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Values Taken By Linear Combinations of Cosine Functions*

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Abstract

For $\theta, \beta \in [-\pi, \pi)$ and $\theta \notin \{-\pi, 0\}$, it is shown that as n runs through the nonnegative integers, the nonzero sequence $(\cos(n\theta + \beta))$ takes infinitely many positive and negative values; and if $\theta = s/t$ is a rational multiple of π, the sequence is purely periodic whose least period is equal to t when s is even and equal to $2t$ when s is odd; while if θ is not a rational multiple of π, the range of values of the sequence is dense in the unit interval $(0, 1)$. Any sequence of the form $\left(\sum_r^d \alpha_r \cos\left(2n\pi s_r/t_r\right) + \beta_r\right)$, with rational s_r/t_r belonging to the unit interval $(0, 1)$, is shown to be purely periodic whose least period is equal to the least common multiple of $t_1, \ldots, t_d$.

1 Introduction

Let $\theta, \beta \in [-\pi, \pi)$ and $\theta \notin \{-\pi, 0\}$. The distribution of the values of $\cos(n\theta + \beta)$ as n varies over the nonnegative integers $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ has been of much interest recently. To mention one instance, in the course of establishing the positivity of elements in a binary sequence, it is shown by Halava, Harju and Hirvensalo ([2, Lemma 5], see also [3]) using elementary means that the inequality $\cos(n\theta + \beta) \geq 0$ cannot hold for all $n \in \mathbb{N}_0$. We endeavor here to carry out two particular tasks. In the next section, we improve upon the above-mentioned result of Halava-Harju-Hirvensalo by showing that both positive and negative values occur infinitely often. When θ is not a rational multiple of π, through the use of a theorem of Kronecker in Diophantine approximation, we show even more that the value set of such cosine function is dense in the closed interval $[-1, 1]$.

Our second task arises from a result of Bell and Gerhold, [1, Lemma 8], dealing with a linear combination of cosine functions which states that: If $\theta_1, \ldots, \theta_d$ are rational

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numbers in the open interval $(0, 1)$, and if α_i, β_i are real numbers such that the purely periodic sequence

$$u_n = \sum_{i=1}^{d} \alpha_i \cos(2\pi\theta_i n + \beta_i)$$

is not identically zero, then the sequence (u_n) takes both positive and negative values.

Our next objective is to obtain more information about the periodicity of the sequence (u_n). We show that the least period of the sequence (u_n) is exactly equal to T, the least common multiple of the denominators $t_1, \ldots, t_d$.

2 Positive and Negative Values

We begin with a result on negative values. For an angle γ, it is convenient to use the phrase $\gamma \bmod (-\pi, \pi]$, when we mean an angle $\gamma_0 \in (-\pi, \pi]$ for which $\gamma \equiv \gamma_0 \bmod 2\pi$. Clearly, the method employed here can also be adapted to examine the values taken by other trigonometric functions.

PROPOSITION 2.1. Let $\theta, \beta \in [-\pi, \pi)$ and $\theta \notin \{-\pi, 0\}$. There are infinitely many $n \in \mathbb{N}_0$ such that $\cos(n\theta + \beta) < 0$.

PROOF. Assume that there are only finitely many $n \in \mathbb{N}_0$, say $n_1, \ldots, n_L$, such that

$$\cos(n\theta + \beta) < 0 \quad (n = n_1, \ldots, n_L).$$

Thus,

$$\cos\left(((n_L + 1)\theta + \beta) + k\theta\right) = \cos\left((n_L + 1 + k)\theta + \beta\right) \geq 0$$

for all $k \in \mathbb{N}_0$, which contradicts the result in Lemma 5 of [2] when φ is replaced by $(n_L + 1)\theta + \beta \bmod (-\pi, \pi]$.

PROPOSITION 2.2. Let $\theta, \beta \in [-\pi, \pi)$ and $\theta \notin \{-\pi, 0\}$.

(i) The relation $\cos(n\theta + \beta) \leq 0$ cannot hold for all $n \in \mathbb{N}_0$.

