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On The Concavity Of The First NLPC Transformation Of Unimodal Symmetric Random Variables*

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Abstract

We study the concavity of the first NLPC transformation for symmetric unimodal distributions on bounded domains. We deduce a comparison principle based on the variances of the first NLPC and show a possible application in constructing goodness-of-fit tests.

1 Introduction

Let X be an absolutely continuous random variable (r.v.) with zero mean, finite variance and density f_X having support the closure $\overline{D}$ of an interval D ($\Upsilon(D)$ will denote the set of these r.v.s). As introduced in [6], the first nonlinear principal component (NLPC) of X, if it exists, is the r.v. $\varphi_1(X)$ where φ_1 is defined as

$$\varphi_1 = \underset{u \in \dot{W}_X^{1,2} \setminus \{0\}}{\arg\max}\ \mathbb{E}\left[u(X)^2\right]\left(\mathbb{E}\left[u'(X)^2\right]\right)^{-1}. \tag{1}$$

Here $\dot{W}_X^{1,2} = \{u \in \dot{\mathcal{L}}_X^2 : \ u' \in \mathcal{L}_X^2\}$ and $\dot{\mathcal{L}}_X^2$ (resp. $\mathcal{L}_X^2$) is the separable Hilbert space of centered (resp. not necessarily centered), square integrable functions $u : D \to \mathbb{R}$. We will assume $(1/f_X) \in \mathcal{L}^1_{\text{loc}}(D)$, thus $\dot{W}_X^{1,2}$ is Hilbert too. By (1) φ_1 realizes the equality in the *Poincaré inequality* (see e.g. [2], [3], [4], [5], [7], [8]):

$$\exists C > 0 : \qquad \text{Var}\left[u(X)\right] \leq C\,\mathbb{E}\left[(u'(X))^2\right] \tag{2}$$

and the variance λ_1 of $\varphi_1(X)$ coincides with the optimal Poincaré constant C. Some properties of φ_1 are collected in the following lemma (see [6])

LEMMA 1. Suppose $X \in \Upsilon(D)$ admits NLPCs and let φ_1 be the first NLPC transformation. The following conclusions hold:

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(i) if f_X is even, then φ_1 is odd;
(ii) if $f_X \in C^1(D)$, then φ_1 is strictly monotone;
(iii) if $\varphi_1 \in C^2(D)$ then $f_X = g/\int_D g$, where $g(x) = (\varphi_1'(x))^{-1} \exp\left\{-\xi_1 \int \varphi_1/\varphi_1'\right\}$ and $\xi_1 = 1/\lambda_1$.

Here and in the following $C^k(D)$ denotes as usual the set of $k \geq 0$ times continuously differentiable real functions defined on D.

Note how statement (iii) highlights the central role of φ_1 in characterizing the distribution of X, justifying the interest in deepening the knowledge of its properties.

Here we investigate which assumptions on f_X guarantee that φ_1 has just one change of concavity in D. This behavior seems to be sufficiently general, as some examples in [6] suggest; moreover, it is a crucial ingredient used in [10] to prove a characterization of the uniform distribution among the unimodal symmetric distributions with bounded support. This is the main reason why we settle our analysis in this framework.

We prove that, under sufficiently mild assumptions on the density f_X, transformation φ_1 effectively presents the above mentioned property. Furthermore, thanks to the obtained results, we generalize the comparison result of [10] and show with an example how a class of goodness-of-fit test can be based on this last result.

2 Main Results

We recall that φ_1 is a *weak solution*, with $\xi = \xi_1 = \lambda_1^{-1}$, of the *Sturm-Liouville* problem:

$$\begin{cases} -(f_X u')' = \xi f_X u, & \text{in } D \\ \lim\limits_{x\to a^+} u'(x) f_X(x) = \lim\limits_{x\to b^-} u'(x) f_X(x) = 0 \end{cases} \tag{3}$$

that is $\varphi_1 \in \dot{W}_X^{1,2}$ and $\mathbb{E}[\varphi_1'(X)\, h'(X)] = \xi\mathbb{E}[\varphi_1(X)\, h(X)]$ for all $h \in \dot{W}_X^{1,2}$, whereas φ_1 is a *strong solution* of (3) if $f_X\varphi_1' \in C^0(\overline{D}) \cap C^1(D)$, that is φ_1 satisfies (3) pointwise. We will assume, without loss of generality, $D = (-1,1)$ and $X \in \Upsilon(D)$ such that:
(**H1**) *it admits first NLPC* φ_1;
(**H2**) *its density* $f_X \in C^0[-1,1] \cap C^1(-1,1)$ *is symmetric and unimodal at* 0, *with* $f_X' \leq 0$ *on* $(0,1)$.
For the sake of shortness we will denote by $\mathcal{H}(D)$ the set of such r.v.s.

