Applied Mathematics E-Notes, 10(2010), 275-284 © ISSN 1607-2510
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Schur-Geometric Convexity for Differences of Means*

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Received 24 September 2009

Abstract

The Schur-geometric convexity in $(0,\infty)\times(0,\infty)$ for the difference of some famous means such as arithmetic mean, geometric mean, harmonic mean, root-square mean, etc. is discussed. Some inequalities related to the difference of means are obtained.

1 Introduction

Recently, the following chain of inequalities for the binary means is given in [1]:

$$H(a,b)\le G(a,b)\le N_1(a,b)\le N_3(a,b)\le N_2(a,b)\le A(a,b)\le S(a,b), \tag{1}$$

where

$$A(a,b)=\frac{a+b}{2},\ G(a,b)=\sqrt{ab},\ H(a,b)=\frac{2ab}{a+b},\ S(a,b)=\sqrt{\frac{a^2+b^2}{2}},$$

and

$$N_1(a,b)=\left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)^2=\frac{A(a,b)+G(a,b)}{2},$$

$$N_3(a,b)=\frac{a+\sqrt{ab}+b}{3}=\frac{2A(a,b)+G(a,b)}{3},$$

$$N_2(a,b)=\left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)\left(\sqrt{\frac{a+b}{2}}\right).$$

The means, $A(a,b), G(a,b), H(a,b), S(a,b), N_1(a,b)$ and $N_3(a,b)$ are arithmetic, geometric, harmonic, root-square, square-root and Heron's means respectively. The mean $N_2(a,b)$ can be seen in Taneja [2, 3].

Furthermore the following differences of means are considered in [1]:

$$M_{SA}(a,b)=S(a,b)-A(a,b), \tag{2}$$

*Mathematics Subject Classifications: 26B25, 26E60, 26D20.

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$$M_{SN_2}(a,b) = S(a,b) - N_2(a,b), \tag{3}$$

$$M_{SN_3}(a,b) = S(a,b) - N_3(a,b), \tag{4}$$

$$M_{SN_1}(a,b) = S(a,b) - N_1(a,b), \tag{5}$$

$$M_{SG}(a,b) = S(a,b) - G(a,b), \tag{6}$$

$$M_{SH}(a,b) = S(a,b) - H(a,b), \tag{7}$$

$$M_{AN_2}(a,b) = A(a,b) - N_2(a,b), \tag{8}$$

$$M_{AG}(a,b) = A(a,b) - G(a,b), \tag{9}$$

$$M_{AH}(a,b) = A(a,b) - H(a,b), \tag{10}$$

$$M_{N_2N_1}(a,b) = N_2(a,b) - N_1(a,b), \tag{11}$$

$$M_{N_2G}(a,b) = N_2(a,b) - G(a,b), \tag{12}$$

and the following Theorem is established:

THEOREM A. The differences of means given by (2)-(12) are nonnegative and convex in $\mathrm{R}_+^2 = (0,\infty) \times (0,\infty)$.

In this paper, the following Theorem is proved, and by this Theorem, some inequalities in (1) are strengthened.

THEOREM 1. The differences of means given by (2)-(12) are Schur-geometrically convex in $\mathrm{R}_+^2 = (0,\infty) \times (0,\infty)$.

2 Definitions and Lemma

The Schur-convex function was introduced by I. Schur in 1923, and it has many important applications in analytic inequalities, linear regression, graphs and matrices, combinatorial optimization, information-theoretic topics, Gamma functions, stochastic orderings, reliability, and other related fields (see e.g., [4] and [11]-[20]).

In 2003, X. M. Zhang propose the concept of a "Schur-geometrically convex function" which is an extension of "Schur-convex function" and establish corresponding decision theorem [6]. Since then, Schur-geometric convexity has evoked the interest of of many researchers and numerous applications and extensions have appeared in the literature, see [7]-[10].

In order to verify our Theorems, the following definitions and lemmas are necessary.

DEFINITION 1 ([4, 5]). Let $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n) \in \mathbb{R}^n$.

(*i*) x is said to be majorized by y (in symbols $x \prec y$) if $\sum_{i=1}^k x_{[i]} \le \sum_{i=1}^k y_{[i]}$ for $k = 1, 2, \dots, n-1$ and $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$, where $x_{[1]} \ge \cdots \ge x_{[n]}$ and $y_{[1]} \ge \cdots \ge y_{[n]}$ are rearrangements of x and y in a descending order.

