Iterative Methods For Mixed Quasi Bifunction Variational Inequalities*

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Abstract

In this paper, we introduce and consider a new class of variational inequalities, which is called the mixed quasi bifunction variational inequality. We use the auxiliary principle technique in conjunction with the Bregman function to suggest and analyze a three-step predictor-corrector method for solving mixed quasi bifunction variational inequalities. We also study the convergence criteria of this new method under some mild conditions. As special cases, we obtain various new and known methods for solving variational inequalities and related optimization problems.

1 Introduction

Variational inequalities are being used to study a wide class of diverse unrelated problems arising in various branches of pure and applied sciences in a unified framework. Various generalizations and extensions of variational inequalities have been considered in different directions using novel and innovative technique, see [1-31] and the references therein. A useful and important generalization of the variational inequalities is called the bifunction variational inequality. Crespi et al [2-5], Fang and Hu [6], Lalitha and Mehra [11] and Noor [18] have studied some aspects of the bifunction variational inequalities. Inspired and motivated by the research going on in this fascinating field, we introduce and consider a new class of variational inequalities, which is called the mixed quasi bifunction variational inequality. This variational inequality includes the bifunction variational inequalities [2-6, 11, 18], variational inequalities and optimization problems as special cases. We note that there are substantial number of numerical methods including projection technique and its variant forms, Wiener-Hopf equations, auxiliary principle and resolvent equations methods for solving variational inequalities. However, it is known that projection, Wiener-Hopf equations and resolvent equations techniques can not be extended and generalized to suggest and analyze similar iterative methods for solving bifunction variational inequalities due to its very special structure.

This fact motivated to use the auxiliary principle technique which is due to Glowinski, Lions and Tremolieres [9]. In this paper, we again use the auxiliary principle technique in conjunction with the Bregman function to suggest and analyze a three-step iterative algorithms for solving mixed quasi bifunction variational inequalities. It is shown that the convergence of this method requires partially relaxed strongly monotonicity, which is a weaker condition than monotonicity. Our results can be considered as a novel and important application of the auxiliary principle technique.

2   Prelimianries

Let H be a real Hilbert space, whose inner product and norm are denoted by $\langle .,. \rangle$ and $\|.\|$ respectively. Let K be a nonempty closed and convex set in H.

For given nonlinear operator $T(.,.) : K \times K \longrightarrow H$ and continuous bifunction $\varphi(.,.) : K \times K \longrightarrow R \cup \{\infty\}$, we consider the problem of finding $u \in K$ such that

$$T(u, v - u) + \varphi(v, u) - \varphi(u, u) \geq 0, \quad \forall v \in K. \tag{1}$$

Inequality of type (1) is called the *mixed quasi variational inequality involving the bifunction.* One can easily show that the minimum of a sum of directional differentiable convex function and a nondifferentiable convex function can be characterized by the mixed quasi bifunction variational inequality of the type (1). It can be shown that a wide class of problems, which arise in pure and applied sciences, can be studied in the unified framework of the mixed quasi bifunction variational inequalities of the type (1), see [2-6, 11, 20].

We note that, if $T(u, v - u) = \langle Au, v - u \rangle$, where A is a nonlinear operator, then problem (1) is equivalent to finding $u \in K$ such that

$$\langle Au, v - u \rangle + \varphi(v, u) - \varphi(u, u) \geq 0, \quad \forall v \in K, \tag{2}$$

which is known as a mixed quasi variational inequality. It has been shown [1, 7-10,12-24] that a wide class of problems arising in elasticity, fluid flow through porous media and optimization can be studied in the general framework of problems (1) and (2).

In particular, if a function $\varphi(.,.) = 0$, then problem (1) is equivalent to finding $u \in K$ such that

$$T(u, v - u) \geq 0, \quad \forall v \in K, \tag{3}$$

which is called the bifunction variational inequality. For the formulation and other aspects of the bifunction variational inequalities and related optimization problems, see [2-6,11-30] and the references therein. For suitable and appropriate choice of the operators $T(.,.), \quad \varphi(.,.)$ and spaces H, one can obtain several classes of variational inequalities as special cases of problem (1). This shows that problem (1) is quite general and unifying ones and has important applications in various fields of pure and applied sciences.

DEFINITION 2.1. The operator $T(.,.) : K \times K \longrightarrow H$ said to be *partially relaxed strongly monotone,* iff, there exists a constant $\alpha > 0$ such that

$$T(u, v - u) + T(z, u - v) \leq \alpha \|z - u\|^2, \quad \forall u, v, z \in K.$$

Note that for $z = v$ partially relaxed strongly monotonicity reduces to monotonicity of the operator $T(.,)$.

