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Well-Posedness Of A Cauchy Problem Involving Nonlinear Fractal Dissipative Equations*

Yuwen Luo†

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Abstract

This paper studies the well-posedness of fractal dissipative equations with the initial data u_0 in Lebesgue spaces. For $u_0 \in L^r(R^n)$, $r > n/(2\alpha - 1)$ and $\alpha > 1/2$ we prove that the fractal dissipative equation has a unique local solution. For the critical case $r = n/(2\alpha - 1)$, the equation has a unique local solution in $C([0,T]; L^r(R^n)) \cap C([0,T]; L^p(R^n))$ for each $T > 0$. Furthermore, if the L^r norm of the initial data is small enough, then the solution is global.

1 Introduction

In this paper, we study the well-posedness for nonlinear fractional dissipative equation

$$\begin{cases} u_t + (-\Delta)^\alpha u + \sum\limits_{i=1}^{n} u u_i = 0, \\ u(0) = u_0, \end{cases} \tag{1}$$

in the domain $R^n \times R^+$. We use the notifications $u_i = \frac{\partial u}{\partial x_i}$, $L^p(R^n) = L^p$ and $\|\cdot\|_{L^p} = \|\cdot\|_{L^p(R^n)}$. The fractal Laplacian is defined as ([8]):

$$\widehat{(-\Delta)^\alpha} f = |\xi|^{2\alpha} \hat{f},$$

here $\hat{f}$ denote the Fourier transform of f. The fractal Laplacian $(-\Delta)^\alpha$ arise from real physical problems, such as overdriven detonations in gases (see [1]), anomalous diffusion in semiconductor growth ([10]) and mathematical models in finance ([7]). Equation (1) models several classical equations, such as

(i) the generalized Navier-Stokes equations

$$u_t + (-\Delta)^\alpha u - u \cdot \nabla u + \nabla P = 0, \quad \nabla u = 0,$$

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†School of Mathematics & Physics, Chongqing University of Technology, Chongqing 400050, P. R. China

(ii) the generalized convection-diffusion equation

$$u_t + (-\Delta)^\alpha u = a \cdot \nabla(|u|^b u), \quad a \in R^n/\{0\},$$

(iii) the dissipative quasi-geostrophic equation

$$\begin{cases} \theta_t + u \cdot \nabla\theta + \kappa(-\Delta)^\alpha \theta = 0, & (t,x) \in R^+ \times R^2, \\ u = (u_1, u_2) = \nabla^\perp \psi, \quad (-\Delta)^\alpha \psi = \theta, & \end{cases}$$

where $1/2 < \alpha \leq 1$.

When $\alpha = 1$, equation (1) becomes a nonlinear heat equation, there are plenty of literatures for well-posedness of equation (1), the reader may refer to [9] and its references therein. For general α, well-posedness of quasi-geostrophic equations has been studied in Lebesgue spaces [11], Sobolev spaces [12], Besov spaces [14, 17] and Hölder spaces [15]. Well-posedness of generalized Navier-Stokes equations has been studied in Lebesgue spaces [4], Besov spaces [13] and Q spaces [3, 16]. For general fractional power dissipative equation, please refer to [5].

In this paper, we restrict ourselves to the initial data u_0 belonging to L^p spaces. The method is standard. By means of the operator semi-group $S_\alpha(t) = e^{-t(-\Delta)^\alpha}$, we can prove that the mapping

$$\mathcal{T}u = S_\alpha(t)u_0 + \int_0^t S_\alpha(t-s)F(u(s))ds$$

has a unique fixed point in a suitable Banach space. Then the existence and uniqueness follow.

The basic estimates we used in this paper are

$$\|e^{-t(-\Delta)^\alpha}\varphi\|_{L^p} \leqslant CT^{-\frac{n}{2\alpha}(\frac{1}{r}-\frac{1}{p})}\|\varphi\|_{L^r},$$

and

$$\|\nabla e^{-t(-\Delta)^\alpha}\varphi\|_{L^p} \leqslant CT^{-\frac{1}{2\alpha}-\frac{n}{2\alpha}(\frac{1}{r}-\frac{1}{p})}\|\varphi\|_{L^r},$$

These estimates was stated in Miao et al. [5] but detailed proof is not provided. For a similar proof, please refer to Wu [11].

Now we rewrite equation (1) in the abstract form

$$u_t + (-\Delta)^\alpha u + F(u) = 0,\ u(0) = u_0,$$

where $F(u) = \sum_{i=1}^n uu_i$.