(ii) There are infinitely many $n \in \mathbb{N}_0$ such that $\cos(n\theta + \beta) > 0$.

PROOF. (i) Assume that $\cos(n\theta + \beta) \leq 0$ for all $n \in \mathbb{N}_0$, which implies that for each $n \in \mathbb{N}_0$,

$$n\theta + \beta \in \left[\frac{-3\pi}{2} + k2\pi, \frac{-\pi}{2} + k2\pi\right],$$

for some $k \in \mathbb{Z}$. Then $\beta \in [-\pi, -\pi/2]$ or $\beta \in [\pi/2, \pi)$.

If $\beta \in [-\pi, -\pi/2]$, since $|\theta| < \pi$, we have

$$\theta + \beta \in \left[\frac{-3\pi}{2}, \frac{-\pi}{2}\right].$$

Inductively, we obtain $n\theta + \beta \in [-3\pi/2, -\pi/2]$ for all $n \in \mathbb{N}_0$. Since $\theta \neq 0$, this implies $|n\theta + \beta| \to \infty \ \ (n \to \infty)$, which is a contradiction.

If $\beta \in [\pi/2, \pi)$, since $|\theta| < \pi$, we have

$$\theta + \beta \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right].$$

Inductively, we obtain $n\theta+\beta\in[\pi/2,3\pi/2]$ for all $n\in\mathbb{N}_0$. Since $\theta\neq 0$, this implies $|n\theta+\beta|\to\infty \quad (n\to\infty)$, which is again a contradiction.

(ii) Assume that there are only finitely many $n\in\mathbb{N}_0$, say $n_1,\ldots,n_L$, such that

$$\cos(n\theta+\beta)>0 \quad (n=n_1,\ldots,n_L).$$

Thus,

$$\cos\left(((n_L+1)\theta+\beta)+k\theta\right)=\cos\left((n_L+1+k)\theta+\beta\right)\leq 0$$

for all $k\in\mathbb{N}_0$, which contradicts the preceding lemma when β is replaced by $(n_L+1)\theta+\beta$ mod $(-\pi,\pi]$.

Next, we derive more information by invoking upon a result in Diophantine approximation known as the Dirichlet approximation theorem.

THEOREM 2.3. Let $\theta,\beta\in[-\pi,\pi)$ with $\theta\notin\{-\pi,0\}$.

(a) If $\theta=s\pi/t$ $(s\in\mathbb{Z},t\in\mathbb{N},\ \gcd(s,t)=1)$ is a rational multiple of π, then the sequence $(\cos(n\theta+\beta))_{n\in\mathbb{N}_0}$ is purely periodic. Moreover, if s is even, then the least period of the sequence $(\cos(n\theta+\beta))_{n\in\mathbb{N}_0}$ is t, while if s is odd, then its least period is $2t$.

(b) If θ is not a rational multiple of π, then as n varies over $\mathbb{N}_0$ the range of values of $\cos(n\theta+\beta)$ is dense in $[-1,1]$.

PROOF. (a) Since the cosine function is periodic of period 2π, the sequence

$$\cos(n\theta+\beta)=\cos\left(\frac{ns\pi}{t}+\beta\right) \quad (n\in\mathbb{N}_0)$$

is purely periodic with at most $2t$ values in a period, corresponding to $n=0,1,2,\ldots,2t-1$, namely,

$$\cos\beta,\ \cos\left(\frac{s\pi}{t}+\beta\right),\ \cos\left(\frac{2s\pi}{t}+\beta\right),\ldots,\ \cos\left(\frac{s(2t-1)\pi}{t}+\beta\right).$$

If $t=1$, then either $\theta=2k\pi$ or $\theta=(2k+1)\pi$ for some $k\in\mathbb{Z}$. In either case, $\theta\notin[-\pi,\pi)\setminus\{-\pi,0\}$, and so there is nothing to consider.

