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Existence Of Solution For Third-Order m-Point Boundary Value Problem*

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Abstract

In this paper, we consider the following nonlinear third-order m-point boundary value problem

$$\begin{cases} u'''(t) + f(t, u(t), u'(t), u''(t)) = 0, \ t \in [0, 1], \\ u(0) = \Sigma_{i=1}^{m-2} a_i u(\eta_i), \ u'(1) = u''(0) = 0, \end{cases}$$

where $0 < \eta_1 < \eta_2 < \cdots < \eta_{m-2} < 1$, $a_i \geq 0$ $(i = 1, 2, \cdots, m-2)$ and $\Sigma_{i=1}^{m-2} a_i < 1$. By imposing some conditions on the nonlinear term f, we construct a lower solution and an upper solution and prove the existence of solution to the above boundary value problem. Our main tools are upper and lower solution method and Schauder fixed point theorem.

1 Introduction

Third-order differential equations arise in a variety of different areas of applied mathematics and physics, e.g., in the deflection of a curved beam having a constant or varying cross section, a three-layer beam, electromagnetic waves or gravity driven flows and so on [5]. Recently, third-order two-point or three-point boundary value problems (BVPs for short) have received much attention from many authors, see [1, 2, 4, 6, 7, 9, 11] and the references therein. Although there are many excellent results on third-order two-point or three-point BVPs, few works have been done for more general third-order m-point BVPs [3, 10].

Motivated greatly by [2, 8], in this paper, we investigate the following nonlinear third-order m-point BVP

$$\begin{cases} u'''(t) + f(t, u(t), u'(t), u''(t)) = 0, \ t \in [0, 1], \\ u(0) = \Sigma_{i=1}^{m-2} a_i u(\eta_i), \ u'(1) = u''(0) = 0. \end{cases} \quad (1)$$

Throughout this paper, we always assume that $0 < \eta_1 < \eta_2 < \cdots < \eta_{m-2} < 1$, $a_i \geq 0$ $(i = 1, 2, ..., m-2)$, $\Sigma_{i=1}^{m-2} a_i < 1$ and $f : [0, 1] \times R^3 \to R$ is continuous. By imposing some conditions on the nonlinear term f, we construct a lower solution and an upper solution and prove the existence of solution to the BVP (1). Our main tools are upper and lower solution method and Schauder fixed point theorem.

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2 Preliminary

In this section, we will present some fundamental definitions and lemmas.

DEFINITION 1. If $x \in C^3[0,1]$ satisfies

$$\begin{cases} x'''(t) + f(t, x(t), x'(t), x''(t)) \geq 0, \ t \in [0,1], \\ x(0) = \sum_{i=1}^{m-2} a_i x(\eta_i), \ x'(1) = 0, \ x''(0) \geq 0, \end{cases} \tag{2}$$

then x is called a lower solution of the BVP (1).

DEFINITION 2. If $y \in C^3[0,1]$ satisfies

$$\begin{cases} y'''(t) + f(t, y(t), y'(t), y''(t)) \leq 0, \ t \in [0,1], \\ y(0) = \sum_{i=1}^{m-2} a_i y(\eta_i), \ y'(1) = 0, \ y''(0) \leq 0, \end{cases} \tag{3}$$

then y is called an upper solution of the BVP (1).

LEMMA 1. Let $\sum_{i=1}^{m-2} a_i \neq 1$. Then for any $h \in C[0,1]$, the second-order m-point BVP

$$\begin{cases} -u''(t) = h(t), \ t \in [0,1], \\ u(0) = \sum_{i=1}^{m-2} a_i u(\eta_i), \ u'(1) = 0 \end{cases} \tag{4}$$

has a unique solution

$$u(t) = \int_0^1 G(t,s) h(s) ds, \ t \in [0,1],$$

where

$$G(t,s) = K(t,s) + \frac{1}{1 - \sum_{i=1}^{m-2} a_i} \sum_{i=1}^{m-2} a_i K(\eta_i, s), \ (t,s) \in [0,1] \times [0,1],$$

here

$$K(t,s) = \begin{cases} s, \ 0 \leq s \leq t \leq 1, \\ t, \ 0 \leq t \leq s \leq 1 \end{cases}$$

