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# Values Of Sequences Of Purely Periodic Functions*

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**Abstract**

Let $f : \mathbb{R} \to \mathbb{R}$ be a nonzero purely periodic function with least period $P$. For $\theta$ $(\neq 0)$ and $b$ both in the interval $[0, P)$, it is shown that when $n$ runs through the nonnegative integers, the nonzero sequence $(f(n\theta + b))$ is purely periodic if $\theta$ is a rational multiple of $P$. While if $\theta$ is not a rational multiple of $P$ and $f$ is continuous, the sequence $(f(n\theta + b))$ is dense in the range of $f$. Moreover, under appropriate conditions, a sequence of the form $\left(\sum_{r=1}^{d} \alpha_r f\left(Pns_r/t_r + b_r\right)\right)$ with rationals $s_r/t_r$ is shown to be purely periodic with least period being the least common multiple of $t_1, \ldots, t_d$.

## 1 Introduction

Let $f : \mathbb{R} \to \mathbb{R}$ be a nonzero purely periodic function with least period $P$. For $\theta$ $(\neq 0)$ and $b$ both belonging to $[0, P)$, the distribution of the values of $f(n\theta + b)$ for certain specific functions $f$, as $n$ varies over the nonnegative integers $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, has been of much interest recently. To mention one instance, in the course of establishing the decidability of the positivity problem for binary sequences, Halava, Harju and Hirvensalo have shown that the inequality $\cos(n\theta + b) \geq 0$ cannot hold for all $n \in \mathbb{N}_0$ ([2, Lemma 5], [3]). In [4], it is shown that the sequence $(\cos(n\theta + b))_{n\in\mathbb{N}_0}$ takes infinitely many positive and negative values.

Here, we derive similar information about the values of $f(n\theta + b)$ for general $f$. First, when $\theta$ is a rational multiple of $P$, we show that the sequence $(f(n\theta + b))_{n\in\mathbb{N}_0}$ is purely periodic. When $\theta$ is not a rational multiple of $P$ and $f$ is continuous, through the use of a theorem of Kronecker in Diophantine approximation, we show that the set of values of $f(n\theta+b)$ is dense in the range of $f$. Next, we deal with a linear combination of $f(n\theta + b)$'s with rational $\theta$. Let $\theta_1 = s_1/t_1, \ldots, \theta_d = s_d/t_d$ be rational numbers, written in reduced fractions, in the open interval $(0, 1)$, and let $\alpha_i \in \mathbb{R},\ b_i \in [0, P)$ be such that the purely periodic sequence $(u_n)_{n\in\mathbb{N}_0}$ defined via

$$u_n = \sum_{i=1}^{d} \alpha_i f(P\theta_i n + b_i)$$

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is not identically zero. When $f$ is the cosine function, the least period of $(u_n)_{n\in\mathbb{N}_0}$ is shown in [4] to be equal to $T$, the least common multiple of the denominators of $t_1,\dots,t_d$. This is, however, not true in general as seen in the following example.

Take $f(x)=\cos^2 x$, a function with least period $\pi$. The sequence

$$(f(\pi n/2)+f(\pi n/2+\pi/2))_{n\in\mathbb{N}_0}$$

is constant and so is purely periodic of period $1\neq 2=\text{l.c.m.}(2,2)$. It is thus natural to ask when the least period of $(u_n)_{n\in\mathbb{N}_0}$ is equal to $T$. Under the condition

$$\sum_{n=0}^{\tau-1} f\left(\frac{Pn}{\tau}+\beta\right)=0, \tag{1}$$

for all $\tau\in\mathbb{N}$, $\tau\geq 2$ and all $\beta\in[0,P)$, we show that the least period of the sequence $(u_n)_{n\in\mathbb{N}_0}$ is exactly equal to $T$. The same conclusion holds if

$$\sum_{n=0}^{T-1} f\left(\frac{Pn}{\tau}+\beta\right)=0 \tag{2}$$

for all $\tau\in\mathbb{N}$, $\tau\geq 2$, $\tau\mid T$ and all $\beta\in[0,P)$. Note that (1) is a condition on $f$ itself whereas (2) depends on both the function $f$ and the rational numbers $\theta_1,\theta_2,\dots,\theta_d$. In the last section, examples are given to illustrate the use of our main results.