Let now $t\geq 2$. If s is even, say $s=2k$, then t must be odd and the sequence

$$\cos(n\theta+\beta)=\cos\left(\frac{2nk\pi}{t}+\beta\right) \quad (n\in\mathbb{N}_0)$$

is periodic of period t with values in each period being

$$\cos\beta,\ \cos\left(\frac{2k\pi}{t}+\beta\right),\ \cos\left(\frac{4k\pi}{t}+\beta\right),\ldots,\ \cos\left(\frac{2k(t-1)\pi}{t}+\beta\right).$$

If t is not the least period, let its least period be ℓ. Thus $\ell\mid t,\ \ell<t$ and

$$\cos\beta=\cos\left(\frac{2k\ell\pi}{t}+\beta\right),\ \ \cos\left(\frac{2k\pi}{t}+\beta\right)=\cos\left(\frac{2k(\ell+1)\pi}{t}+\beta\right),\ldots$$

The first equality yields two possibilities

$$\frac{2k\ell\pi}{t} + \beta = 2N\pi \pm \beta \quad \text{for some } N \in \mathbb{Z}.$$

The possibility $2k\ell\pi/t + \beta = 2N\pi + \beta$ yields $N\pi = k\ell\pi/t$, which is impossible because $t \nmid k\ell,\ t \geq 3$ and so only the other possibility can hold which yields $\beta = N\pi - k\ell\pi/t$. Similarly, the second inequality yields $\beta = M\pi - 2k\pi/t - k\ell\pi/t$ for some $M \in \mathbb{Z}$. Equating the two values of β gives $(N-M)\pi = -2k\pi/t$ which is not tenable because $t \nmid (2k),\ t \geq 3$. Consequently, t is the least period.

If $s = 2k+1$ is odd, then the sequence

$$\cos(n\theta + \beta) = \cos\left(\frac{n(2k+1)\pi}{t} + \beta\right) \quad (n \in \mathbb{N}_0)$$

is periodic of period $2t$ with values in each period being

$$\cos\beta,\ \cos\left(\frac{(2k+1)\pi}{t} + \beta\right),\ \cos\left(\frac{2(2k+1)\pi}{t} + \beta\right), \ldots, \cos\left(\frac{(2t-1)(2k+1)\pi}{t} + \beta\right).$$

There remains to show that $2t$ is the least period in this case. If $2t$ is not the least period, let its least period be m. Thus $m \mid 2t,\ m < 2t$ and

$$\cos\beta = \cos\left(\frac{(2k+1)m\pi}{t} + \beta\right),\ \cos\left(\frac{(2k+1)\pi}{t} + \beta\right) = \cos\left(\frac{(2k+1)(m+1)\pi}{t} + \beta\right),$$

etc. The first equality yields two possibilities

$$\frac{(2k+1)m\pi}{t} + \beta = 2N\pi \pm \beta \quad \text{for some } N \in \mathbb{Z}.$$

The possibility $(2k+1)m\pi/t + \beta = 2N\pi + \beta$ yields $N\pi = (2k+1)m\pi/2t$, which is impossible because $2t \nmid (2k+1)m,\ t \geq 2$ and so only the other possibility can hold which yields $2\beta = 2N\pi - (2k+1)m\pi/t$. Similarly, the second inequality yields $2\beta = 2M\pi - 2(2k+1)\pi/t - (2k+1)m\pi/t$ for some $M \in \mathbb{Z}$. Equating the two values of β gives $(N-M)\pi = -(2k+1)\pi/t$ which is not tenable because $t \nmid (2k+1),\ t \geq 2$. Consequently, $2t$ is the least period.

(b) Assume that θ is not a rational multiple of π, say $\theta = \vartheta \cdot 2\pi$, where $\vartheta \in \mathbb{R} \setminus \mathbb{Q}$. Writing

$$\vartheta = [\vartheta] + \xi,$$

where $[\vartheta]$ denotes its integer part, and $\xi := \{\vartheta\} \in (0,1)$ denotes its fractional part which must be irrational, for $n \in \mathbb{N}_0$ we have

$$\cos(n\theta + \beta) = \cos(n\vartheta 2\pi + \beta) = \cos(2n\xi\pi + \beta) = \cos\left(\{n\xi\}\, 2\pi + \beta\right).$$

Since ξ is irrational, by the Kronecker's approximation theorem, see e.g. Corollary 6.4 on page 75 of [4], we know the set $\{\{n\xi\}\ ;\ k \in \mathbb{N}\}$ is dense in [0,1]. Consequently, the set $\{\{n\xi\}2\pi\ ;\ n \in \mathbb{N}\}$ is dense in $[0, 2\pi]$, implying that the range of values of

$\cos(\{n\xi\} 2\pi + \beta) \quad (n \in \mathbb{N})$ is dense in $[-1, 1]$.