is the Green's function of the second-order two-point BVP

$$\begin{cases} -u''(t) = 0, \ t \in [0,1], \\ u(0) = u'(1) = 0. \end{cases}$$

PROOF. If u is a solution of the BVP (4), then we may suppose that

$$u(t) = \int_0^1 K(t,s) h(s) ds + c_1 t + c_2, \ t \in [0,1].$$

By the boundary conditions in (4), we know that

$$c_1 = 0 \text{ and } c_2 = \frac{1}{1 - \sum_{i=1}^{m-2} a_i} \sum_{i=1}^{m-2} a_i \int_0^1 K(\eta_i, s) h(s) ds.$$

Therefore, the unique solution of the BVP (4)

$$\begin{aligned} u(t) &= \int_0^1 K(t,s)h(s)ds + \frac{1}{1-\sum_{i=1}^{m-2} a_i} \sum_{i=1}^{m-2} a_i \int_0^1 K(\eta_i, s)h(s)\,ds \\ &= \int_0^1 G(t,s)\,h(s)\,ds,\ \ t \in [0,1]. \end{aligned}$$

For $G(t,s)$, we have the following obvious result.

LEMMA 2. Let $a_i \geq 0$ $(i = 1, 2, \cdots, m-2)$ and $\Sigma_{i=1}^{m-2} a_i < 1$. Then $0 \leq G(t,s) \leq G(s,s)$ for $(t,s) \in [0,1] \times [0,1]$.

3 Main Result

For convenience, we let $\gamma = \int_0^1 G(s,s)\,sds$. Obviously, $\gamma > 0$. Our main result is the following theorem.

THEOREM 1. If there exist two constants M and N with $M \leq 0 \leq N$ and $N \geq -M$ such that

$$M \leq f(t,s,r,l) \leq 0 \text{ for } (t,s,r,l) \in [0,1] \times [\gamma M, 0] \times [\frac{M}{2}, 0] \times [0, -M] \tag{5}$$

and

$$0 \leq f(t,s,r,l) \leq N \text{ for } (t,s,r,l) \in [0,1] \times [0, \gamma N] \times [0, \frac{N}{2}] \times [-N, 0], \tag{6}$$

then the BVP (1) has a solution u_0, which satisfies

$$x(t) \leq u_0(t) \leq y(t) \text{ and } y''(t) \leq u_0''(t) \leq x''(t) \text{ for } t \in [0,1],$$

where $x(t) = M \int_0^1 G(t,s)sds$ and $y(t) = N \int_0^1 G(t,s)sds,\ t \in [0,1]$.

PROOF. Let $E = C[0,1]$ be equipped with the norm $\|v\|_\infty = \max_{t\in[0,1]} |v(t)|$ and

$$K = \{v \in E : v(t) \geq 0 \text{ for } t \in [0,1]\}.$$

Then K is a cone in E and (E, K) is an ordered Banach space.

Define operators A and $B : E \to E$ as follows:

$$(Av)(t) = \int_0^1 G(t,s)\,v(s)\,ds,\ \ t \in [0,1]$$

and

$$(Bv)(t) = \int_t^1 v(s)\,ds,\ \ t \in [0,1].$$

Obviously, A and B are increasing on E.

If we let $v(t) = -u''(t)$, $t \in [0,1]$, then the BVP (1) is equivalent to the following problem

$$\begin{cases} v'(t) = f(t, (Av)(t), (Bv)(t), -v(t)),\ \ t \in [0,1], \\ v(0) = 0. \end{cases} \tag{7}$$

Now, we divide our proof into four steps.

Step 1. We assert that x and y are, respectively, a lower and an upper solution of the BVP (1).

In fact, if we let $\alpha(t) = -x''(t) = Mt$ and $\beta(t) = -y''(t) = Nt,\ t \in [0,1]$, then it follows from (5) and (6) that

$$\begin{cases} \alpha'(t) - f(t,(A\alpha)(t),(B\alpha)(t),-\alpha(t)) \le 0,\ t \in [0,1]\,, \\ \alpha(0) = 0 \end{cases}$$

and

$$\begin{cases} \beta'(t) - f(t,(A\beta)(t),(B\beta)(t),-\beta(t)) \ge 0,\ t \in [0,1]\,, \\ \beta(0) = 0, \end{cases}$$

which implies that x and y are, respectively, a lower and an upper solution of the BVP (1).