(*ii*) $\Omega \subseteq \mathbb{R}^n$ is called a convex set if $(\alpha x_1 + \beta y_1, \dots, \alpha x_n + \beta y_n) \in \Omega$ for every x and $y \in \Omega$, where α and $\beta \in [0,1]$ with $\alpha + \beta = 1$.

(*iii*) Let $\Omega \subseteq \mathbb{R}^n$. The function φ: $\Omega \to \mathbb{R}$ be said to be a Schur-convex function on Ω if $x \prec y$ on Ω implies $\varphi(x) \le \varphi(y)$. φ is said to be a Schur-concave function on Ω if and only if $-\varphi$ is Schur-convex.

DEFINITION 2 ([6]). Let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n) \in \mathbb{R}^n_+$.

(*i*) $\Omega \subseteq \mathbb{R}^n_+$ is called a geometrically convex set if $(x_1^\alpha y_1^\beta, \ldots, x_n^\alpha y_n^\beta) \in \Omega$ for all x and $y \in \Omega$, where α and $\beta \in [0,1]$ with $\alpha + \beta = 1$.

(*ii*) Let $\Omega \subseteq \mathbb{R}^n_+$. The function φ: $\Omega \to \mathbb{R}_+$ is said to be Schur-geometrically convex function on Ω if $(\ln x_1, \ldots, \ln x_n) \prec (\ln y_1, \ldots, \ln y_n)$ on Ω implies $\varphi(x) \le \varphi(y)$. The function φ is said to be a Schur-geometrically concave on Ω if and only if $-\varphi$ is Schur-geometrically convex.

DEFINITION 3 ([4, 5]).

(*i*) $\Omega \subseteq \mathbb{R}^n$ is called symmetric set, if $x \in \Omega$ implies $Px \in \Omega$ for every $n \times n$ permutation matrix P.

(*ii*) The function $\varphi : \Omega \to \mathbb{R}$ is called symmetric if for every permutation matrix P, $\varphi(Px) = \varphi(x)$ for all $x \in \Omega$.

DEFINITION 4 ([4, 5]). Let $\Omega \subseteq \mathbb{R}^n$, $\varphi : \Omega \to \mathbb{R}$ is a symmetric and convex function. Then φ is Schur convex on Ω.

REMARK 1. It is obvious that the difference of means given by (2)-(12) are symmetric, so by Theorem A and Lemma 1, it follows that those differences are all Schur-convex in $\mathrm{R}^2_+ = (0,\infty) \times (0,\infty)$.

LEMMA 1 ([6]). Let $\Omega \subseteq \mathbb{R}^n_+$ be symmetric with a nonempty interior geometrically convex set, and let $\varphi : \Omega \to \mathbb{R}_+$ be continuous on Ω and differentiable in Ω^0. If φ is symmetric on Ω and

$$(\ln x_1 - \ln x_2)\left(x_1 \frac{\partial \varphi}{\partial x_1} - x_2 \frac{\partial \varphi}{\partial x_2}\right) \ge 0 (\le 0) \tag{13}$$

holds for any $x = (x_1, \cdots, x_n) \in \Omega^0$, then φ is a Schur-geometrically convex (Schur-geometrically concave) function.

LEMMA 2 ([7]). Let $a \le b, u(t) = ta + (1-t)b, v(t) = tb + (1-t)a$. If $1/2 \le t_2 \le t_1 \le 1$ or $0 \le t_1 \le t_2 \le 1/2$, then

$$\left(\frac{a+b}{2}, \frac{a+b}{2}\right) \prec (u(t_2), v(t_2)) \prec (u(t_1), v(t_1)) \prec (a, b). \tag{14}$$

