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New Class Of Inequalities Associated with Hilbert-Type Double Series Theorem*

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Abstract

Considering a Dirichlet series associated with the Hilbert's bilinear double series a new class of Hilbert–type double series theorems is obtained in the case of a non–homogeneous kernel.

1 Introduction

Let ℓ_p be the space of all complex sequences $\boldsymbol{x} = (x_n)_{n=1}^\infty$ endowed with the finite norm $\|\boldsymbol{x}\|_p := \big(\sum_{n=1}^\infty |x_n|^p\big)^{1/p}$. Assume $\boldsymbol{a} = (a_n)_{n=1}^\infty \in \ell_p,\ \boldsymbol{b} = (b_n)_{n=1}^\infty \in \ell_q$ are nonnegative sequences and $1/p + 1/q = 1,\ p > 1$. The celebrated Hilbert's double series theorem (in other words, discrete Hilbert inequality) reads:

$$\sum_{m,n\geq 1} \frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)} \|\boldsymbol{a}\|_p \|\boldsymbol{b}\|_q \,. \tag{1}$$

Here the constant $\pi/\sin(\pi/p)$ is best possible [2, p. 253]. An overview of another kind of Hilbert's double series theorems is given in [1], see also the references listed therein.

Let us consider the infinite bilinear form

$$\mathfrak{H}_K^{\boldsymbol{a},\boldsymbol{b}} := \sum_{m,n\geq 1} K(m,n)\, a_m b_n \,, \tag{2}$$

where $\boldsymbol{a}, \boldsymbol{b}$ are nonnegative; $K(\cdot,\cdot)$ is called a kernel function of the double series (2). Hilbert–type inequalities are sharp upper bounds for $\mathfrak{H}_K^{\boldsymbol{a},\boldsymbol{b}}$ in terms of (weighted) ℓ_p–norms of $\boldsymbol{a}, \boldsymbol{b}$. We note that for the first time after (1) Hilbert–type inequality in terms of $\|\boldsymbol{a}\|_p, \|\boldsymbol{b}\|_q$ was given by Draščić Ban and the author himself in [1, Theorem 1]. They considered the case

$$K(m,n) = (\lambda_m + \rho_n)^{-\mu} \qquad \mu > 0, \tag{3}$$

where $\lambda, \rho\colon \mathbb{R}_+ \mapsto \mathbb{R}_+$ are monotone increasing functions such that

$$\lim_{x\to\infty} \lambda(x) = \lim_{x\to\infty} \rho(x) = \infty \,, \tag{4}$$

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and $\lambda\big|_{\mathbb{N}} = \boldsymbol{\lambda} = (\lambda_n)_{n=1}^\infty, \rho\big|_{\mathbb{N}} = \boldsymbol{\rho} = (\rho_n)_{n=1}^\infty$, that is

$$\mathfrak{H}_{\boldsymbol{\lambda},\boldsymbol{\rho}}^{\boldsymbol{a},\boldsymbol{b}} = \sum_{m,n\geq 1} \frac{a_m b_n}{(\lambda_m + \rho_n)^\mu}\,. \tag{5}$$

Of course, such $K(m,n)$ is in general not homogeneous.

In this short note one extends the results of [1] to a new class of inequalities by introducing and studying an auxiliary function. We consider the Dirichlet–series

$$\mathfrak{H}_{\boldsymbol{\lambda},\boldsymbol{\rho}}^{\boldsymbol{a},\boldsymbol{b}}(\mu;x) := \sum_{m,n\geq 1} \frac{a_m b_n}{(\lambda_m + \rho_n)^\mu}\, x^{\lambda_m+\rho_n} \qquad x>0 \tag{6}$$

associated with the Hilbert's bilinear double series (5).

