & 0 \\ 0 & c_2 \end{bmatrix}Q^{-1}\right), \tag{8}$$

where c_1, c_2 are both arbitrary nonzero complex numbers, when $A \in \mathcal{A}_2$; (iii) infinitely many bijective meromorphic solutions

$$\phi(z) = \tilde{h}\left(Q\begin{bmatrix} c_1 & c_2 \\ 0 & c_1 \end{bmatrix}Q^{-1}\right) \tag{9}$$

where c_1, c_2 are both arbitrary complex numbers and $c_1 \neq 0$, when $A \in \mathcal{A}_3$.

PROOF. Case (i). When $Q^{-1}AQ = J_1$, $f(z) = z$, which commutes with arbitrary functions.

Case (ii). $Q^{-1}AQ = J_2$. All matrices commuting with J_2 are of the form

$$C_2 = \begin{bmatrix} c_1 & 0 \\ 0 & c_2 \end{bmatrix},$$

where c_1, c_2 are both arbitrary constants such that C_2 is invertible, i.e., $c_1c_2 \neq 0$. It implies that all matrices commuting with $A = QJ_2Q^{-1}$ are of the form $P_2 = QC_2Q^{-1}$. Thus $X = P_2$ is the general solution of Eq.(6). By Lemma 3, the result (8) follows.

Case (iii). $Q^{-1}AQ = J_3$. All matrices commuting with J_3 are of the form

$$C_3 = \begin{bmatrix} c_1 & c_2 \\ 0 & c_1 \end{bmatrix},$$

where c_1, c_2 are both arbitrary constants such that C_3 is invertible, i.e., $c_1 \neq 0$. It implies that all matrices commuting with $A = QJ_3Q^{-1}$ are of the form $P_3 = QC_3Q^{-1}$. Thus $X = P_3$ is the general solution of Eq.(6). By Lemma 3, the result (9) follows.

We give an example to illustrate the use of the formulae obtained above.

EXAMPLE 1. Consider $f(z) = \frac{7z-3}{18z-8}$, which corresponds to

$$A = \begin{bmatrix} 7 & -3 \\ 18 & -8 \end{bmatrix}.$$

Choosing

$$Q = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix},$$

we have

$$Q^{-1}AQ = \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then $A \in \mathcal{A}_2$. From (8), we see that Eq.(3) has infinitely many bijective meromorphic solutions

$$\begin{aligned}\phi(z) &= \tilde{h}\left(Q\begin{bmatrix} c_1 & 0 \\ 0 & c_2\end{bmatrix}Q^{-1}\right)\\ &= \tilde{h}\left(\begin{bmatrix} -2c_1+3c_2 & c_1-c_2 \\ -6c_1+6c_2 & 3c_1-2c_2\end{bmatrix}\right)\\ &= \frac{(-2c_1+3c_2)z+c_1-c_2}{(-6c_1+6c_2)z+3c_1-2c_2}\\ &= \frac{(-2+3\mu)z+1-\mu}{(-6+6\mu)z+3-2\mu},\end{aligned}$$

where $\mu := c_2/c_1$ and $\mu \in \mathbb{C}$ is an arbitrary nonzero constant.

We consider normal forms of the Möbius group under $\mathrm{Aut}(\widehat{\mathbb{C}})$-conjugacy.

LEMMA 5. Under $\mathrm{Aut}(\widehat{\mathbb{C}})$-conjugacy, the Möbius group has only three normal forms:

(1) $e_1(z) = z$;
(2) $e_2(z) = \lambda z$, $\lambda \neq 0, 1$;
(3) $e_3(z) = z + 1$.

PROOF. Suppose that $\ell \in \mathrm{Aut}(\widehat{\mathbb{C}})$ corresponds to a matrix A which can be transformed into one of the three Jordan canonical forms in (7). So assume that the Jordan canonical form of A is J_i for some i. Then there exists a nonsingular 2×2 matrix Q such that $J_i = Q^{-1}AQ$. By Lemma 2, we have

$$\begin{aligned}e_i(z) &= \tilde{h}(J_i) = \tilde{h}(Q^{-1}AQ) = \tilde{h}(Q^{-1})\circ\tilde{h}(A)\circ\tilde{h}(Q)\\ &= \tilde{h}^{-1}(Q)\circ\tilde{h}(A)\circ\tilde{h}(Q) = \tilde{h}^{-1}(Q)\circ\ell(z)\circ\tilde{h}(Q).\end{aligned}$$

Since $\tilde{h}^{-1}(Q), \tilde{h}(Q) \in \mathrm{Aut}(\widehat{\mathbb{C}})$, $\ell(z)$ is conjugate to $e_i(z)$ under $\mathrm{Aut}(\widehat{\mathbb{C}})$-conjugacy.