PROPOSITION 1. Let $X \in \mathcal{H}(D)$ with $f_X \in C^2(-1,1)$ and assume

$$A(x) := -\frac{d^2}{dx^2}\ln(f_X(x)) - \xi_1 \qquad x \in [0,1) \tag{4}$$

is such that (i) $A(0) < 0$; (ii) A has at most one zero in $(0,1)$ in which it changes sign. Then φ_1 is concave in $[0,1]$.

PROOF. By (**H1**) and (**H2**) φ_1 is a strong solution of (3). Since $f_X \in C^1(-1,1)$ we obtain $\varphi_1 \in C^2(-1,1)$; moreover by $f_X \in C^2(-1,1)$ and (3) and it follows $\varphi_1 \in C^3(-1,1)$. Differentiating in (3), we get

$$\varphi_1''(x) = -\frac{f_X'(x)}{f_X(x)}\varphi_1'(x) - \xi_1\varphi_1(x), \quad \forall x \in [0,1). \tag{5}$$

Since $f_X'(0) = 0$, $\varphi_1(0) = 0$ and $\varphi_1'(0) > 0$, we have $\varphi_1''(0) = 0$. Differentiating in (5) we obtain

$$\varphi_1'''(x) = \varphi_1'(x)A(x) - \varphi_1''(x)\frac{d}{dx}(\ln(f_X(x)) \quad \forall x \in [0,1), \tag{6}$$

from which $\varphi_1'''(0) = \varphi_1'(0)A(0) < 0$. Thus $\varphi_1''(x) < 0$ in a right neighborhood of 0 (recall that $\varphi_1'' \in C^0(-1,1)$).

Assume first $A(x) < 0$ in $(0,1)$. Since $\varphi_1'(x) > 0$ in $(-1,1)$, if there exists $x_1 \in (0,1)$ such that $\varphi_1''(x_1) = 0$ from (6) it follows $\varphi_1'''(x_1) < 0$, a contradiction; thus $\varphi_1''(x) < 0$ in $(0,1)$ and we conclude.

Assume now that there exists (a unique) $\overline{x} \in (0,1)$ such that $A(\overline{x}) = 0$ and $A(x) > 0$ in $(\overline{x}, 1)$, hence

$$\frac{d^2}{dx^2}\ln(f_X(x)) \leq -\xi_1 \quad \text{for all } x \in [\overline{x}, 1). \tag{7}$$

We show first that

$$\limsup_{x\to 1^-} \varphi_1''(x) < 0. \tag{8}$$

If $\limsup_{x\to 1^-} -f_X'(x)/f_X^2(x) = c \in [0,+\infty)$, since $\lim_{x\to 1^-} f_X(x)\varphi_1'(x) = 0$, (8) easily follows from (5).

Suppose $\limsup_{x\to 1^-} -f_X'(x)/f_X^2(x) = +\infty$. Condition (7) assures that the function $f_X'(x)/f_X(x)$ is strictly decreasing in $[\overline{x}, 1)$, hence it exists $\lim_{x\to 1^-} f_X'(x)/f_X(x) = \alpha$ with $\alpha \in [-\infty, 0)$. With some computations one deduces $\lim_{x\to 1^-} f_X^2(x)/f_X'(x) = 0$. Then, we get