Since

$$\big|\mathfrak{H}_{\boldsymbol{\lambda},\boldsymbol{\rho}}^{\boldsymbol{a},\boldsymbol{b}}(\mu;x)\big| \leq \mathfrak{H}_{\boldsymbol{\lambda},\boldsymbol{\rho}}^{\boldsymbol{a},\boldsymbol{b}}(\mu;|x|)\,,$$

it is sufficient to study the Dirichlet–series (6) for $x > 0$, bearing in mind that $\mathfrak{H}_{\boldsymbol{\lambda},\boldsymbol{\rho}}^{\boldsymbol{a},\boldsymbol{b}}(\mu;1) \equiv \mathfrak{H}_{\boldsymbol{\lambda},\boldsymbol{\rho}}^{\boldsymbol{a},\boldsymbol{b}}$ converges. Now, extension of a bounding inequality for $\mathfrak{H}_{\boldsymbol{\lambda},\boldsymbol{\rho}}^{\boldsymbol{a},\boldsymbol{b}}(\mu;\zeta)$ to the case $\zeta\in\mathbb{C}$ is straightforward.

We obtain our pricipal result by Hölder inequality with non–conjugated parameters, i.e. when $p,p'>1$ and $1/p+1/p'\geq 1$, letting

$$\Delta := \frac{1}{p'} + \frac{1}{q'} - 1\,.$$

The derived Hilbert–type inequality class consists of sharp inequalities when the Hölder exponents are conjugated.

2 Main Result

First, using the Gamma–function formula $\Gamma(\mu)\xi^{-\mu} = \int_0^\infty x^{\mu-1}e^{-\xi x}\,\mathrm{d}x$, valid for all $\Re\{\mu\}>0$, we transform the double series (6). After that, we separate the kernel function into a product of two Dirichlet series, we estimate them by the Hölder inequality with non–conjugated exponents p, p', $\min\{p,p'\}>1$, $1/p+1/p'\geq 1$ and q, q', $\min\{q,q'\}>1$, $1/q+1/q'\geq 1$, respectively. These transformations result in

$$\begin{aligned}\mathfrak{H}_{\boldsymbol{\lambda},\boldsymbol{\rho}}^{\boldsymbol{a},\boldsymbol{b}}(\mu;x) &= \frac{1}{\Gamma(\mu)}\int_0^\infty t^{\mu-1}\Bigg(\sum_{m\geq 1} a_m \big(x\mathrm{e}^{-t}\big)^{\lambda_m}\Bigg)\Bigg(\sum_{n\geq 1} b_n \big(x\mathrm{e}^{-t}\big)^{\rho_n}\Bigg)\mathrm{d}t \\ &\leq \frac{\|\boldsymbol{a}\|_p\|\boldsymbol{b}\|_q}{\Gamma(\mu)}\int_0^\infty t^{\mu-1}\Bigg(\sum_{m\geq 1}\big(x\mathrm{e}^{-t}\big)^{\lambda_m p'}\Bigg)^{1/p'}\Bigg(\sum_{n\geq 1}\big(x\mathrm{e}^{-t}\big)^{\rho_n q'}\Bigg)^{1/q'}\mathrm{d}t\,. \end{aligned} \tag{7}$$

By the Laplace integral representation of the Dirichlet series [5, §5] we conclude

$$\mathcal{D}_\lambda(\xi) = \sum_{m\geq 1} a_m e^{-\lambda_m\xi} = \xi\int_0^\infty e^{-\xi t}\sum_{m=1}^{[\lambda^{-1}(t)]} a_m\,\mathrm{d}t \qquad \Re\{\xi\}>0 \tag{8}$$

for positive monotone increasing $(\lambda_n)_{n=1}^\infty$ satisfying (4), with $[w]$ being the integer part of some w. Applying (8) to the inner–most Dirichlet series in (7) we get

$$\mathcal{D}_\lambda(p't) = \sum_{m\ge 1} x^{\lambda_m p'} \mathrm{e}^{-\lambda_m(p't)} = p't \int_0^\infty \mathrm{e}^{-p'tu} \sum_{m:\, \lambda_m\le u} x^{\lambda_m p'}\,\mathrm{d}u$$

$$= p't \int_{\lambda_1}^\infty \mathrm{e}^{-p'tu} \sum_{m=1}^{[\lambda^{-1}(u)]} x^{\lambda_m p'}\,\mathrm{d}u\,.$$