When θ is a rational multiple of π, the values in a period can be distinct or otherwise as seen in the next example.

EXAMPLE 1. Take $\theta = s\pi/t = 2\pi/3,\ \beta = 0$. The sequence

$$(u_n)_{n\in\mathbb{N}_0} = \left(\cos\left(\frac{2n\pi}{3}\right)\right)_{n\in\mathbb{N}_0}$$

is periodic of length 3 with each period being

$$\cos(0) = 1,\ \ \cos(2\pi/3) = -1/2,\ \ \cos(4\pi/3) = -1/2,$$

showing that there are exactly $2 < 3 = t$ distinct sequence values.

EXAMPLE 2. Take $\theta = s\pi/t = 2\pi/3,\ \beta = \pi/2$. The sequence

$$(u_n)_{n\in\mathbb{N}_0} = \left(\cos\left(\frac{\pi}{2} + \frac{2n\pi}{3}\right)\right)_{n\in\mathbb{N}_0}$$

is periodic of length 3 with each period being

$$\cos\left(\frac{\pi}{2}\right) = 0,\ \ \cos\left(\frac{\pi}{2} + \frac{2\pi}{3}\right) = -\frac{\sqrt{3}}{2},\ \ \cos\left(\frac{\pi}{2} + \frac{4\pi}{3}\right) = \frac{\sqrt{3}}{2},$$

showing that there are exactly $3 = t$ distinct sequence values.

3 Linear Combination of Cosine Functions

In this section, we consider a linear combination of cosine functions. Such expression is always periodic as we now see.

THEOREM 3.1. Let $\theta_1 = s_1/t_1, \ldots, \theta_d = s_d/t_d$ $(s_r, t_r \in \mathbb{N},\ \gcd(s_r, t_r) = 1)$ be rational numbers in the open interval $(0, 1)$, and let α_r, β_r be real numbers such that the purely periodic sequence

$$u_n = \sum_{r=1}^{d} \alpha_r \cos(2\pi\theta_r n + \beta_r)$$

is not identically zero. Let $T := l.c.m.(t_1, \ldots, t_d)$. Then the least period of the sequence (u_n) is a divisor of T.

PROOF. As seen in the proof of Theorem 2.3, the sequence

$$u_n = \sum_{r=1}^{d} \alpha_r \cos\left(2n\pi\frac{s_r}{t_r} + \beta_r\right) \quad (n \in \mathbb{N}_0)$$

is purely periodic with a period T and the values in such a period correspond to $n \in \{0, 1, 2, \ldots, T-1\}$, namely,

$$\sum_{r=1}^{d} \alpha_r \cos \beta_r,\ \sum_{r=1}^{d} \alpha_r \cos\left(2\pi \frac{s_r}{t_r} + \beta_r\right),\ \sum_{r=1}^{d} \alpha_r \cos\left(4\pi \frac{s_r}{t_r} + \beta_r\right), \ldots,$$
$$\sum_{r=1}^{d} \alpha_r \cos\left(2(T-1)\pi \frac{s_r}{t_r} + \beta_r\right).$$

From Theorem 3.1, there then arises a natural question of determining when T is the least period. A rather surprising fact that T is always the least period of such sequence is now proved. To proceed further, let us note the following useful fact. Since

$$\sum_{n=0}^{T-1} \exp\left(2i\pi n s_r / t_r\right) = 0,$$

it follows that $\sum_{n=0}^{T-1} u_n = 0$.