3 Proofs of Main Results

1) For

$$M_{SA}(a,b)=S(a,b)-A(a,b)=\sqrt{\frac{a^2+b^2}{2}}-\frac{a+b}{2},$$

we have

$$\frac{\partial M_{SA}}{\partial a}=\frac{a}{2}\left(\frac{a^2+b^2}{2}\right)^{-1/2}-\frac{1}{2},$$

$$\frac{\partial M_{SA}}{\partial b}=\frac{b}{2}\left(\frac{a^2+b^2}{2}\right)^{-1/2}-\frac{1}{2},$$

and then

$$\begin{aligned}\Lambda &:=(\ln a-\ln b)\left(a\frac{\partial M_{SA}}{\partial a}-b\frac{\partial M_{SA}}{\partial b}\right)\\ &=(\ln a-\ln b)\left[\left(\frac{a^2+b^2}{2}\right)^{-1/2}\frac{a^2-b^2}{2}-\frac{a-b}{2}\right]\\ &=\frac{(\ln a-\ln b)\,(a-b)}{2}\left[(a+b)\left(\frac{a^2+b^2}{2}\right)^{-1/2}-1\right].\end{aligned}$$

Since $\ln x$ is increasing, we have $(\ln a-\ln b)\,(a-b)\geq 0$, and $(a+b)\left(\frac{a^2+b^2}{2}\right)^{-1/2}-1\geq 0$ is equivalent to $a^2+b^2\leq 2a^2+2b^2+4ab$, which is ture obviously, so $\Lambda\geq 0$. By the Lemma 1, it follows that $M_{SA}(a,b)$ is Schur-geometrically convex in $\mathrm{R}_+^2=(0,\infty)\times(0,\infty)$.

2) For

$$M_{AN_2}(a,b)=A(a,b)-N_2(a,b)=\frac{a+b}{2}-\left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)\left(\sqrt{\frac{a+b}{2}}\right),$$

we have

$$\frac{\partial M_{AN_2}}{\partial a}=\frac{1}{2}-\frac{1}{4\sqrt{a}}\sqrt{\frac{a+b}{2}}-\frac{1}{4}\left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)\left(\frac{a+b}{2}\right)^{-1/2},$$

$$\frac{\partial M_{AN_2}}{\partial b}=\frac{1}{2}-\frac{1}{4\sqrt{b}}\sqrt{\frac{a+b}{2}}-\frac{1}{4}\left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)\left(\frac{a+b}{2}\right)^{-1/2},$$

and then

$$\begin{aligned}\Lambda &=(\ln a-\ln b)\left(a\frac{\partial M_{AN_2}}{\partial a}-b\frac{\partial M_{AN_2}}{\partial b}\right)\\ &=(\ln a-\ln b)\left[\frac{a-b}{2}-\frac{1}{4}\sqrt{\frac{a+b}{2}}\left(\sqrt{a}-\sqrt{b}\right)-\frac{1}{4}\left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)\left(\frac{a+b}{2}\right)^{-1/2}(a-b)\right]\end{aligned}$$

$$= \frac{(\ln a - \ln b)(a-b)}{2}\left[1 - \frac{1}{2}\sqrt{\frac{a+b}{2}}\left(\sqrt{a}+\sqrt{b}\right)^{-1} - \frac{1}{2}\left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)\left(\frac{a+b}{2}\right)^{-1/2}\right].$$

It is easy to check that

$$1 - \frac{1}{2}\sqrt{\frac{a+b}{2}}\left(\sqrt{a}+\sqrt{b}\right)^{-1} - \frac{1}{2}\left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)\left(\frac{a+b}{2}\right)^{-1/2} \geq 0$$

is equivalent to

$$(a+b)^2 + 2(a+b)\sqrt{ab} \geq ab,$$

so $\Lambda \geq 0$. By the Lemma 1, it follows that $M_{AN_2}(a,b)$ is Schur-geometrically convex in R^2_+.

3) For

$$M_{SN_2}(a,b) = S(a,b) - N_2(a,b) = \sqrt{\frac{a^2+b^2}{2}} - \left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)\left(\sqrt{\frac{a+b}{2}}\right),$$

notice that

$$M_{SN_2}(a,b) = M_{SA}(a,b) + M_{AN_2}(a,b),$$

by the definition of the Schur-geometrically convex function, it follows that the sum of two Schur-geometrically convex function is also the Schur-geometrically convex, so $M_{SN_2}(a,b)$ is Schur-geometrically convex in R^2_+.