$$\begin{aligned}
\limsup_{x\to 1^-} \frac{-f_X'(x)\,\varphi_1'(x)}{f_X(x)} &= \limsup_{x\to 1^-} \frac{-\varphi_1'(x)f_X(x)}{f_X^2(x)(f_X'(x))^{-1}} \leq \limsup_{x\to 1^-} \frac{-(\varphi_1'(x)f_X(x))'}{(f_X^2(x)(f_X'(x))^{-1})'} \\
&\leq \limsup_{x\to 1^-} \xi_1\varphi_1(x)\left[1 - \frac{d^2}{dx^2}\ln(f_X(x))\left(\frac{d}{dx}\ln(f_X(x))\right)^{-2}\right]^{-1} \\
&\leq \limsup_{x\to 1^-} \xi_1\varphi_1(x)\left[1 + \xi_1\left(\frac{d}{dx}\ln(f_X(x))\right)^{-2}\right]^{-1} \\
&< \xi_1 \lim_{x\to 1^-} \varphi_1(x),
\end{aligned}$$

where again we use (7). By this, (8) follows.

Now suppose by contradiction that φ_1'' changes sign in $(0,1)$ and let x_1, $x_2 \in (0,1)$, with $x_1 < x_2$, be its "first and last" zeroes, respectively. By (6) and (8) we get

$$0 \leq \varphi_1'''(x_1) = \varphi_1'(x_1)A(x_1) \qquad \text{and} \qquad 0 \geq \varphi_1'''(x_2) = \varphi_1'(x_2)A(x_2).$$

Since $\varphi_1'(x) > 0$ in $(-1,1)$, it must be $A(x_1) \geq 0$ and $A(x_2) \leq 0$. By this we deduce that $x_1 \geq \overline{x}$ but this produces the contradiction $A(x_2) > 0$.

The basic idea in the proof of Proposition 1 is to study the sign of the φ_1'' expression that can be deduced from (3). A direct inspection of this expression shows that if $f_X'(x) > 0$ for all $x \in (0,1)$, then φ_1 is concave in $(0,1)$. This also tells us that the concavity study of φ_1 in the unimodal case presents all the main difficulties that one could find in the multimodal one.

Hypotheses (i) and (ii) of Proposition 1 requiring an a priori estimate of ξ_1 are, in general, difficult to handle. Here we state a sufficient condition for their validity.

PROPOSITION 2. Let $X \in \mathcal{H}(D)$ and suppose there exists $n_0 \geq 4$ (even) such that f_X is differentiable n_0 times in $(-1,1)$, and

$$\frac{d^3}{dx^3}\ln(f_X(0)) = \cdots = \frac{d^{n_0-1}}{dx^{n_0-1}}\ln(f_X(0)) = 0; \qquad \frac{d^{n_0}}{dx^{n_0}}\ln(f_X(0)) \neq 0.$$

If $\dfrac{d^3}{dx^3}\ln(f_X(x)) < 0$ in $(0,1)$, then φ_1 is concave in $[0,1]$.

PROOF. We show that function A in (4) satisfies (i) and (ii) of Proposition 1. The assumption $\dfrac{d^3}{dx^3}\ln(f_X(x)) < 0$ implies that the function A is strictly increasing in $(0,1)$. This readily implies (ii) of Proposition 1.

To check (i), we assume by contradiction that $A(0) \geq 0$. Thus, by the monotonicity of A and from (5) in the proof of Proposition 1, the first NLPC φ_1 associated to X satisfies

$$\text{if } x_1 \in (0,1) : \varphi_1''(x_1) = 0 \quad \Rightarrow \quad \varphi_1'''(x_1) > 0. \tag{9}$$

Furthermore, we have that $\varphi_1''(0) = 0$ and $\limsup_{x\to 1^-} \varphi_1''(x) < 0$.
If $A(0) > 0$, then $\varphi_1'''(0) > 0$ hence $\varphi_1''(x) > 0$ in a right neighborhood of $x = 0$. Hence, since $\limsup_{x\to 1^-} \varphi_1''(x) < 0$, (9) gives a contradiction.
Assume now that $A(0) = 0$, then $\varphi_1'''(0) = 0$. Differentiating in (6) we get $\varphi_1^i(0) = 0$ for $i = 2, ..., n_0$ and

$$\varphi_1^{n_0+1}(0) = \varphi_1'(0)A^{(n_0-2)}(0) = -\varphi_1'(0)\frac{d^{n_0}}{dx^{n_0}}\ln(f_X(x))(0) > 0,$$

where the fact that $A^{(n_0-2)}(0) = -\dfrac{d^{n_0}}{dx^{n_0}}\ln(f_X(x))(0) > 0$ follows from the monotonicity of A. We conclude that $\varphi_1''(x)$ is positive in a left neighborhood of $x = 0$ and the contradiction comes arguing as for the case $A(0) > 0$.