Introducing

$$Q_N^\varphi(z) = z^{\varphi_1} + \cdots + z^{\varphi_N}$$

we have

$$\mathcal{D}_\lambda(p't) = p't \int_{\lambda_1}^\infty \mathrm{e}^{-p'tu}\, Q^\lambda_{[\lambda^{-1}(u)]}\big(x^{p'}\big)\,\mathrm{d}u\,, \tag{9}$$

and analogously

$$\mathcal{D}_\rho(q't) = q't \int_{\rho_1}^\infty \mathrm{e}^{-q'tv}\, Q^\rho_{[\rho^{-1}(v)]}\big(x^{q'}\big)\,\mathrm{d}v\,. \tag{10}$$

Substituting (9) and (10) into (7) we deduce a new class of Hilbert–type inequalities with the scaling parameter $x>0$ such that it corresponds to the auxiliary Dirichlet–series (6).

THEOREM 1. Let $p, q > 1$; p', q' be non–conjugated Hölder exponents to p and q, respectively. $\mu > 0$, $\boldsymbol{a} = (a_n)_{n\in\mathbb{N}} \in \ell_p$, $\boldsymbol{b} = (b_n)_{n\in\mathbb{N}} \in \ell_q$ are nonnegative sequences and λ, ρ are positive monotone increasing functions satisfying (4). Then

$$\mathfrak{H}^{\boldsymbol{a},\boldsymbol{b}}_{\boldsymbol{\lambda},\boldsymbol{\rho}}(\mu; x) \le \mathsf{C}^\mu_{p',q'}(\lambda,\rho;x)\,\|\boldsymbol{a}\|_p \|\boldsymbol{b}\|_q\,, \tag{11}$$

where the constant

$$\mathsf{C}^\mu_{p',q'}(\lambda,\rho;x) = \frac{{p'}^{1/p'}{q'}^{1/q'}}{\Gamma(\mu)} \int_0^\infty t^{\mu+\Delta} \left(\int_{\lambda_1}^\infty \mathrm{e}^{-p'tu}\, Q^\lambda_{[\lambda^{-1}(u)]}\big(x^{p'}\big)\,\mathrm{d}u \right)^{1/p'}$$

$$\times \left(\int_{\rho_1}^\infty \mathrm{e}^{-q'tv} Q^\rho_{[\rho^{-1}(v)]}\big(x^{q'}\big)\,\mathrm{d}v \right)^{1/q'} \mathrm{d}t\,. \tag{12}$$

The equality in (11) appears for $\lambda(x) = \rho(x) = \mathcal{I}(x) = x$ and conjugated Hölder exponents $p = q = 2$ when

$$a_m\, b_n^{-1} = C\,\delta_{mn} \qquad m, n \in \mathbb{N}, \tag{13}$$

at $x = 1$. Here C is an absolute constant and δ_{mn} stands for Kronecker's delta.

PROOF. There remains only equality analysis in (11). Making use of $a_m\, b_n = C\,\delta_{mn}$, we get

$$\mathfrak{H}^{\boldsymbol{a},\boldsymbol{b}}_{\boldsymbol{\lambda},\boldsymbol{\rho}}(\mu; x) = \sum_{m,n\ge 1} \frac{a_m/b_n\, \cdot b_n^2}{(m+n)^\mu} = C \sum_{m,n\ge 1} \frac{\delta_{mn} b_n^2}{(m+n)^\mu}$$

$$= C \sum_{m,n\ge 1} \frac{b_n^2}{(2m)^\mu} = \frac{C\zeta(\mu)}{2^\mu}\,\|\boldsymbol{b}\|_2^2\,,$$

such that it coincides with the right–hand side expression in (11), when (13) is fulfilled. Indeed, for $p' = q' = 2$ and $\Delta = 0$, constant (12) becomes

$$\begin{aligned}\mathsf{C}^{\mu}_{2,2}(\mathcal{I},\mathcal{I};1) &= \frac{2}{\Gamma(\mu)}\int_0^\infty\int_1^\infty t^\mu\,\mathrm{e}^{-2tu}\left(\sum_{n=1}^{[u]}1\right)\mathrm{d}t\mathrm{d}u\\ &= \frac{2}{\Gamma(\mu)}\int_1^\infty\left(\int_0^\infty t^\mu \mathrm{e}^{-2ut}\mathrm{d}t\right)[u]\,\mathrm{d}u\\ &= \frac{\mu}{2^\mu}\int_1^\infty \frac{[u]}{u^{\mu+1}}\,\mathrm{d}u\,.\end{aligned}$$