4) For

$$M_{SN_3}(a,b) = S(a,b) - N_3(a,b) = \sqrt{\frac{a^2+b^2}{2}} - \frac{a+\sqrt{ab}+b}{3},$$

we have

$$\frac{\partial M_{SN_3}}{\partial a} = \frac{a}{2}\left(\frac{a^2+b^2}{2}\right)^{-1/2} - \frac{1}{3}\left(1 + \frac{b}{2\sqrt{ab}}\right),$$

$$\frac{\partial M_{SN_3}}{\partial b} = \frac{b}{2}\left(\frac{a^2+b^2}{2}\right)^{-1/2} - \frac{1}{3}\left(1 + \frac{a}{2\sqrt{ab}}\right),$$

and then

$$\Lambda = (\ln a - \ln b)\left(a\frac{\partial M_{SN_3}}{\partial a} - b\frac{\partial M_{SN_3}}{\partial b}\right)$$
$$= (\ln a - \ln b)(a-b)\left[\left(\frac{a^2+b^2}{2}\right)^{-1/2}\left(\frac{a+b}{2}\right) - \frac{1}{3}\right],$$

notice that

$$\left(\frac{a^2+b^2}{2}\right)^{-1/2}\left(\frac{a+b}{2}\right) - \frac{1}{3} \geq 0 \Leftrightarrow 9(a+b)^2 \geq 2\left(a^2+b^2\right),$$

we have $\Lambda \geq 0$, so $M_{SN_3}(a,b)$ is Schur-geometrically convex in R_+^2.

5) For

$$M_{N_2N_1}(a,b) = N_2(a,b) - N_1(a,b) = \left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)\left(\sqrt{\frac{a+b}{2}}\right) - \frac{a+b}{4} - \frac{\sqrt{ab}}{2},$$

we have

$$\frac{\partial M_{N_2N_1}}{\partial a} = \frac{1}{4\sqrt{a}}\sqrt{\frac{a+b}{2}} + \frac{1}{4}\left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)\left(\frac{a+b}{2}\right)^{-1/2} - \frac{1}{4} - \frac{b}{4\sqrt{ab}},$$

$$\frac{\partial M_{N_2N_1}}{\partial b} = \frac{1}{4\sqrt{b}}\sqrt{\frac{a+b}{2}} + \frac{1}{4}\left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)\left(\frac{a+b}{2}\right)^{-1/2} - \frac{1}{4} - \frac{a}{4\sqrt{ab}},$$

and then

$$\begin{aligned}\Lambda &= (\ln a - \ln b)\left(a\frac{\partial M_{N_2N_1}}{\partial a} - b\frac{\partial M_{N_2N_1}}{\partial b}\right)\\ &= (\ln a - \ln b)\left[\frac{1}{4}\sqrt{\frac{a+b}{2}}\left(\sqrt{a}-\sqrt{b}\right) + \frac{1}{4}\left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)\left(\frac{a+b}{2}\right)^{-1/2}(a-b) - \frac{1}{4}(a-b)\right]\\ &= \frac{1}{4}(\ln a - \ln b)(a-b)\left[\sqrt{\frac{a+b}{2}}\left(\sqrt{a}+\sqrt{b}\right)^{-1} + \left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)\left(\frac{a+b}{2}\right)^{-1/2} - 1\right].\end{aligned}$$

By the AM-GM inequality, we have

$$\begin{aligned}&\sqrt{\frac{a+b}{2}}\left(\sqrt{a}+\sqrt{b}\right)^{-1} + \left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)\left(\frac{a+b}{2}\right)^{-1/2} - 1\\ &\geq 2\left[\sqrt{\frac{a+b}{2}}\left(\sqrt{a}+\sqrt{b}\right)^{-1}\cdot\left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)\left(\frac{a+b}{2}\right)^{-1/2}\right]^{1/2} - 1 = \sqrt{2}-1 \geq 0,\end{aligned}$$

so $M_{N_2N_1}(a,b)$ is Schur-geometrically convex in R_+^2.

6) For

$$M_{SN_1}(a,b) = S(a,b) - N_1(a,b) = \sqrt{\frac{a^2+b^2}{2}} - \left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)^2,$$

notice that

$$M_{SN_1}(a,b) = M_{SN_2}(a,b) + M_{N_2N_1}(a,b),$$

i.e. $M_{SN_1}(a,b)$ is the sum of two Schur-geometrically convex function, so $M_{SN_2}(a,b)$ is Schur-geometrically convex in R_+^2.