## 2 A Single Function

We begin with some consequences of the two conditions (1) and (2).

LEMMA 1. Let $f:\mathbb{R}\to\mathbb{R}$ be a nonzero purely periodic function with least period $P$ and let $b\in[0,P)$. If $s,t\in\mathbb{N}$, $\gcd(s,t)=1$, then

$$\sum_{n=0}^{t-1} f\left(\frac{Pn}{t}+b\right)=\sum_{n=0}^{t-1} f\left(\frac{Psn}{t}+b\right).$$

PROOF. Since $\gcd(s,t)=1$, the set $\{0,s,2s,\dots,s(t-1)\}$ is a complete residue system modulo $t$. By the periodicity of $f$, it follows that

$$\{f(b),f(P/t+b),\dots,f((t-1)P/t+b)\}=\{f(b),f(sP/t+b),\dots,f((t-1)sP/t+b)\}$$

for any $s\in\mathbb{N}$ with $\gcd(s,t)=1$, and the result follows.

We remark that if (1) holds for some fixed $\tau=t$ and fixed $\beta=b$, then Lemma 1 and the periodicity of $f$ together show that

$$\sum_{n=0}^{kt-1} f\left(\frac{Psn}{t}+b\right)=0 \tag{3}$$

for any $k \in \mathbb{N}$.

LEMMA 2. Let $f : \mathbb{R} \to \mathbb{R}$ be a nonzero purely periodic function with least period $P$, $b \in [0, P)$ and $t, T \in \mathbb{N}$. If

$$\sum_{n=0}^{T-1} f\left(\frac{Pn}{t} + b\right) = 0$$

holds for some fixed $t \geq 2$ and $t \mid T$, then

$$\sum_{n=0}^{T-1} f\left(\frac{Psn}{t} + b\right) = 0 \tag{4}$$

holds for all $s \in \mathbb{N}$ with $\gcd(s, t) = 1$.

PROOF. Let $T = tm \quad (m \in \mathbb{N})$. By the periodicity of $f$, we have

$$m \sum_{n=0}^{t-1} f\left(\frac{Pn}{t} + b\right) = \sum_{n=0}^{T-1} f\left(\frac{Pn}{t} + b\right) = 0.$$

Lemma 1 thus implies that $\sum_{n=0}^{t-1} f\left(Psn/t + b\right) = 0$ where $s \in \mathbb{N}$ with $\gcd(s, t) = 1$. The desired result follows at once by the preceding remark.

We now state and prove our first main theorem.

THEOREM 3. Let $f : \mathbb{R} \to \mathbb{R}$ be a nonzero purely periodic function with least period $P$ and let $b \in [0, P)$.

(a) If $\theta = sP/t$ $(s \in \mathbb{Z},\ t \in \mathbb{N},\ \gcd(s, t) = 1)$ is a rational multiple of $P$, then the sequence $(f(n\theta + b))_{n \in \mathbb{N}_0}$ is purely periodic.

Moreover, if $f$ satisfies the condition

$$\sum_{n=0}^{\tau-1} f\left(\frac{Pn}{\tau} + \beta\right) = 0, \tag{5}$$

for all $\tau \in \mathbb{N}$, $\tau \geq 2$ and all $\beta \in [0, P)$, and the sequence $(f(n\theta + \beta))$ is not identically zero, then its least period is equal to $t$.

(b) If $\theta$ is not a rational multiple of $P$ and $f$ is continuous, then as $n$ varies over $\mathbb{N}_0$, the set of values of $f(n\theta + \beta)$ is dense in the range of $f$.