We present now two families of distributions to which Proposition 2 applies.

EXAMPLE 1. For the one parameter family of centered, scaled and symmetric beta $(css\beta(r))$ on $D = (-1,1)$

$$f_X(x,r) = K_r\left(1-x^2\right)^r \qquad r \in (0,+\infty),\ K_r = \left[\int_{-1}^{1}\left(1-x^2\right)^r dx\right]^{-1} \tag{10}$$

assumption (**H1**) has been tested in [6, Example 15] and (**H2**) holds. Some computations give for all $x \in (0,1)$

$$\frac{d^3}{dx^3}\ln(f_X(x,r)) = \frac{-4rx(x^2+3)}{(1-x^2)^3} < 0; \quad \frac{d^4}{dx^4}\ln(f_X(x,r)) = \frac{-12r\left(x^4+6x^2+1\right)}{\left(x^2-1\right)^4} \neq 0.$$

Hence, Proposition 2 and, in turn, Proposition 1 applies.

Another family of distributions to which Proposition 2 applies is the Generalized Normal truncated distribution on $D = (-1, 1)$:

$$f_X(x) = K_m e^{-x^{2m}}, \qquad m \in \mathbb{N},\, m \geq 2,\, K_m > 0.$$

Here, (**H1**) follows from [6, Theorem 5] and (**H2**) holds.

Next example shows that the assumptions of Proposition 2 are not necessary.

EXAMPLE 2. Consider the "Logistic truncated distribution":

$$f_X(x) = \frac{(e+1)\,e^x}{(e-1)\,(1+e^x)^2}, \qquad x \in [-1, 1]. \tag{11}$$

Since $\frac{d^3}{dx^3}\ln(f_X(x)) > 0$, Proposition 2 does not apply. Anyway, as $f_X(1) \neq 0$ and $\varphi_1 \in \dot{W}^{1,2}$, it holds:

$$\xi_1^{-1} = \frac{\int_{-1}^{1} \varphi_1^2\,(x)\, f_X\,(x)\,dx}{\int_{-1}^{1} (\varphi_1')^2\,(x)\, f_X\,(x)\,dx} \leq \frac{f_X\,(0)}{f_X(1)} \max_{u \in \dot{W}^{1,2}} \frac{\int_{-1}^{1} u^2\,(x)\,dx}{\int_{-1}^{1} (u')^2\,(x)\,dx} = \frac{f_X\,(0)}{f_X(1)}\frac{4}{\pi^2}$$

hence $\xi_1 \geq e\pi^2/\left(e+1\right)^2$. In turn, this implies

$$A(0) = -\frac{d^2}{dx^2}\ln(f_X(0)) - \xi_1 \leq \frac{1}{2} - \frac{e\pi^2}{\left(e+1\right)^2} < 0$$

and, jointly with the fact that $A'(x) < 0$ in $(0, 1)$, it allows to apply Proposition 1. Similarly one can treat the Standard Normal truncated distribution:

$$f_X(x) = K\,e^{-x^2/2}, \qquad K > 0,\; x \in [-1, 1]$$

having zero third logarithmic derivative. Note that for the above distributions assumption (**H1**) follows from [6, Theorem 5], while (**H2**) is easily verified.

Under the assumptions of Proposition 1 we are able to obtain a comparison principle for unimodal symmetric distributions, extending a result obtained in [10] for the uniform one. We note that this result does not seem easily extendible to the asymmetric case.

PROPOSITION 3. Let X and Y be in $\mathcal{H}(D)$. If X satisfies the assumptions of Proposition 1, f_X intersects f_Y once in $(0, 1)$ and $f_X(0) > f_Y(0)$, then $\lambda_1^{f_X} < \lambda_1^{f_Y}$ where $\lambda_1^{f_X}$ and $\lambda_1^{f_Y}$ are the variances of the first NLPC of X and Y, respectively.