7) For

$$M_{AG}(a,b) = A(a,b) - G(a,b) = \frac{a+b}{2} - \sqrt{ab},$$

we have

$$\frac{\partial M_{AG}}{\partial a}=\frac{1}{2}-\frac{b}{2\sqrt{ab}},\frac{\partial M_{AG}}{\partial b}=\frac{1}{2}-\frac{a}{2\sqrt{ab}},$$

and then

$$\Lambda=(\ln a-\ln b)\left(a\frac{\partial M_{AG}}{\partial a}-b\frac{\partial M_{AG}}{\partial b}\right)=\frac{1}{2}(\ln a-\ln b)(a-b)\geq 0,$$

so $M_{AG}(a,b)$ is Schur-geometrically convex in R^2_+.

8) For

$$M_{SG}(a,b)=S(a,b)-G(a,b)=\sqrt{\frac{a^2+b^2}{2}}-\sqrt{ab},$$

notice that

$$M_{SG}(a,b)=M_{SA}(a,b)+M_{AG}(a,b),$$

i.e. $M_{SG}(a,b)$ is the sum of two Schur-geometric convex function, so $M_{SG}(a,b)$ is Schur-geometrically convex in R^2_+.

9) For

$$M_{AH}(a,b)=A(a,b)-H(a,b)=\frac{a+b}{2}-\frac{2ab}{a+b},$$

we have

$$\frac{\partial M_{AH}}{\partial a}=\frac{1}{2}-\frac{2b^2}{(a+b)^2},\frac{\partial M_{AH}}{\partial b}=\frac{1}{2}-\frac{2a^2}{(a+b)^2},$$

and then

$$\begin{aligned}\Lambda&=(\ln a-\ln b)\left(a\frac{\partial M_{AH}}{\partial a}-b\frac{\partial M_{AH}}{\partial b}\right)\\&=(\ln a-\ln b)(a-b)\left[\frac{1}{2}+\frac{2ab}{(a+b)^2}\right]\geq 0,\end{aligned}$$

so $M_{AH}(a,b)$ is Schur-geometrically convex in R^2_+.

10) For

$$M_{SH}(a,b)=S(a,b)-H(a,b)=\sqrt{\frac{a^2+b^2}{2}}-\frac{2ab}{a+b},$$

notice that

$$M_{SH}(a,b)=M_{SA}(a,b)+M_{AH}(a,b),$$

i.e. $M_{SH}(a,b)$ is the sum of two Schur-geometrically convex function, so $M_{SH}(a,b)$ is Schur-geometrically convex in R^2_+.

11) For

$$M_{N_2G}(a,b)=N_2(a,b)-G(a,b)=\left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)\left(\sqrt{\frac{a+b}{2}}\right)-\sqrt{ab},$$

we have

$$\frac{\partial M_{N_2G}}{\partial a}=\frac{1}{4\sqrt{a}}\left(\sqrt{\frac{a+b}{2}}\right)+\frac{1}{4}\left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)\left(\frac{a+b}{2}\right)^{-1/2}-\frac{b}{2\sqrt{ab}},$$

$$\frac{\partial M_{N_2G}}{\partial b} = \frac{1}{4\sqrt{b}}\left(\sqrt{\frac{a+b}{2}}\right) + \frac{1}{4}\left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)\left(\frac{a+b}{2}\right)^{-1/2} - \frac{a}{2\sqrt{ab}},$$

and then

$$\begin{aligned}\Lambda &= (\ln a - \ln b)\left(a\frac{\partial M_{N_2G}}{\partial a} - b\frac{\partial M_{N_2G}}{\partial b}\right)\\ &= (\ln a - \ln b)\left[\frac{1}{4}\left(\sqrt{\frac{a+b}{2}}\right)\left(\sqrt{a}-\sqrt{b}\right) + \frac{1}{4}\left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)\left(\frac{a+b}{2}\right)^{-1/2}(a-b)\right]\\ &= \frac{1}{4}(\ln a - \ln b)(a-b)\left[\left(\sqrt{\frac{a+b}{2}}\right)\left(\sqrt{a}+\sqrt{b}\right)^{-1} + \frac{1}{4}\left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)\left(\frac{a+b}{2}\right)^{-1/2}\right] \geq 0,\end{aligned}$$

so $M_{N_2G}(a,b)$ is Schur-geometrically convex in R_+^2.