PROOF. (a) Let $u_n = f(nsP/t + b)$. For each $k, m \in \mathbb{N}_0$, since

$$u_{kt+m} = f\left(\frac{(kt + m)sP}{t} + b\right) = f\left(\frac{msP}{t} + b\right) = u_m,$$

the sequence $(u_n)_{n \in \mathbb{N}_0}$ is purely periodic with period $t$.

Let $\ell$ be the least period of $(u_n)_{n\in\mathbb{N}_0}$ and assume that $\ell < t,\ t \geq 2$. Being the least period, it is easily checked that $\ell \mid t$ and since $\gcd(t/\ell, s) = 1$, the validity of (5) implies that of (3) which in turn shows that

$$tu_j = \sum_{k=0}^{t-1} u_{k\ell+j} = \sum_{k=0}^{t-1} f\left(\frac{(k\ell+j)Ps}{t} + b\right) = \sum_{k=0}^{t-1} f\left(\frac{kPs}{t/\ell} + \frac{jPs}{t} + b\right) = 0$$

for any $j \in \mathbb{N}_0$, contradicting the fact that the sequence is nonzero.

(b) Assume now that $\theta$ is not a rational multiple of $P$, and put $\theta = \nu P$ so that $\nu \in \mathbb{R} \setminus \mathbb{Q}$. Writing

$$\nu = [\nu] + \xi,$$

where $[\nu]$ denotes its integer part, and $\xi := \{\nu\} \in (0, 1)$ denotes its fractional part which must be irrational, for $n \in \mathbb{N}_0$ we have

$$f(n\theta + b) = f(n\nu P + b) = f(n\xi P + b) = f(\{n\xi\}P + b).$$

Since $\xi$ is irrational, by Kronecker's approximation theorem, [5, Cor. 6.4, p. 75], the set $\{\{n\xi\};\ n \in \mathbb{N}_0\}$ is dense in $[0, 1]$. Consequently, the set $\{\{n\xi\}P + b\ ;\ n \in \mathbb{N}_0\}$ is dense in the interval $[b, P + b)$. By the continuity and periodicity of $f$, the set $\{f(\{n\xi\}P + b)\ ;\ n \in \mathbb{N}_0\}$ is thus dense in the range of $f$.

Regarding the result about least period in the last part of Theorem 3(a), apart from the condition (5), there are also other condition(s) ensuring the least period being $t$. We give another one in the next theorem.

THEOREM 4. Let $f : \mathbb{R} \to \mathbb{R}$ be a nonzero purely periodic function with least period $P,\ b \in [0, P)$ and let $s, t \in \mathbb{N}$ with $\gcd(s, t) = 1$. Assume that for each $y$ in the range of $f$, there are only finitely many, say $x_1, \ldots, x_R \in [0, P)$ such

$$f(x_i) = y \quad (i = 1, \ldots, R) \tag{6}$$

and that $x_i - x_j$ is not a rational multiple of $P$ for all $i, j \in \{1, \ldots, R\},\ i \neq j$. If the sequence $(f(nsP/t + b))_{n\in\mathbb{N}_0}$ is not identically zero, then it is purely periodic with least period equal to $t$.

PROOF. That the sequence is purely periodic with period $t$ follows directly from the proof of the first part of Theorem 3(a). To show that $t$ is the least period, suppose that its least period $t'(\in \mathbb{N})$ is $< t,\ t \geq 2$. Thus, $f(t'sP/t + b) = f(b)$. Writing

$$st'/t = [st'/t] + \{st'/t\},$$

where $[st'/t]$ denotes its integer part, and $\{st'/t\} \in (0, 1) \cap \mathbb{Q}$ denotes its fractional part, we obtain

$$f\left(\{st'/t\}P + b\right) = f(b).$$

By hypothesis, let $f$ take the value $f(b)$ at $x \in \{b = b_1, \ldots, b_R\} \subset [0, P)$. Since $\{st'/t\}P + b \in (0, 2P)$, we deduce that

$$\{st'/t\}P + b \in \{b_1, \ldots, b_R, P + b_1 \ldots, P + b_R\},$$

which is a contradiction because $\{t's/t\}P \notin \{0, P\}$ and each $b_i - b_j$ $(i \neq j)$ is not a rational multiple of $P$.