THEOREM 3.2. For $r \in \{1, 2, \ldots, d\}$, let s_r/t_r $(s_r, t_r \in \mathbb{N},\ \gcd(s_r, t_r) = 1)$ be rational numbers in the unit interval $(0, 1)$, and let $\alpha_r (\neq 0), \beta_r$ be real numbers. If the purely periodic, non-identically zero sequence $(u_n)_{n \in \mathbb{N}_0}$ is defined by

$$u_n = \sum_{r=1}^{d} \alpha_r \cos\left(\frac{2\pi n s_r}{t_r} + \beta_r\right),$$

then its least period is equal to $T := l.c.m.(t_1, t_2, \ldots, t_d)$.

PROOF. Let ℓ be the least period of (u_n). Since $\ell \mid T$, let $T = \ell L$ for some $L \in \mathbb{N}$. We proceed to prove the theorem by induction on d.

The case $d = 1$ is contained in Part (a) of Theorem 2.3. Assume now that $d \geq 2$ and that the theorem holds up to $d-1$.

If $t_r \mid \ell$ for every $r \in \{1, 2, \ldots, d\}$, then $\ell = T$ and we are done. Suppose then that not all of the t_r's divide ℓ. If there are some t_r's that divide ℓ and some t_r's that do not divide ℓ. Without loss of generality, assume that ℓ is divisible by $t_1, t_2, \ldots, t_m$, but not divisible by $t_{m+1}, t_{m+2}, \ldots, t_d$ for some $m \in \{1, 2, \ldots, d-1\}$. For $k, j \in \mathbb{N}_0$, we have

$$\begin{aligned}
Lu_j &= \sum_{k=0}^{L-1} u_{k\ell+j} = \sum_{k=0}^{L-1} \sum_{r=1}^{d} \alpha_r \cos\left(\frac{2\pi s_r}{t_r}(k\ell + j) + \beta_r\right) \\
&= \sum_{r=1}^{m} \alpha_r \sum_{k=0}^{L-1} \cos\left(\frac{2\pi s_r}{t_r} j + \beta_r\right) + \sum_{r=m+1}^{d} \alpha_r \mathrm{Re}\left(\sum_{k=0}^{L-1} \exp\left(\frac{2i\pi s_r}{t_r}(k\ell + j) + i\beta_r\right)\right) \\
&= L \sum_{r=1}^{m} \alpha_r \cos\left(\frac{2\pi s_r}{t_r} j + \beta_r\right) \\
&\quad + \sum_{r=m+1}^{d} \alpha_r \mathrm{Re}\left(\exp\left(\frac{2i\pi s_r}{t_r} j + i\beta_r\right) \frac{1 - \exp\left(2i\pi \ell L s_r / t_r\right)}{1 - \exp\left(2i\pi \ell s_r / t_r\right)}\right)
\end{aligned}$$

$$= L \sum_{r=1}^{m} \alpha_r \cos\left(\frac{2\pi s_r}{t_r} j + \beta_r\right).$$

Thus, $u_j = \sum_{r=1}^{m} \alpha_r \cos\left(\frac{2\pi s_r}{t_r} j + \beta_r\right)$, which shows that u_n has $m \leq d-1$ terms, the induction hypothesis finishes this case.

If none of the t_r's divides ℓ, the same arguments as in the last steps shows that $u_n \equiv 0$, which is untenable.

We remark that the above proof can clearly be applied, with appropriate adjustments, to certain other trigonometric functions and/or periodic functions.

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References

[1] J. P. Bell and S. Gerhold, On the positivity set of a linear recurrence sequence, Israel J. Math. 157(2007), 333-345.

[2] V. Halava, T. Harju and M. Hirvensalo, Positivity of second order linear recurrent sequences, Discrete Applied Math., 154(2006), 447-451.

[3] V. Laohakosol and P. Tangsupphathawat, Positivity of third order linear recurrence sequences, Discrete Applied Mathematics, 157(2009), 3239-3248.

[4] I. Niven, Irrational Numbers, The Carus Mathematical Monographs Number 11, The Mathematical Association of America, 1967.