DEFINITION 2.2. The bifunction $\varphi(.,.) : H \times H \longrightarrow R \cup \{+\infty\}$ is called *skew-symmetric,* iff,

$$\varphi(u, u) - \varphi(u, v) - \varphi(v, u) + \varphi(v, v) \geq 0, \quad \forall u, v \in H.$$

Clearly if the skew-symmetric bifunction $\varphi(.,.)$ is bilinear, then

$$\varphi(u, u) - \varphi(u, v) - \varphi(v, u) + \varphi(v, v) = \varphi(u - v, u - v) \geq 0, \quad \forall u, v \in H.$$

3 Main Resulrs

In this section, we use the auxiliary principle technique of Glowinski et al [9], as developed by Noor [15-21] to suggest and analyze a three-step iterative algorithm for solving mixed quasi bifunction variational inequalities (1).

For a given $u \in K$, consider the problem of finding $z \in K$ such that

$$\rho T(u, v - z) + \langle E'(z) - E'(u), v - z \rangle \geq \rho\varphi(z, z) - \rho\varphi(v, z), \quad \forall v \in K, \qquad (4)$$

where $E'(u)$ is the differential of a strongly convex function $E(u)$ and $\rho > 0$ is a constant. Problem (4) has a unique solution due to the strongly convexity of the function $E(u)$, see [7-9, 11].

REMARK 3.1. The function $B(z, u) = E(z) - E(u) - \langle E'(u), z - u \rangle$ associated with the convex function $E(u)$ is called the generalized Bregman function. For the applications of the Bregman function for solving variational inequalities and complementarity problems, see [19, 31].

We remark that if $z = u$, then z is a solution of the mixed quasi bifunction variational inequality (1). On the basis of this observation, we suggest and analyze the following iterative algorithm for solving (1) as long as (4) is easier to solve than (1).

ALGORITHM 3.1. For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative schemes:

$$\rho T(w_n, v - u_{n+1}) + \langle E'(u_{n+1}) - E'(w_n), v - u_{n+1} \rangle \geq \rho\varphi(u_{n+1}, u_{n+1}) - \rho\varphi(v, u_{n+1}), \forall v \in K \tag{5}$$

$$\nu(Ty_n, v - w_n) + \langle E'(w_n) - E'(y_n), v - w_n \rangle \geq \nu\varphi(w_n, w_n) - \nu\varphi(v, w_n), \quad \forall v \in K \tag{6}$$

$$\mu T(u_n, v - y_n) + \langle E'(y_n) - E'(u_n), v - y_n \rangle \geq \mu\varphi(y_n, y_n) - \mu\varphi(v, y_n), \quad \forall v \in K, \tag{7}$$

where E' is the differential of a strongly convex function E. Here $\rho > 0, \nu > 0$ and $\mu > 0$ are constants.

Algorithm 3.1 is called the three-step predictor-corrector iterative method for solving the mixed quasi bifunction variational inequalities (1). If $T(u, v - u) = \langle Au, v - u \rangle$, then Algorithm 3.1 reduces to:

ALGORITHM 3.2. For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme

$$\langle \rho Aw_n + E'(u_{n+1}) - E'(w_n), v - u_{n+1} \rangle \geq \rho\varphi(u_{n+1}, u_{n+1}) - \rho\varphi(v, u_{n+1}), \forall v \in K$$

$$\langle \nu A y_n + E'(w_n) - E'(y_n), v - w_n \rangle \geq \nu\varphi(w_n, w_n) - \nu\varphi(v, w_n), \quad \forall v \in K$$
$$\langle \mu A u_n + E'(y_n) - E'(u_n), v - y_n \geq \mu\varphi(y_n, y_n) - \mu\varphi(v, y_n), \quad \forall v \in K,$$

where E' is the differential of a strongly convex function E.

Algorithm 3.2 is known as three-step iterative method for solving mixed quasi variational inequalities (2), see [15, 17, 19]. For appropriate and suitable choice of the operators $T(.,), \varphi(.,.)$ and the space H, on can obtain several new and known three-step, two-step and one-step iterative methods for solving various classes of variational inequalities and related optimization problems.

We now study the convergence analysis of Algorithm 3.1, which is the main motivation of our next result.

THEOREM 3.1. Let E be strongly differentiable convex function with modulus β. Let the bifunction $\varphi(.,.)$ be skew-symmetric. If the operator $T(.,.)$ is partially relaxed strongly monotone with constant $\alpha > 0$, then the approximate solution obtained from Algorithm 3.1 converges to a solution $u \in K$ of (1) for $\rho < \frac{\beta}{\alpha}$, $\nu < \frac{\beta}{\alpha}$ and $\mu < \frac{\beta}{\alpha}$.

PROOF. Let $u \in K$ be a solution of (1). Then

$$\rho\{T(u, v - u) + \varphi(v, u) - \varphi(u, u)\} \geq 0 \quad \forall v \in K \tag{8}$$
$$\mu\{T(u, v - u) + \varphi(v, u) - \varphi(u, u)\} \geq 0 \quad \forall v \in K \tag{9}$$
$$\nu\{T(u, v - u) + \varphi(v, u) - \varphi(u, u)\} \geq 0 \quad \forall v \in K, \tag{10}$$

where $\rho > 0$, $\mu > 0$ and $\nu > 0$ are constants.