# 3 Linear Combination of Functions

Our next theorem deals with a linear combination of $f(n\theta+b)$'s, generalizing Theorem 3(a).

THEOREM 5. Let $f:\mathbb{R}\to\mathbb{R}$ be a nonzero purely periodic function with least period $P$. Assume $f$ satisfies the condition

$$\sum_{n=0}^{\tau-1} f\left(\frac{Pn}{\tau}+\beta\right)=0, \tag{7}$$

for all $\tau\in\mathbb{N}$, $\tau\geq 2$ and all $\beta\in[0,P)$. For $d\in\mathbb{N}$, $r\in\{1,2,\ldots,d\}$, let $s_r/t_r$ $(s_r,t_r\in\mathbb{N},\ \gcd(s_r,t_r)=1)$ be rational numbers in the unit interval $(0,1)$, and let $\alpha_r\in\mathbb{R}\setminus\{0\}$, $b_r\in[0,P)$. If the purely periodic, non-identically zero sequence $(u_n)_{n\in\mathbb{N}_0}$ is defined by

$$u_n=\sum_{r=1}^{d}\alpha_r f\left(\frac{Pns_r}{t_r}+b_r\right),$$

then its least period is equal to $T:=l.c.m.(t_1,t_2,\ldots,t_d)$.

PROOF. Clearly, the sequence $(u_n)$ is purely periodic with period $T$. Let $\ell$ be its least period so that $\ell\mid T$. We proceed to prove the theorem by induction on $d$. The case $d=1$ is contained in Theorem 3(a). Assume now that $d\geq 2$ and that the theorem holds up to $d-1$.

If $t_r\mid\ell$ for every $r\in\{1,2,\ldots,d\}$, then $\ell=T$ and we are done.

Suppose then that not all of the $t_r$'s divide $\ell$. We treat first the case where there are some $t_r$'s that divide $\ell$ and some $t_r$'s that do not divide $\ell$. Without loss of generality, assume that $\ell$ is divisible by $t_1,t_2,\ldots,t_m$, but not divisible by $t_{m+1},t_{m+2},\ldots,t_d$ for some $m\in\{1,2,\ldots,d-1\}$. For $k,j\in\mathbb{N}_0$, we have

$$\begin{aligned}
Tu_j=\sum_{k=0}^{T-1}u_{k\ell+j}&=\sum_{k=0}^{T-1}\sum_{r=1}^{d}\alpha_r f\left(\frac{Ps_r}{t_r}(k\ell+j)+b_r\right)\\
&=\sum_{r=1}^{m}\alpha_r\sum_{k=0}^{T-1}f\left(\frac{Ps_r}{t_r}j+b_r\right)+\sum_{r=m+1}^{d}\alpha_r\sum_{k=0}^{T-1}f\left(\frac{Ps_r}{t_r}(k\ell+j)+b_r\right)\\
&=T\sum_{r=1}^{m}\alpha_r f\left(\frac{Ps_r}{t_r}j+b_r\right)+\sum_{r=m+1}^{d}\alpha_r\sum_{k=0}^{T-1}f\left(\frac{Ps_r}{t_r}(k\ell+j)+b_r\right).
\end{aligned}\tag{8}$$