Thus the proof of Theorem 1 is complete.

4 Applications

As an application of our main result, we have the following.

THEOREM 2. Let $0 < a \leq b$. If $1/2 \leq t \leq 1$ or $0 \leq t \leq 1/2$, then

$$0 \leq \sqrt{\frac{a^{t^2}b^{(1-t)^2} + a^{(1-t)^2}b^{t^2}}{2}} - \frac{a^t b^{1-t} + a^{1-t}b^t}{2} \leq \sqrt{\frac{a^2+b^2}{2}} - \frac{a+b}{2}, \tag{15}$$

$$\begin{aligned}0 &\leq \sqrt{\frac{a^{t^2}b^{(1-t)^2} + a^{(1-t)^2}b^{t^2}}{2}} - \left(\frac{\sqrt{a^t b^{1-t}} + \sqrt{a^{1-t}b^t}}{2}\right)\left(\sqrt{\frac{a^t b^{1-t} + a^{1-t}b^t}{2}}\right)\\ &\leq \sqrt{\frac{a^2+b^2}{2}} - \left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)\left(\sqrt{\frac{a+b}{2}}\right),\end{aligned} \tag{16}$$

$$0 \leq \sqrt{\frac{a^{t^2}b^{(1-t)^2} + a^{(1-t)^2}b^{t^2}}{2}} - \frac{a^t b^{1-t} + \sqrt{ab} + a^{1-t}b^t}{3} \leq \sqrt{\frac{a^2+b^2}{2}} - \frac{a+\sqrt{ab}+b}{3}, \tag{17}$$

$$\begin{aligned}0 &\leq \frac{a^{t^2}b^{(1-t)^2} + a^{(1-t)^2}b^{t^2}}{2} - \left(\frac{\sqrt{a^t b^{1-t}} + \sqrt{a^{1-t}b^t}}{2}\right)\left(\sqrt{\frac{a^t b^{1-t} + a^{1-t}b^t}{2}}\right)\\ &\leq \frac{a+b}{2} - \left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)\left(\sqrt{\frac{a+b}{2}}\right),\end{aligned} \tag{18}$$

$$0 \leq \left(\frac{\sqrt{a^t b^{1-t}} + \sqrt{a^{1-t}b^t}}{2}\right)\left(\sqrt{\frac{a^t b^{1-t} + a^{1-t}b^t}{2}}\right) - \left(\frac{\sqrt{a^t b^{1-t}} + \sqrt{a^{1-t}b^t}}{2}\right)^2$$

$$\leq \left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)\left(\sqrt{\frac{a+b}{2}}\right)-\left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)^2. \tag{19}$$

PROOF. From Lemma 2, we have

$$\left(\ln\sqrt{ab},\ln\sqrt{ab}\right)\prec\left(\ln(b^ta^{1-t}),\ln(a^tb^{1-t})\right)\prec(\ln a,\ln b),$$

and by Theorem 1, the difference of two means in (2)

$$M_{SA}(a,b)=S(a,b)-A(a,b)=\sqrt{\frac{a^2+b^2}{2}}-\frac{a+b}{2},$$

is Schur-geometrically convex in R_+^2, so we have

$$M_{SA}(\sqrt{ab},\sqrt{ab})\leq M_{SA}(a^tb^{1-t},a^{1-t}b^t)\leq M_{SA}(a,b),$$

i.e. (15) holds.

Similarly, by Schur-geometric convexity of the difference of two means in (3), (4), (8) and (11), from (20) it follows that (16), (17), (18) and (19) hold respectively.

The proof of Theorem 2 is complete.

REMARK 2. (15) is the sharpening of the inequality $A(a,b)\leq S(a,b)$ in (1), and (16) is the sharpening of the inequality $N_2(a,b)\leq A(a,b)$ in (1).

Acknowledgment. The authors are indebted to the referees for their helpful suggestions. This work was supported in part by the Scientific Research Common Program of Beijing Municipal Commission of Education (KM201011417013).

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