REMARK 1. The hypothesis of Proposition 3 can be relaxed assuming that $f_X(x) \geq f_Y(x)$ for every $x \in [0, x_1]$, being x_1 the intersection point. Furthermore, a similar statement holds if f_X intersects f_Y $(2N+1)$ times in $(0, 1)$ $(N \geq 0)$. More precisely, named x_i $(i = 1, ..., 2N+1)$ the intersection points, if $f_X(0) > f_Y(0)$ and $\int_{x_{2k}}^{x_{2k+2}} f_X = \int_{x_{2k}}^{x_{2k+2}} f_Y, \forall\, 0 \leq k \leq N$, where $x_0 = 0$ and $x_{2N+2} = 1$, then one still gets the comparison principle.

PROOF. By the last assumption, there must exist $x_1 \in (0, 1)$ such that $f_X(x) > f_Y\,(x)$ on $[0, x_1)$, and $f_X(x) < f_Y\,(x)$ on $(x_1, 1)$. Let $\varphi_1 \in \dot{W}_X^{1,2}$ be the first NLPCs transformation associated to f_X. Since $\varphi_1 \in C^1(-1,1)$ is concave in $(0,1)$ its first derivative φ_1' is decreasing there. Thus there exists $\lim_{x\to 1^-} \varphi_1'(x)$ which, being φ_1' positive, must be finite and, in particular, $\varphi_1 \in \dot{W}^{1,2}$. By this, $\lim_{x\to 1^-} \varphi_1(x)$ is finite too. We have $\varphi_1 \in \dot{W}^{1,2} \subset \dot{W}_Y^{1,2}$, where the embedding is due to the boundedness of f_Y. The strict monotonicity of φ_1 (see Lemma 1), by which $\varphi_1^2(x)$ is strictly increasing on $[0,1]$, gives

$$\begin{aligned}\int_{-1}^{1} \varphi_1^2(x)(f_X(x)-f_Y(x))dx &= 2\int_0^1 \varphi_1^2(x)(f_X(x)-f_Y(x))dx \\ &= 2\int_0^{x_1} \varphi_1^2(x)(f_X(x)-f_Y(x))dx + 2\int_{x_1}^1 \varphi_1^2(x)(f_X(x)-f_Y(x))dx \\ &< 2\int_0^{x_1} \varphi_1^2(x_1)(f_X(x)-f_Y(x))dx + 2\int_{x_1}^1 \varphi_1^2(x_1)(f_X(x)-f_Y(x))dx \\ &= \varphi_1^2(x_1)\int_{-1}^1 (f_X(x)-f_Y(x))dx = 0\end{aligned}$$

that is

$$\int_{-1}^{1} \varphi_1^2(x)\ f_X(x)\,dx < \int_{-1}^{1} \varphi_1^2(x)\ f_Y(x)\,dx. \tag{12}$$

Since by Proposition 1 transformation φ_1 is concave on $[0,1]$, it follows that $(\varphi_1'(x))^2$ is decreasing on $[0,1]$. Thus, in a completely analogous way, we deduce

$$\int_{-1}^{1} (\varphi_1'(x))^2\ f_X(x)\,dx \geq \int_{-1}^{1} (\varphi_1'(x))^2\ f_Y(x)\,dx. \tag{13}$$

By (12) and (13), we finally conclude that

$$\lambda_1^{f_X} = \frac{\int_{-1}^1 \varphi_1^2(x)\ f_X(x)dx}{\int_{-1}^1 (\varphi_1'(x))^2\ f_X(x)\,dx} < \max_{\varphi\in\dot{W}_Y^{1,2}} \frac{\int_{-1}^1 \varphi^2(x)\ f_Y(x)\,dx}{\int_{-1}^1 (\varphi'(x))^2\ f_Y(x)\,dx} = \lambda_1^{f_Y}.$$

Since, under the assumptions of Proposition 3, it holds $\mathbb{E}\left[X^2\right] < \mathbb{E}\left[Y^2\right]$ we conclude that for the set of unimodal symmetric distributions considered, the variance ordering is preserved passing to the corresponding first NLPCs.