In the second sum on the right hand side of the last expression, writing

$$\sum_{k=0}^{T-1}f\left(\frac{Ps_r}{t_r}(k\ell+j)+b_r\right)=\sum_{k=0}^{T-1}f\left(\frac{Ps_rk\ell'_r}{t'_r}+\gamma_r\right), \tag{9}$$

where $\gcd(\ell'_r,t'_r)=1$, $t'_r\mid T$ (as $t_r\mid T$) and $\gamma_r$ is equal to $\frac{Ps_rj}{t_r}+b_r$ reduced mod $P$. The condition (7) implies that (3) holds which in turn implies

$$\sum_{k=0}^{T-1}f\left(\frac{Ps_rk\ell'_r}{t'_r}+\gamma_r\right)=0.$$

Thus, (8) gives $u_j = \sum_{r=1}^{m} \alpha_r f\left(\frac{Ps_r}{t_r} j + b_r\right)$, which shows that $(u_j)_{j\in\mathbb{N}_0}$ has $m \le d-1$ terms, the induction hypothesis finishes this case.

If none of the $t_r$'s divides $\ell$, the same arguments as in the last steps show that $u_n \equiv 0$, which is untenable.

Apart from (1), there are other sufficient conditions for the least period to be $T$. We provide one in the next theorem.

THEOREM 6. Let $f : \mathbb{R} \to \mathbb{R}$ be a nonzero purely periodic function with least period $P$. For $d \in \mathbb{N}$, $r \in \{1, 2, \ldots, d\}$, let $s_r/t_r$ ($s_r, t_r \in \mathbb{N}$, $\gcd(s_r, t_r) = 1$) be rational numbers in the unit interval $(0, 1)$, $\alpha_r \in \mathbb{R} \setminus \{0\}$, $b_r \in [0, P)$ and $T := l.c.m.(t_1, t_2, \ldots, t_d)$. Assume $f$ satisfies the condition

$$\sum_{n=0}^{T-1} f\left(\frac{Pn}{\tau} + \beta\right) = 0 \tag{10}$$

for all $\tau \in \mathbb{N}$, $\tau \ge 2$, $\tau \mid T$ and all $\beta \in [0, P)$. If the purely periodic, non-identically zero sequence $(u_n)_{n\in\mathbb{N}_0}$ is defined by

$$u_n = \sum_{r=1}^{d} \alpha_r f\left(\frac{Pns_r}{t_r} + b_r\right),$$

then its least period is equal to $T$.

PROOF. Proceed exactly as in the proof of Theorem 5 up to (9). Now use the condition (10) and Lemma 2 to deduce that (4) holds which in turn gives

$$\sum_{k=0}^{T-1} f\left(\frac{Ps_r k\ell'_r}{t'_r} + \gamma_r\right) = 0.$$

The same conclusion as at the end of the proof of Theorem 5 finishes the proof.

## 4 Examples

In this section, we work out some examples to illustrate the use and the necessity of the conditions of our results. The first shows that analogous result in [4] is a special case when the condition (1) is fulfilled.

EXAMPLE 1. Let $d \in \mathbb{N}$ ; $\rho_1, \rho_2 \in \mathbb{R}$ ; $\lambda \in \mathbb{R}^+ := \{x \in \mathbb{R} ; x > 0\}$ and $\gamma_1, \gamma_2 \in [0, 2\pi/\lambda)$. Consider

$$f(x) = \rho_1 \cos(\lambda x + \gamma_1) + \rho_2 \sin(\lambda x + \gamma_2) \not\equiv 0,$$

which is a periodic continuous function with least period $2\pi/\lambda$. Define the sequence $(u_n)_{n\in\mathbb{N}_0}$ by

$$u_n = \sum_{r=1}^{d} \alpha_r f\left(\frac{2\pi ns_r}{\lambda t_r} + b_r\right),$$

where $s_r, t_r \in \mathbb{N},\ t_r \geq 2,\ \gcd(s_r, t_r) = 1,\ \alpha_r \in \mathbb{R} \setminus \{0\}\ \ (r = 1, \ldots, d),\ b_r \in [0, 2\pi/\lambda)$ and $T := l.c.m.(t_1, t_2, \ldots, t_d)$. By Theorem 5, to see that the sequence $(u_n)_{n\in\mathbb{N}_0}$ has least period $T$, we need only check that $f$ satisfies the condition (1), i.e.,