Step 2. We consider the following auxiliary problem

$$\begin{cases} v'(t) = F(t,(Av)(t),(Bv)(t),-v(t)),\ t \in [0,1]\,, \\ v(0) = 0, \end{cases} \tag{8}$$

where

$$F(t,s,r,l) = \begin{cases} f_1(t,s,r,-\alpha(t)),\ l > -\alpha(t), \\ f_1(t,s,r,l),\ \ -\beta(t) \le l \le -\alpha(t), \\ f_1(t,s,r,-\beta(t)),\ l < -\beta(t), \end{cases}$$

$$f_1(t,s,r,l) = \begin{cases} f_2(t,s,(B\beta)(t),l),\ r > (B\beta)(t), \\ f_2(t,s,r,l),\ (B\alpha)(t) \le r \le (B\beta)(t), \\ f_2(t,s,(B\alpha)(t),l),\ r < (B\alpha)(t) \end{cases}$$

and

$$f_2(t,s,r,l) = \begin{cases} f(t,(A\beta)(t),r,l),\ s > (A\beta)(t), \\ f(t,s,r,l),\ (A\alpha)(t) \le s \le (A\beta)(t), \\ f(t,(A\alpha)(t),r,l),\ s < (A\alpha)(t). \end{cases}$$

If we define an operator $T : E \to E$ by

$$(Tv)(t) = \int_0^t F(s,(Av)(s),(Bv)(s),-v(s))ds,\ t \in [0,1],$$

then it is obvious that fixed points of T are solutions of the auxiliary problem (8). Now, we will apply Schauder fixed point theorem to prove that the operator T has a fixed point.

Let $B_N = \{v \in E :\ \|v\|_\infty \le N\}$. Then B_N is a bounded, closed and convex set. First, we prove that $T : B_N \to B_N$. For any $v \in B_N$, we consider the following four cases:

Case 1. $\beta(t) < v(t) \le N,\ t \in [0,1]$;

Case 2. $0 \le v(t) \le \beta(t),\ t \in [0,1]$;

Case 3. $\alpha(t) \le v(t) \le 0,\ t \in [0,1]$;

Case 4. $-N \leq v(t) < \alpha(t),\ t \in [0,1]$.
We can verify that

$$0 \leq F(t,(Av)(t),(Bv)(t),-v(t)) \leq N \text{ in Case 1 and Case 2} \tag{9}$$

and

$$M \leq F(t,(Av)(t),(Bv)(t),-v(t)) \leq 0 \text{ in Case 3 and Case 4.} \tag{10}$$

Since the proof is similar, we only consider Case 1. In this case, by the definition of F, we obtain

$$\begin{aligned} F(t,(Av)(t),(Bv)(t),-v(t)) &= f_1(t,(Av)(t),(Bv)(t),-\beta(t)) \\ &= f_2(t,(Av)(t),(B\beta)(t),-\beta(t)) \\ &= f(t,(A\beta)(t),(B\beta)(t),-\beta(t)), \end{aligned}$$

which together with (6) indicates that (9) is fulfilled. Since $N \geq -M$, it follows from (9) and (10) that for any $v \in B_N$,

$$|F(t,(Av)(t),(Bv)(t),-v(t))| \leq N,\ t \in [0,1],$$

which implies that

$$\begin{aligned} |(Tv)\,(t)| &= \left|\int_0^t F(s,(Av)(s),(Bv)(s),-v(s))ds\right| \\ &\leq \int_0^1 |F(s,(Av)(s),(Bv)(s),-v(s))|\,ds \\ &\leq N,\ t \in [0,1]. \end{aligned}$$

This shows that $T : B_N \to B_N$.

