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The Sine-Cosine Method For The Davey-Stewartson Equations*

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Abstract

In this paper, we establish exact solutions for nonlinear Davey-Stewartson equations. The sine-cosine method is used to construct periodic and solitary wave solutions.

1 Introduction

The sine-cosine method (see e.g. [1-4]) has been used to to solve different types of nonlinear systems of PDEs. The higher-dimensional nonlinear wave fields have richer phenomena than one-dimensional ones, since various localized solitons may be considered in higher-dimensional space.

The Davey-Stewartson equation (DSE) was introduced in [6] to describe the evolution of a three-dimensional wave-packet on water of finite depth. It is a system of partial differential equations for a complex (wave-amplitude) field $q(t, x, y)$ and a real (mean-flow) field $\phi(t, x, y)$:

$$\begin{array}{c} iq_t + \frac{1}{2}\sigma^2 \left(q_{xx} + \sigma^2 q_{yy}\right) + \lambda \left|q\right|^2 q - \phi_x q = 0, \\ \phi_{xx} - \sigma^2 \phi_{yy} - 2\lambda \left(\left|q\right|^2\right)_x = 0, \end{array} \tag{1}$$

where $\lambda = \pm 1$ and $\sigma^2 = \pm 1$. The case $\sigma = 1$ is called the DSI equation, while the case $\sigma = i$ is called the DSII equation. The parameter λ characterizes the focusing or defocusing case [5]. The DS equation has four kinds of soliton solutions: the conventional line, algebraic, periodic and lattice solitons. The conventional line soliton has an essentially one-dimensional structure. On the other hand, the algebraic, periodic and lattice solitons have a two-dimensional localized structure.

The DSI and DSII equations are two well-known examples of integrable equations in two spacial dimensions, which arise as higher dimensional generalizations of the nonlinear Schrodinger equation, as well as from physical considerations [6-7]. Indeed,

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they appear in many applications, for example in the description of gravity-capillarity surface wave packets in the limit of shallow water. Therefore it is of interests to derive explicit solutions of the DS equation.

During the past decades, quite a few methods for obtaining explicit traveling and solitary wave solutions of nonlinear evolution equations have been proposed, such as the inverse scattering method, bilinear transformation, the tanh-sech method, extended tanh method and homogeneous balance method.

Concepts like solitons, peakons, kinks, breathers, cusps and compactons are being thoroughly investigated in the scientific literature (see e.g. [8-10]).

2 Sine-Cosine Method

We introduce the wave variable $\xi = x - ct$ into the PDE

$$P(u, u_t, u_x, , u_{tt}, u_{xx}, ...) = 0, \tag{2}$$

where $u(x,t)$ is a traveling wave solution. This enables us to use the following changes of variables:

$$\frac{\partial}{\partial t} = -c\frac{\partial}{\partial\ \xi}, \quad \frac{\partial^2}{\partial t\ ^2} = c^2\frac{\partial^2}{\partial \xi^2}, \quad \frac{\partial}{\partial x} = \frac{\partial}{\partial \xi}\frac{\partial^2}{\partial \xi^2}, \ldots\ . \tag{3}$$

One can immediately reduce the nonlinear PDE (2) into a nonlinear ODE

$$Q(u, u_\xi, u_{\xi\xi}, , u_{\xi\xi\xi}, ...) = 0. \tag{4}$$

The ordinary differential equation (4) is then integrated as long as all terms contain derivatives, where we neglect the integration constants.

The solutions of many nonlinear equations can be expressed in the form

$$u(x,t) = \begin{cases} \lambda_1 \sin^\beta(\mu\xi), & |\xi| \le \frac{\pi}{\mu} \\ 0, & \text{otherwise} \end{cases}, \tag{5}$$

or in the form

$$u(x,t) = \begin{cases} \lambda_1 \cos^\beta(\mu\xi), & |\xi| \le \frac{\pi}{2\mu} \\ 0 & \text{otherwise} \end{cases}, \tag{6}$$

where λ, μ and β are parameters to be determined, μ and c are the wave number and the wave speed, respectively [2]. We use

$$\begin{aligned} u(\xi) &= \lambda_1 \sin^\beta(\mu\xi), & (7)\\ u^n(\xi) &= \lambda_1^n \sin^{n\beta}(\mu\xi), & (8)\\ (u^n)_\xi &= n\mu\beta\lambda_1^n \cos(\mu\xi)\sin^{n\ \beta-1}(\mu\xi), & (9)\\ (u^n)_{\xi\xi} &= -n^2\mu^2\beta^2\lambda_1^n \sin^{n\beta}(\mu\xi) + n\mu^2\lambda_1^n\beta(n\beta-1)\sin^{n\beta-2}(\mu\xi), & (10) \end{aligned}$$

and their derivatives.