$$\sum_{n=0}^{\tau-1} \left\{ \rho_1 \cos\left(\frac{2n\pi}{\tau} + \lambda\beta + \gamma_1\right) + \rho_2 \sin\left(\frac{2n\pi}{\tau} + \lambda\beta + \gamma_1\right) \right\} = 0 \tag{11}$$

for all $\tau \geq 2$, and $\beta \in [0, 2\pi/\lambda)$. Since

$$\sum_{n=0}^{\tau-1} \cos\left(\frac{2n\pi}{\tau} + \lambda\beta + \gamma_1\right) = \sum_{n=0}^{\tau-1} \Re\left\{\exp\ i\left(\frac{2n\pi}{\tau} + \lambda\beta + \gamma_1\right)\right\} = 0,$$

and similarly, $\sum_{n=0}^{\tau-1} \sin\left(\frac{2n\pi}{\tau} + \lambda\beta + \gamma_1\right) = 0$, the condition (11) is verified.

Moreover, the function $f$ in Example 1 satisfies the conditions of both Theorems 3 and 4.

Our second example provides an uncountable class of well-behaved functions satisfying the condition (2).

EXAMPLE 2. Let $f : \mathbb{R} \to \mathbb{R}$ be continuously differentiable and periodic with least period $P$. Then, $f$ possesses a Fourier series of the form ([1, Theorem 10.3])

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} \left( a_m \cos\frac{2m\pi x}{P} + b_m \sin\frac{2m\pi x}{P} \right) \qquad (x \in \mathbb{R}). \tag{12}$$

Define the sequence $(u_n)_{n\in\mathbb{N}_0}$ via

$$u_n = \sum_{r=1}^{d} \alpha_r f\left(\frac{Pns_r}{t_r} + B_r\right),$$

where $d, s_r, t_r \in \mathbb{N},\ t_r \geq 2,\ \gcd(s_r, t_r) = 1,\ \alpha_r \in \mathbb{R} \setminus \{0\},\ B_r \in [0, P)$. Let $T := l.c.m.(t_1, t_2, \ldots, t_d)$. Consider the set, $S$, of such functions $f$ for which their Fourier coefficients are subject to the conditions

$$a_0 = a_m = b_m = 0 \ \text{ for all } m \mid T,\ m \in \mathbb{N}.$$

For $f \in S$, the condition (2) is fulfilled, and so Theorem 6 confirms that the least period of the sequence $(u_n)_{n\in\mathbb{N}_0}$ is $T$.

The next illustration gives an example where the condition (1) does not hold but the condition (2) does.

EXAMPLE 3. Let $A$ and $P$ be positive real numbers and consider

$$f(x) = \begin{cases} 4Ax/P & if \ \ 0 \leq x < P/4, \\ -4Ax/P + 2A & if \ \ P/4 \leq x < 3P/4, \\ 4Ax/P - 4A & if \ \ 3P/4 \leq x < P \end{cases}$$

The function $f$ is continued throughout $\mathbb{R}$ by periodicity with least period $P$. For $\alpha_1, \alpha_2 \in \mathbb{R} \setminus \{0\}$ and $b_1, b_2 \in [0, P)$, define the sequence $(u_n)_{n\in\mathbb{N}_0}$ by

$$u_n = \alpha_1 f\left(\frac{Pn}{2} + b_1\right) + \alpha_2 f\left(\frac{3Pn}{4} + b_2\right).$$

Since $\sum_{n=0}^{2} f(Pn/3 + P/2) \neq 0$, the function $f$ does not satisfy the condition (1). To verify whether the condition (2) holds, we need only prove that $\sum_{n=0}^{3} f\left(Pn/t + \beta\right) = 0$ when $t = 2$ or $t = 4$ and $\beta \in [0, P)$. For $t = 2$, we have

$$\sum_{n=0}^{3} f\left(\frac{Pn}{2} + \beta\right) = 2\left(f(\beta) + f\left(\frac{P}{2} + \beta\right)\right) = 0.$$