The integral representation formula of the Riemann Zeta function [5, Corollary 6] reads

$$\zeta(s) = s\int_1^\infty \frac{[x]}{x^{s+1}}\,\mathrm{d}x \qquad s > 1\,.$$

Hence $\mathsf{C}^{\mu}_{2,2}(\mathcal{I},\mathcal{I};1) = \zeta(\mu)2^{-\mu}$. On the other hand, $\|\boldsymbol{a}\|_p\|\boldsymbol{b}\|_q = C\|\boldsymbol{b}\|_2^2$. The proof is complete.

3 Discussions

A. An advantage of the inequality class described in Theorem 1 is the fact that Hölder exponents p, q are *non–conjugated and independent*. For the sake of simplicity, taking $p = q$ and $p' = q'$ we arrive at the following result.

COROLLARY 1. Let all other assumptions of the Theorem 1 hold. Then we get the following sharp inequality:

$$\mathfrak{H}^{\boldsymbol{a},\boldsymbol{b}}_{\boldsymbol{\lambda},\boldsymbol{\rho}}(\mu;x) \le \mathsf{C}^{\mu}_{p'}(\lambda,\rho;x)\,\|\boldsymbol{a}\|_p\|\boldsymbol{b}\|_p\,, \tag{14}$$

where the constant $\mathsf{C}^{\mu}_{p'}(\lambda,\rho;x)$ becomes

$$\frac{{p'}^{2/p'}}{\Gamma(\mu)}\int_0^\infty t^{\mu+\Delta}\left(\int_{\lambda_1}^\infty\int_{\rho_1}^\infty \mathrm{e}^{-p't(u+v)}\,Q^{\lambda}_{[\lambda^{-1}(u)]}\big(x^{p'}\big)\,Q^{\rho}_{[\rho^{-1}(v)]}\big(x^{p'}\big)\,\mathrm{d}u\,\mathrm{d}v\right)^{1/p'}\mathrm{d}t\,.$$

B. Choosing $p' = q' = 2$ one deduces the following consequence of the Theorem 1, that is of considerable interest.

COROLLARY 2. Let $p' = q' = 2$ be conjugated Hölder exponents and let other assumptions of the Theorem 1 hold. For $\lambda = \rho$ we have a sharp inequality:

$$\mathfrak{H}^{\boldsymbol{a},\boldsymbol{b}}_{\boldsymbol{\lambda},\boldsymbol{\lambda}}(\mu;x) \le \mathsf{C}^{\mu}_{2,2}(\lambda;x)\,\|\boldsymbol{a}\|_p\|\boldsymbol{b}\|_q\,, \tag{15}$$

where

$$\mathsf{C}^{\mu}_{2,2}(\lambda;x) = \frac{\mu}{2^\mu}\int_{\lambda_1}^\infty \frac{Q^{\lambda}_{[\lambda^{-1}(u)]}\big(x^2\big)}{u^{\mu+1}}\,\mathrm{d}u\,.$$

REMARK. First, here p, q remain independent; we can only say that their range is $p, q \in (1, 2]$. Second, the equality of kernel sequences $\boldsymbol{\lambda} = \boldsymbol{\rho}$ does not mean that the kernel of the Dirichlet–series $\mathfrak{H}^{\boldsymbol{a},\boldsymbol{b}}_{\boldsymbol{\lambda},\boldsymbol{\lambda}}(\mu; x)$ is homogeneous.

Finally, note that Corollary 2 cannot be deduced from Corollary 1 by some specification.

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