Taking $v = u_{n+1}$ in (8) and $v = u$ in (5), we have

$$\rho\{T(u, u_{n+1} - u) + \varphi(u_{n+1}, u) - \varphi(u, u)\} \geq 0 \tag{11}$$

$$\rho T(w_n, u - u_{n+1}) + \langle E'(u_{n+1}) - E'(w_n), u - u_{n+1} \rangle \geq \rho\{\varphi(u_{n+1}, u_{n+1}) - \varphi(u, u_{n+1})\}. \tag{12}$$

Consider the Bregman function,

$$B(z) := B(u, z) = E(u) - E(z) - \langle E'(z), u - z \rangle \geq \beta\|u - z\|^2, \tag{13}$$

since the function $E(u)$ is strongly convex with modulus $\beta > 0$.

From (11), (12) and (13), we have

$$
\begin{aligned}
B(u, w_n) - B(u, u_{n+1}) &= E(u_{n+1}) - E(w_n) - \langle E'(w_n), u - w_n \rangle \\
&\quad + \langle E'(u_{n+1}), u - u_{n+1} \rangle \\
&= E(u_{n+1}) - E(w_n) - \langle E'(w_n) - E'(u_{n+1}), u - u_{n+1} \rangle \\
&\quad - \langle E'(w_n), u_{n+1} - w_n \rangle \\
&\geq \beta\|u_{n+1} - w_n\|^2 + \langle E'(u_{n+1}) - E'(w_n), u - u_{n+1} \rangle \\
&\geq \beta\|u_{n+1} - w_n\|^2 - \rho T(w_n, u - u_{n+1}) \\
&\quad + \rho\{\varphi(u_{n+1}, u_{n+1}) - \varphi(u, u_{n+1})\} \\
&\geq \beta\|u_{n+1} - w_n\|^2 + \rho\{\varphi(u_{n+1}, u_{n+1}) - \rho\varphi(u, u_{n+1}) \\
&\quad - \varphi(u_{n+1}, u) + \varphi(u, u)\}
\end{aligned}
$$

$$-\rho\{T(w_n, u - u_{n+1}) + T(u, u_{n+1} - u)\}$$
$$\geq \quad \beta\|u_{n+1} - w_n\|^2 - \alpha\rho\|u_{n+1} - w_n\|^2$$
$$= \quad \{\beta - \rho\alpha\}\|u_{n+1} - w_n\|^2,$$

where we have used the fact that the bifunction $\varphi(.,.)$ is skew-symmetric and the operator T is a partially relaxed strongly monotone with constant $\alpha > 0$.

In a similar way, we have

$$B(u, y_n) - B(u, w_n) \quad \geq \quad \{\beta - \nu\alpha\}\|w_n - y_n\|^2$$
$$B(u, u_n) - B(u, y_n) \quad \geq \quad \{\beta - \mu\alpha\}\|y_n - u_n\|^2.$$

If $u_{n+1} = w_n = y_n =_n$, then clearly u_n is a solution of the mixed quasi bifunction variational inequality (1). Otherwise, for $\rho < \frac{\beta}{\alpha,}$ $\nu < \frac{\beta}{\alpha}$ and $\mu < \frac{\beta}{\alpha}$, the sequences $B(u, w_n) - B(u, u_{n+1})$, $B(u, y_n) - B(u, w_n)$ and $B(u, u_n) - B(u, y_n)$ are nonnegative and we must have

$$\lim_{n\longrightarrow\infty}\|u_{n+1} - w_n\| = 0, \quad \lim_{n\longrightarrow\infty}\|w_n - y_n\| = 0 \quad \text{and} \quad \lim_{n\longrightarrow\infty}\|y_n - u_n\| = 0.$$

Thus

$$\lim_{n\longrightarrow\infty}\|u_{n+1} - u_n\| = \lim_{n\longrightarrow\infty}\|u_{n+1} - w_n\| + \lim_{n\longrightarrow\infty}\|w_n - y_n\| + \lim_{n\longrightarrow\infty}\|y_n - u_n\| = 0.$$

From which, it follows that the sequence $\{u_n\}$ is bounded. Let $\bar{u} \in K$ be a cluster point of the sequence $\{u_n\}$ and let the subsequence $\{u_{n_i}\}$ of the sequence converge to $\bar{u} \in K$, which is a solution of the mixed quasi bifunction variational inequality (1). Now following the technique of Zhu and Marcotte [31], if we replace u by $\bar{u}$ in (13), the analysis remains the same for $\bar{u}$ and its associated Bregman function $\breve{B}$.The sequence $\{\bar{u}\}$ still strictly decreases and we have

$$\bar{B}(u_m) \leq \beta\|\bar{u} - u_n\|^2.$$

this shows that the sequence $\bar{B}(u_n)$ converges to zero. This result, together with the inequality

$$\bar{B}(u_n) \geq \beta\|u_n - \bar{u}\|^2,$$

enables us to conclude the entire sequence $\{u_n\}$ converges to the cluster point $\overline{u}$ satisfying the mixed quasi bifunction variational inequality (1).

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