Applying this to the case $t = 4$, we obtain

$$\sum_{n=0}^{3} f\left(\frac{Pn}{4} + \beta\right) = \left(f(\beta) + f\left(\frac{P}{2} + \beta\right)\right) + \left(f\left(\frac{P}{4} + \beta\right) + f\left(\frac{3P}{4} + \beta\right)\right) = 0.$$

By Theorem 6, the least period of the sequence $(u_n)_{n\in\mathbb{N}_0}$ is equal to l.c.m.$(2, 4) = 4$.

Observe that the function $f$ in Example 3 satisfies neither the condition (5) of Theorem 3(a) nor the finite value-attainment condition related to (6) of Theorem 4.

Our final illustration provides an example of a wildly behaved non-continuous function.

EXAMPLE 4. Let

$$f(x) = \begin{cases} \sin 2\pi x & if \;\; x \in \mathbb{Q}, \\ 0 & if \;\; x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Clearly, $f(x + 1) = f(x) \quad (x \in \mathbb{R})$. To show that 1 is the least period of $f$, we first note that any irrational number $\gamma \in \mathbb{R} \setminus \mathbb{Q}$ cannot be a period of $f$ because if we take $x = 1/3 - \gamma$, then $f(x + \gamma) = \sin 2\pi/3 \neq 0 = f(x)$. Next, let $\alpha \in \mathbb{Q}$ and $0 < \alpha < 1$ and suppose that $f(x + \alpha) = f(x)$. Then, for each $x \in \mathbb{Q}$, we get

$$\sin(2\pi x + 2\pi\alpha) = \sin 2\pi x. \tag{13}$$

Taking $x = 0$, we must have $\alpha = 1/2$. Substituting $\alpha = 1/2$ into (13) shows $\sin 2\pi x = 0$ for each $x \in \mathbb{Q}$, which is a contradiction. We conclude then that 1 is the least period of $f$.

Consider the sequence $(u_n)_{n\in\mathbb{N}_0}$ defined by

$$u_n = \sum_{r=1}^{d} \alpha_r f\left(\frac{ns_r}{t_r} + b_r\right),$$

where $s_r, t_r \in \mathbb{N}$, $t_r \geq 2$, $\gcd(s_r, t_r) = 1$, $\alpha_r \in \mathbb{R} \setminus \{0\}$, $b_r \in [0, 1)$. Let $T := l.c.m.(t_1, t_2, \ldots, t_d)$. To see that the sequence $(u_n)_{n\in\mathbb{N}_0}$ has the least period $T$, we verify the condition (1), i.e., $\sum_{n=0}^{\tau-1} f\left(n/\tau+\beta\right) = 0$, for all $\tau \in \mathbb{N}$, $\tau \geq 2$, and all $\beta \in [0,1)$. If $\beta \in \mathbb{Q}$, then

$$\sum_{n=0}^{\tau-1} f\left(\frac{n}{\tau}+\beta\right) = \sum_{n=0}^{\tau-1} \sin\left(\frac{2\pi n}{\tau}+2\pi\beta\right) = \sum_{n=0}^{\tau-1} \Im\left(\exp i\left(\frac{2\pi n}{\tau}+2\pi\beta\right)\right) = 0.$$

If $\beta \in \mathbb{R} \setminus \mathbb{Q}$, then by the definition of $f$, each term $f\left(n/\tau+\beta\right)$ is 0. Thus,

$$\sum_{n=0}^{\tau-1} f\left(n/\tau+\beta\right) = 0.$$

By Theorem 5, the least period of $(u_n)_{n\in\mathbb{N}_0}$ is equal to $T$.

This last example also gives us a function satisfying the condition (5) of Theorem 3(a) but does not satisfy the condition (6) of Theorem 4.

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