By Duhamel's principle, the solution of equation (1) can then be written as

$$u(x,t) = e^{-t(-\Delta)^a}u_0 + \int_0^t e^{-(t-s)(-\Delta)^\alpha}F(u(s))ds \triangleq e^{-t(-\Delta)^a}u_0 + Gu,$$

where the operator semi-group $e^{-t(-\Delta)^\alpha}$ is defined by the Fourier transform

$$e^{-t(-\Delta)^\alpha}u_0 = \mathcal{F}^{-1}(e^{-t|\xi|^{2\alpha}}\mathcal{F}u_0) = (\mathcal{F}^{-1}e^{-t|\xi|^{2\alpha}}) * u_0.$$

We write $K_t(x) = \mathcal{F}^{-1}e^{-t|\xi|^{2\alpha}}$, which is the kernel function of the operator semi-group $S_\alpha(t)$. Here $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the Fourier transform and its inverse transform. As usual, we write $\hat{f} = \mathcal{F}(f), \check{f} = \mathcal{F}^{-1}(f)$.

Using the basic estimates, we will prove that the mapping

$$\mathcal{T}u = e^{-t(-\Delta)^a}u_0 + Gu \tag{2}$$

is a contraction mapping from a suitable Banach space to itself. Then by contraction mapping principle, we get the existence and uniqueness of solution in this space.

2 Well-posedness for $r > n/(2\alpha - 1)$

We have the following preparatory result.

THEOREM 1. Let $r > n/(2\alpha - 1)$ and $\alpha > 1/2$. If the initial data $u_0 \in L^r(R^n)$, then (1) has a unique solution u having the property

$$u \in C([0,T]; L^r)$$

PROOF. We introduce the notation

$$X = C(I; L^r),$$

where $I = [0,T]$ and consider a fixed point of the mapping $\mathcal{T}$ in the following complete metric space

$$\mathcal{X} = \{u \in C(I; L^r) : u(0) = u_0, \|u\|_X \leqslant C\|u_0\|_{L^r}\}.$$

Note that

$$\|S_\alpha(t)u_0\|_X \leqslant C\|u_0\|_{L^p}$$

from the basic estimates. Then by Banach contraction mapping principle, we only need to prove that $\mathcal{T}$ is a contraction mapping from X to X. Next we prove this fact.

By direct computation, we get

$$\begin{aligned} \|S_\alpha(t-s)F(u(s))\|_{L^p} &= \|\mathcal{F}^{-1}(e^{-(t-s)|\xi|^{2\alpha}}\mathcal{F}(\sum_{i=1}^{n} uu_i))\|_{L^p} \\ &= \|\nabla K_{t-s}(x) * (u^2)\|_{L^p} \\ &\leqslant C(t-s)^{-\frac{1}{2\alpha}-\frac{n}{2\alpha}(\frac{2}{p}-\frac{1}{p})}\|u\|^2_{L^p}. \end{aligned} \tag{3}$$

Then

$$\begin{aligned} \|Gu\|_X &= \sup_{t\in I}\int_0^t \|e^{-(t-s)(-\Delta)^\alpha}F(u(s))\|_{L^p}ds \\ &\leqslant CT^{1-\frac{1}{2\alpha}-\frac{n}{2\alpha p}}\sup_{t\in I}\|u\|^2_{L^p} \\ &\leqslant CT^{1-\frac{1}{2\alpha}-\frac{n}{2\alpha p}}\|u_0\|^2_{L^p}. \end{aligned} \tag{4}$$

From these two inequalities, we get

$$\|\mathcal{T}u\| \leqslant C\|u_0\|_{L^p} + CT^{1-\frac{1}{2\alpha}-\frac{n}{2\alpha p}}\|u_0\|^2_{L^p},$$

and

$$d(\mathcal{T}u, \mathcal{T}v) \leqslant CT^{1-\frac{1}{2\alpha}-\frac{n}{2\alpha p}}\|u_0\|_{L^p}d(u,v).$$

If we choose T small enough such that $CT^{1-\frac{1}{2\alpha}-\frac{n}{2\alpha p}} < 1$, then $\mathcal{T}$ is a contraction mapping from X to X. This completes the proof.

3 Well-posedness for $p = n/(2\alpha - 1)$

The case $p = n/(2\alpha-1)$ is more complicated than $p > n/(2\alpha-1)$. The class of spaces we worked in is no longer fitted for this case. So we need another Banach space to work in. First, we give some definitions.