Next, we prove that $T : B_N \to B_N$ is completely continuous. Since the continuity of T is obvious, we only need to prove that T is compact. Let X be a bounded subset in B_N. Then $T(X) \subseteq B_N$, which implies that $T(X)$ is uniformly bounded. Now, we shall prove that $T(X)$ is equicontinuous. For any $\epsilon > 0$, we choose $\delta = \frac{\epsilon}{N+1}$. Then for any $\omega \in T(X)$ (there exists a $v \in X$ such that $\omega = Tv$) and $t_1, t_2 \in [0,1]$ with $|t_1 - t_2| < \delta$, we have

$$\begin{aligned} |\omega(t_1) - \omega(t_2)| &= |(Tv)(t_1) - (Tv)(t_2)| \\ &= \left|\int_0^{t_1} F(s,(Av)(s),(Bv)(s),-v(s))ds \right. \\ &\quad \left. - \int_0^{t_2} F(s,(Av)(s),(Bv)(s),-v(s))ds\right| \\ &\leq \left|\int_{t_2}^{t_1} |F(s,(Av)(s),(Bv)(s),-v(s))|\,ds\right| \\ &\leq N\,|t_1 - t_2| \\ &< \epsilon, \end{aligned}$$

which shows that $T(X)$ is equicontinuous. By the Arzela-Ascoli theorem, we know that $T : B_N \to B_N$ is a compact mapping.

It is now immediate from the Schauder fixed point theorem that the operator T has a fixed point v_0, which solves the auxiliary problem (8).

Step 3. We prove that v_0 is a solution of the problem (7). To this end, we only need to verify that $\alpha(t) \le v_0(t) \le \beta(t)$ for $t \in [0,1]$. Since the proof of $v_0(t) \le \beta(t)$ for $t \in [0,1]$ is similar, we only prove $\alpha(t) \le v_0(t)$ for $t \in [0,1]$.

Suppose on the contrary that there exists $\overline{t} \in [0,1]$ such that $v_0(\overline{t}) < \alpha(\overline{t})$. Obviously, $\overline{t} \in (0,1]$. By the continuity of v_0 and α and $v_0(0) = 0 = \alpha(0)$, we know that there exists $t^* \in [0,\overline{t})$ such that $v_0(t^*) = \alpha(t^*)$ and $v_0(t) < \alpha(t)$ for $t \in (t^*,\overline{t}]$. Therefore,

$$\begin{aligned} v_0'(t) &= F(t,(Av_0)(t),(Bv_0)(t),-v_0(t)) \\ &= f_1(t,(Av_0)(t),(Bv_0)(t),-\alpha(t)) \\ &= f_2(t,(Av_0)(t),(B\alpha)(t),-\alpha(t)) \\ &= f(t,(A\alpha)(t),(B\alpha)(t),-\alpha(t)), \end{aligned} \tag{11}$$

for $t \in \left(t^*,\overline{t}\right]$. Let $m(t) = v_0(t) - \alpha(t),\ t \in \left[t^*,\overline{t}\right]$. Since x is a lower solution of the BVP (1), one has

$$\begin{aligned} \alpha'(t) &= -x'''(t) \le f\left(t,x(t),x'(t),x''(t)\right) \\ &= f\left(t,(A\alpha)(t),(B\alpha)(t),-\alpha(t)\right),\ t \in [0,1]. \end{aligned} \tag{12}$$

In view of (11) and (12), we have $m'(t) = v_0'(t) - \alpha'(t) \ge 0$ for $t \in \left(t^*,\overline{t}\right]$, which together with $m(t^*) = 0$ implies that $m(t) \ge 0$ for $t \in \left[t^*,\overline{t}\right]$, that is, $v_0(t) \ge \alpha(t)$ for $t \in \left[t^*,\overline{t}\right]$. This is a contradiction. Thus, $\alpha(t) \le v_0(t)$ for $t \in [0,1]$.

Step 4. We claim that the BVP (1) has a solution.

In fact, if we let $u_0(t) = \int_0^1 G(t,s)v_0(s)ds,\ t \in [0,1]$, then u_0 is a desired solution of the BVP (1) satisfying $x(t) \le u_0(t) \le y(t)$ and $y''(t) \le u_0''(t) \le x''(t)$ for $t \in [0,1]$.

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