EXAMPLE 3. Consider the $css\beta(r)$ family (10). A direct inspection of K_r gives $r_2 > r_1$ if and only if $K_{r_2} > K_{r_1}$, $r_1, r_2 \in \mathbb{R}_+$. Thus $f_X(0,r) = K_r$ is increasing with respect to r. Furthermore, when r varies, the $f_X(x,r)$ intersect themselves once. On the other hand, by Example 1, we know that $f_X(x,r)$ satisfies the assumptions of Proposition 1 for all r. Hence Proposition 3 applies and, setting $\lambda_1^r := \lambda_1^{f_X(x,r)}$, we get $r_2 > r_1$ if and only if $\lambda_1^{r_2} < \lambda_1^{r_1}$, $\forall r \in \mathbb{R}_+$.

3 An Application

In [6] a goodness-of-fit test for uniform distributions against unimodal distributions, based on a comparison result proved in [10], was given. Proposition 3 and Remark

1 allow to characterize all the distributions involved only by the knowledge of λ_1, permitting to generalize such a test procedure.

As explanatory example, we test $X \in \Upsilon([-1,1])$ is Wigner (that is $css\beta\,(1/2)$, see (10)) against any other unimodal symmetric distribution and we state the hypothesis $\mathcal{H}_0 : \lambda_1 = \lambda_1^W$ against $\mathcal{H}_1 : \lambda_1 \neq \lambda_1^W$, where $\lambda_1^W = 0.28096$ is the variance of the first NLPC of a Wigner distribution on $[-1,1]$ computed by the package SLEIGN2 ([11]). This last computation is theoretically supported by the following

PROPOSITION 4. A Wigner r.v. X admits NLPCs $\varphi_j(X) = ce_j(\arccos(X), q_j)$, $j \in \mathbb{N}\setminus\{0\}$ where the $ce_j(\theta, q)$ are *Mathieu functions* (see [1] and [9]). Furthermore $\lambda_1 = (2a_1(q_1))^{-1}$, where $a_1(q)$ is a *characteristic value* and q_1 is the unique solution of $a_1(q) = 2q$.

PROOF. We recall that (see [1] and [9]) the 2π-periodic even solutions of the *Mathieu equation*:

$$z''(\theta) + (a - 2q\cos(2\theta))\; z(\theta) = 0 \qquad a, \theta, q \in \mathbb{R} \tag{14}$$

are called (*even*) *Mathieu functions*, usually indicated with $ce_j(\theta, q)$, $j \geq 1$. They can be expressed in uniformly convergent Fourier series of cosines where the coefficients can be determined only when a belongs to the set of the so called *characteristic value* $a_j(q)$ of the Mathieu equation. For the Wigner distribution, problem (3) can be written as

$$\begin{cases} \left(x^2 - 1\right) u''(x) + x u'(x) = \xi\left(1 - x^2\right)\, u(x) \qquad x \in (-1,1), \quad \xi \in \mathbb{R}_+ \\ \lim\limits_{x \to -1^+} u'(x)(1-x^2)^{1/2} = \lim\limits_{x \to 1^-} u'(x)(1-x^2)^{1/2} = 0\;. \end{cases} \tag{15}$$

By setting $x = \cos(\theta)$ and $z(\theta) = u(\cos(\theta))$, the equation in (15) becomes (14), but with $\theta \in (0, \pi)$ and $a = 2q = \xi/2$. Each solution of (15) can be extended to $\mathbb{R}$ in a $2\pi-$periodic even way, hence becoming one of the Mathieu functions $ce_j(\theta, q)$ (if $z(\theta)$ solves (14) the same holds for $z(\theta + k\pi)$, $k \in \mathbb{Z}$). We prove that, fixed j, for each family $ce_j(\theta, q)$, depending on $q \in \mathbb{R}_+$, there exists a unique value q_j such that $ce_j(\arccos(x), q_j)$, with $x \in (-1,1)$, solves problem (15). By construction, the $ce_j(\arccos(x), q_j)$ satisfy the boundary conditions in (15), for every $j \geq 1$ and $q \in \mathbb{R}_+$. Furthermore, by the continuity of $a_j(q)$ and $a_j(0) = j^2$, $a_j(q) \sim -2q + O(q^{1/2})$ as $q \to +\infty$, we get the existence, for every $j \geq 1$, of at least a solution q_j of $a_j(q) = 2q$. To each q_j it corresponds a solution $ce_j(\arccos(x), q_j)$ of (14) with $\xi = \xi_j = 2a_j(q_j)$. Recalling that each $ce_j(\theta, q)$ has exactly j zeros in $(0, \pi)$, independently on q (see [9], p. 234), the uniqueness of q_j, for every $j \geq 1$, follows by the simplicity of each ξ_j combined with the fact that two eigenfunctions can not have the same number of zeroes in $(-1,1)$. Finally, the completeness in $\dot{W}^{1,2}_{f_{\mathbf{X}}}$ of the set $ce_j(\arccos(x), q_j)\}_{j\geq 1}$ follows by standard theory of compact operators on Hilbert spaces.