DEFINITION 1. We say that (p, q, r) is an admissible triplet if

$$\frac{1}{q} = \frac{n}{2\alpha}\left(\frac{1}{r} - \frac{1}{p}\right)$$

where

$$1 < r \leqslant p \leqslant \begin{cases} \frac{nr}{n-2\alpha}, & \text{for } n > 2\alpha \\ \infty, & \text{for } n \leqslant 2\alpha \end{cases}.$$

DEFINITION 2. We say that (p, q, r) is a generalized admissible triplet if

$$\frac{1}{q} = \frac{n}{2\alpha}\left(\frac{1}{r} - \frac{1}{p}\right)$$

where

$$1 < r \leqslant p \leqslant \begin{cases} \frac{nr}{n-2\alpha r}, & \text{for } n > 2\alpha r \\ \infty, & \text{for } n \leqslant 2\alpha r \end{cases}.$$

Let B be a Banach space, $\sigma > 0$ and $I = [0, T]$. Define space $\mathcal{C}_\sigma(I; B)$ and its homogeneous space $\dot{\mathcal{C}}_\sigma(I; B)$ as

$$\mathcal{C}_\sigma(I; B) = \{f \in C(I, B) | \|f\|_{C_\sigma(I;B)} = \sup_{t\in I} t^{\frac{1}{\sigma}} \|f\|_B < \infty\},$$

and

$$\dot{\mathcal{C}}_\sigma(I; B) = \{f \in \mathcal{C}_\sigma(I, B) | \lim_{t\to 0^+} t^{\frac{1}{\sigma}} \|f\|_B = 0\}.$$

In this paper, B will be restricted to be $L^p(R^n)$ with $1 < p < \infty$.

With the above definitions, we are ready to construct our work spaces. First, let

$$\Gamma = \{ \text{ all generalized admissible triplet } (p, q, r) \text{ and } p \neq r\},$$
$$\Xi = \{ \text{ all admissible triplet } (p, q, r) \text{ and } p \neq r\}.$$

Then let

$$X(I) = \{u \in C_b(I; L^r) \cap \dot{\mathcal{C}}_{q(p,r)}(I; L^p)\}, \|u\|_X = \max_{(q,p,r)\in\Gamma} \sup_{t\in I} t^{\frac{1}{q}} \|u\|_{L^p} + \sup_{t\in I} \|u\|_{L^r}.$$

Here $C_b(I; L^r)$ denotes the spaces of bounded continuous functions from I to L^r.

Now we look for a fixed point of mapping $\mathcal{T}$ defined in (2) in the following complete metric spaces

$$\mathcal{X}(I) = \{u \in X, \|u\|_X \leqslant C\|u_0\|_{L^r}\}, d(u, v) = \|u - v\|_{X(I)}.$$

Before we are going on, we state a preliminary lemma.

LEMMA 1.

(i) Let (q,p,r) be any admissible triplet and let $\varphi \in L^r(R^n)$ Then $S_a(t)\varphi \in L^q(I;L^p)\cap C_b(I;L^r)$ with the estimate

$$\|S_\alpha(t)\varphi(x)\|_{L^q(I;L^p)} \leqslant C\|\varphi\|_{L^r}, \tag{5}$$

for $0<T\leqslant\infty$, where C is a positive constant.

(ii) Let (q,p,r) be any generalized admissible triplet and let $\varphi \in L^r(R^n)$. Then $S_a(t)\varphi \in \mathcal{C}_q(I;L^p)\cap C_b(I;L^r)$ with the estimate

$$\|S_\alpha(t)\varphi(x)\|_{L^q(I;L^p)} \leqslant C\|\varphi\|_{L^r}, \tag{6}$$

Here $C_b(I;L^r)$ denote the bounded continuous functions from I to L^r.

For details of this lemma, please refer to Miao et al. [5].

THEOREM 2. Let $\alpha > 1/2$ and initial data $u_0 \in L^r$ and (q,p,r) an arbitrary generalized admissible triplet. If $p = n/(2\alpha-1)$, then equation (1) has a unique solution

$$u \in C(I;L^r)\cap \mathcal{C}_{q(p,r)}(I;L^p).$$

Furthermore, if the norm of the initial data $\|u_0\|_n$ is small enough, then the solution is global.

PROOF. We first remark that our proof follows that of Miao and Zhang [6], in which nonlinear heat equations are considered. First, note from Lemma 1 that for $u_0 \in L^r$, we get

$$\|S_\alpha(t)u_0\|_X \leqslant C\|u_0\|_{L^r}.$$

We first consider the special case $(q,p,r) = (4, 2n/(2\alpha-1), n/(2\alpha-1))$. By direct computation and inequality (3), we get

$$\begin{aligned}
\|Gu\|_{\mathcal{C}_{q(p,r)}(I;L^p)} &\leqslant \sup_{t\in I} t^{\frac{1}{q}}\int_0^t |t-s|^{-\frac{1}{2\alpha}-\frac{n}{2\alpha}(\frac{2}{p}-\frac{1}{p})}\|u\|_{L^p}^2 ds\\
&\leqslant CT^{1-\frac{1}{2\alpha}-\frac{n}{2\alpha p}-\frac{1}{q}}\|u\|_{\mathcal{C}q(p,r)(I;L^p)}\int_0^1 |1-s|^{-\frac{1}{2}-\frac{n}{2\alpha p}}s^{-\frac{2}{q}}ds\\
&\leqslant CT^{1-\frac{1}{2\alpha}-\frac{n}{2\alpha r}}\|u\|_{\mathcal{C}q(p,r)(I;L^p)}\\
&\leqslant C\|u\|_{\mathcal{C}q(p,r)(I;L^p)},
\end{aligned} \tag{7}$$