Obviously the three normal forms above are not conjugate to each other under $\mathrm{Aut}(\widehat{\mathbb{C}})$-conjugacy. This completes the proof.

3 Conjugacy Equation

We have the following main result.

THEOREM 1. Suppose that $f, g \in \mathrm{Aut}(\widehat{\mathbb{C}})$. Then there exists a bijective meromorphic solution of Eq.(1) if and only if f and g have the same normal form. Moreover, suppose $\varphi_j \in \mathrm{Aut}(\widehat{\mathbb{C}}), j = 1, 2$ satisfy

$$\varphi_1^{-1}\circ f\circ\varphi_1 = \varphi_2^{-1}\circ g\circ\varphi_2 = e_i \qquad \text{for some } i. \tag{10}$$

Then every bijective meromorphic solutions of Eq.(1) is given by

$$\varphi = \varphi_2\circ\phi\circ\varphi_1^{-1},$$

where ϕ is a bijective meromorphic solution of the equation $\phi\circ e_i = e_i\circ\phi$.

PROOF. By (10), we have $\varphi_1^{-1} \circ f = e_i \circ \varphi_1^{-1}$ and $g \circ \varphi_2 = \varphi_2 \circ e_i$. For any bijective meromorphic solution solution ϕ of the equation $\phi \circ e_i = e_i \circ \phi$, let $\varphi = \varphi_2 \circ \phi \circ \varphi_1^{-1}$. Then

$$\begin{aligned} \varphi \circ f &= \varphi_2 \circ \phi \circ \varphi_1^{-1} \circ f \\ &= \varphi_2 \circ \phi \circ e_i \circ \varphi_1^{-1} \\ &= \varphi_2 \circ e_i \circ \phi \circ \varphi_1^{-1} \\ &= g \circ \varphi_2 \circ \phi \circ \varphi_1^{-1} \\ &= g \circ \varphi. \end{aligned}$$

Conversely, if there exists a bijective meromorphic solution φ of Eq.(1), then $f = \varphi^{-1} \circ g \circ \varphi$. Therefore f and g have the same normal form. This completes the proof.

EXAMPLE 2. Consider Eq.(2). Choosing $\varphi_0(z) = 1/z$, we have $\varphi_0 \circ f = g \circ \varphi_0$. By Lemma 1, it suffices to solve $\phi \circ f = f \circ \phi$. In fact, the general solution of $\phi(z+1) = \phi(z) + 1$ is given by $\phi(z) = \Theta(z) + z$, where $\Theta(z) = \Theta(z+1)$ is an arbitrary periodic function with unit period. By Lemma 1, the general solution of Eq.(2) is given by

$$\varphi(z) = \varphi_0 \circ \phi(z) = \frac{1}{\Theta(z) + z}.$$

Remark that Eq.(2) was discussed in [2, pp.390-391, Theorem 10.1.2]. Their result shows that the only convex or concave solutions $\varphi : (0, \infty) \to \mathbb{R}$ of Eq.(2) are $\varphi = 0$ and $\varphi(x) = 1/(x + d)$, where $d \in \mathbb{R}^+$ is an arbitrary constant. Clearly they are two particular solutions.

EXAMPLE 3. Consider the functional equation

$$\varphi\left(\frac{31z - 12}{70z - 27}\right) = \frac{43\varphi(z) - 24}{70\varphi(z) - 39}. \tag{11}$$

Put

$$f(z) = \frac{31z - 12}{70z - 27}, \quad g(z) = \frac{43z - 24}{70z - 39}.$$

By Lemma 5, choose

$$\varphi_1(z) = \frac{3z + 2}{7z + 5}, \quad \varphi_2(z) = \frac{3z + 4}{5z + 7}.$$

Then $\varphi_1^{-1} \circ f \circ \varphi_1(z) = \varphi_2^{-1} \circ g \circ \varphi_2(z) = 3z$. From Lemma 4, all bijective meromorphic solutions of $\phi(3z) = 3\phi(z)$ are given by $\phi(z) = \mu z$, where $\mu \in \mathbb{C}$ is an arbitrary nonzero constant. By Theorem 1, all bijective meromorphic solutions of Eq.(11) are given by

$$\varphi(z) = \varphi_2 \circ \phi \circ \varphi_1^{-1}(z) = \frac{(15\mu - 28)z - 6\mu + 12}{(25\mu - 49)z - 10\mu + 21}.$$

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