To define the critical region of this test, we introduce the statistic $\delta_n = \sqrt{n}|\widehat{\lambda}_1 - \lambda_1^W|$, where $\widehat{\lambda}_1$ is a suitable estimate of λ_1 from a sample of size n (see [6]). We obtain the critical values by a Monte Carlo calculation based on five hundred replications.

Some numerical experiments to study the level and the power of the test proposed are carried out, having chosen as alternatives the $css\beta\,(r)$ family (10) and the Truncated Normal distribution $\mathcal{N}^T(0, \sigma)$ on $[-1,1]$. Sample sizes $n = 100$, 200 and 500 were considered. Testing at the level $\alpha = 0.1$, results obtained from five hundred simulations, are compared with the ones by the Kolmogorov-Smirnov and the Chi-square test. The substantially good performances of the test based on δ_n can be deduced from Table 1.

Distributions	Wigner	cssβ $(r = 0)$			cssβ $(r = 3/4)$			cssβ $(r = 1)$		
n	δ_n	δ_n	K-S	χ^2	δ_n	K-S	χ^2	δ_n	K-S	χ^2
100	0.094	0.884	0.390	0.590	0.131	0.117	0.152	0.378	0.177	0.280
200	0.098	0.989	0.682	0.875	0.321	0.150	0.191	0.786	0.332	0.473
500	0.104	1.000	0.982	0.998	0.691	0.264	0.325	0.995	0.739	0.847

Distributions	N^T $(0, \sigma = 1)$			N^T $(0, \sigma = 1/2)$		
n	δ_n	K-S	χ^2	δ_n	K-S	χ^2
100	0.440	0.154	0.216	0.585	0.308	0.415
200	0.626	0.212	0.355	0.944	0.604	0.655
500	0.903	0.378	0.716	0.100	0.969	0.964

Table 1: Estimated level and power in comparison with the Kolmogorov-Smirnov (K-S) and the Chi-square (χ^2) test ($\alpha = 0.1$).

References

[1] M. Abramowitz and I. A. Stegun, Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, Dover, New York, 1972.

[2] A. A. Borovkov and S. A. Utev, On an inequality and a related characterization of the normal distribution, Theory Probab. Appl., 28(1983), 209–218.

[3] T. Cacoullos, On upper and lower-bounds for the variance of a function of a random variable, Ann. Probab., 10(3)(1982), 799–809.

[4] L. H. Y. Chen and J. H. Lou, Characterization of probability distributions by Poincaré-type inequalities, Ann. Inst. H. Poincaré Probab. Statist., 23(1)(1987), 91–110.

[5] H. Chernoff, A note on an inequality involving the normal distribution, Ann. Probab., 9(3)(1981), 533–535.

[6] A. Goia and E. Salinelli, Optimal Nonlinear Transformations of Random Variables, to appear in Ann. Inst. H. Poincaré Probab. Statist.

[7] O. Johnson and A. Barron, Fisher information ineaqualities and the Central Limit Theorem, Probab. Theory Related Fields, 129(2004), 391–409.

[8] C. A. J. Klaassen, On an inequality of Chernoff, Ann. Probab., 13(3)(1985), 966–974.

[9] N. W. McLachlan, Theory and Applications of Mathieu Functions, Dover, New York, 1964.

[10] S. Purkayastha and S. K. Bhandari, Characterization of uniform distributions by inequality of Chernoff-type, Sankhya Series A, 52(1990), 376–382.

[11] A. Zettl, Sturm-Liouville Theory, Mathematical Survey and Monographs, 121, American Mathematical Society, Providence, 2005.