where the last inequality follows from the fact $r = n/(2\alpha-1)$. Next, we compute

$$\begin{aligned}
\|Gu\|_{C_b(I;L^r)} &\leqslant \sup_{t\in I}\int_0^t |t-s|^{-\frac{1}{2\alpha}-\frac{n}{2\alpha}(\frac{2}{p}-\frac{1}{r})}\|u\|_{L^p}^2 ds\\
&\leqslant \|u\|^2_{\mathcal{C}_{q(p,r)}(I;L^p)}\sup_{t\in I}\int_0^t |t-s|^{-\frac{1}{2\alpha}-\frac{n}{2\alpha}(\frac{2}{p}-\frac{1}{r})}s^{-\frac{2}{q}}ds\\
&\leqslant C\|u\|^2_{\mathcal{C}_{q(p,r)}(I;L^p)}T^{1-\frac{1}{2\alpha}-\frac{2}{q}}\int_0^1 |1-s|^{-\frac{1}{2}-\frac{n}{2\alpha p}}s^{-\frac{2}{q}}ds\\
&\leqslant C\|u\|^2_{\mathcal{C}_{q(p,r)}(I;L^p)},
\end{aligned} \tag{8}$$

where in the last inequality we used the fact that $1-\frac{1}{2\alpha}-\frac{2}{q}=0$. These two inequalities and Lemma 1 imply that

$$\|\mathcal{T}u\|_X \leqslant C\|u_0\|_{L^r} + C\|u\|^2_{\mathcal{C}_{q(p,r)}(I;L^p)}. \tag{9}$$

Furthermore, by (3), we have

$$t^{\frac{1}{q}}\|\mathcal{T}u\|_p \leqslant t^{\frac{1}{q}}\|S_\alpha(t)u_0\|_{L^p} + C(t^{\frac{1}{q}}\|u\|_{L^r})^2, \tag{10}$$

and

$$d(\mathcal{T}u, \mathcal{T}v) \leqslant C\left[\|u\|_{\mathcal{C}_{q(p,r)}(I,L^p)} + \|v\|_{\mathcal{C}_{q(p,r)}(I,L^p)}\right] d(u,v). \tag{11}$$

Note that

$$\lim_{t\to 0} t^{\frac{1}{q}}\|u\|_{L^p} = 0, \quad \lim_{t\to 0} t^{\frac{1}{q}}\|S_a(t)u_0\|_{L^p} = 0. \tag{12}$$

From (9), (10), (11) and (12), we conclude that if T is small enough, then $\mathcal{T}$ is a contraction mapping from $\mathcal{X}$ into itself. This proves the existence and uniqueness of the solution in the case $(q,p,r) = (4, 2n/(2\alpha-1), n/(2\alpha-1))$. Furthermore, if $\|u_0\|_r$ is small enough, then by (9), (10), (11), the solution is global.

Next we consider an arbitrary generalized triplet. Write $(q,p,r) = (4, 2n/(2\alpha-1), n/(2\alpha-1))$ and an arbitrary generalized triplet as $(\hat{q},\hat{p},r) = (\hat{q},\hat{p},n/(2\alpha-1))$. Then

$$\begin{aligned}
\|Gu\|_{\mathcal{C}_{\hat{q}(\hat{p},r)}(I;L^{\hat{p}})} &\leqslant \sup_{t\in I} t^{\frac{1}{q}}\int_0^t |t-s|^{-\frac{1}{2}-\frac{n}{2\alpha}(\frac{2}{p}-\frac{1}{\hat{p}})}\|u\|_{L^p}^2 ds \\
&\leqslant \sup_{t\in I} t^{\frac{1}{q}}\int_0^t |t-s|^{-\frac{1}{2}-\frac{n}{2\alpha}(\frac{2}{p}-\frac{1}{\hat{p}})} s^{-\frac{2}{q}} s^{\frac{2}{q}}\|u\|_{L^p}^2 ds \\
&\leqslant C\|u\|_{\mathcal{C}q(p,r)(I;L^p)} T^{1-\frac{1}{2\alpha}-\frac{2}{q}}\int_0^1 |1-s|^{-\frac{1}{2\alpha}-\frac{n}{2\alpha}(\frac{2}{p}-\frac{1}{\hat{p}})} s^{-\frac{2}{q}} ds \\
&\leqslant C\|u\|_{\mathcal{C}q(p,r)(I;L^p)} \\
&< \infty.
\end{aligned}$$

The rest of the proof is similar with the case $(q,p,r) = (4, 2n/(2\alpha-1), n/(2\alpha-1))$ and we omit it here. This completes the proof.

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