$$u(\xi) = \lambda_1 \cos^\beta(\mu\xi), \tag{11}$$

$$\begin{aligned} u^n(\xi) &= \lambda_1^n \cos^{n\beta}(\mu\xi), && (12)\\ (u^n)_\xi &= -n\mu\beta\lambda_1^n \sin(\mu\xi)\cos^{n\beta-1}(\mu\xi), && (13)\\ (u^n)_{\xi\xi} &= -n^2\mu^2\beta^2\lambda_1^n \cos^{n\beta}(\mu\xi) + n\mu^2\lambda_1^n\beta(n\beta-1)\cos^{n\beta-2}(\mu\xi) && (14)\end{aligned}$$

and so on. We substitute (7)-(10) or (11)-(14) into the reduced equation (4), balance the terms of the cosine functions when (7)-(10) are used, or balance the terms of the sine functions when (11)-(14) are used, and solve the resulting system of algebraic equations by using computerized symbolic packages. We next collect all terms with the same power in $\cos^k(\mu\xi)$ or $\sin^k(\mu\xi)$ and set to zero their coefficients to get a system of algebraic equations among the unknowns λ, μ and β, and solve the subsequent system.

3 The Davey-Stewartson Equation

In this section, we deal with the Davey–Stewartson equation (1). Take the following transformations of (1)

$$\begin{aligned} q(x,y,t) &= U(\xi)e^{i\theta},\ \phi(x,y,t) = V(\xi), && (15)\\ \xi &= x + y - c\,t,\ \theta = K_1 x + K_2\ y + K_3 t, && (16)\end{aligned}$$

where K_1, K_2 and K_3 are real constants [11]. It is easy to derive from (15), (16) and (1) that

$$c = \sigma^2 k_1 + k_2, \qquad (17)$$

$$\sigma^2\left\{(1+\sigma^2)U''\right\} - \left[2k_3 + \sigma^2(k_1^2+\sigma^2 k_2^2) + 2V'\right]U + 2\lambda U^3 = 0, \qquad (18)$$

$$(1-\sigma^2)V'' - 2\lambda(U^2)' = 0. \qquad (19)$$

Integrating (19) with respect to ξ and setting the constant of integration to zero, we find

$$V' = \frac{2\lambda}{1-\sigma^2}U^2. \qquad (20)$$

Substituting (20) into (18) gives

$$\sigma^2\left\{(1+\sigma^2)U''\right\} - \left[2k_3 + \sigma^2\ (k_1^2 + \sigma^2\ k_2^2)\right]U + 2\lambda\left[\frac{-2}{1-\sigma^2}+1\right]U^3 = 0. \qquad (21)$$

Seeking solutions of the form (5), we get

$$\begin{aligned} &\sigma^2(1+\sigma^2\ )\left[-\mu^2\beta^2\lambda_1\sin^\beta(\mu\xi) + \mu^2\beta^2\lambda_1\beta(\beta-1)\sin^{\beta-2}(\mu\xi)\right] - \\ &\left[2k_3 + \sigma^2\ (k_1^2+\sigma^2\ k_2^2)\right]\lambda_1\sin^\beta(\mu\xi) + 2\lambda\left[\frac{-2}{1-\sigma^2}+1\right]\lambda_1^3\sin^{3\beta}(\mu\xi)\\ =\ & 0. && (22)\end{aligned}$$

Equating the exponents and the coefficients of each pair of the sine functions we find the following algebraic system:

$$\begin{gathered} 3\beta = \beta - 2,\\ \beta - 1 \neq 0,\\ \sigma^2(1+\sigma^2)\mu^2\lambda_1\beta(\beta-1) + 2\lambda\left[1 - \tfrac{2}{1-\sigma^2}\right]\lambda_1^3 = 0,\\ -\sigma^2(1+\sigma^2)\mu^2\beta^2\lambda_1 - \left[2k_3 + \sigma^2(k_1^2+\sigma^2 k_2^2)\right]\lambda_1 = 0.\end{gathered} \qquad (23)$$

By solving the algebraic system (23), we get, when $\frac{2k_3+k_1^2\sigma^2+k_2^2}{\sigma^2} < 0$,

$$\begin{aligned} \beta &= -1, \\ \mu &= \pm\sqrt{-\tfrac{2k_3+k_1^2\sigma^2+k_2^2}{\sigma^2}}, \\ \lambda_1 &= \pm\sqrt{-\tfrac{2k_3+k_1^2\sigma^2+k_2^2}{\lambda}}. \end{aligned} \tag{24}$$

In view of (5), (15), (16) and (24), for $\frac{2k_3+k_1^2\sigma^2+k_2^2}{\sigma^2} < 0$, we obtain the periodic solutions

$$q(x,y,t) = \pm\sqrt{-\frac{2k_3+k_1^2\sigma^2+k_2^2}{\lambda}}e^{i(k_1x+k_2y+k_3t)}\csc\left[\pm\sqrt{-\frac{2k_3+k_1^2\sigma^2+k_2^2}{\sigma^2}}(x+y-ct)\right], \tag{25}$$

where

$$0 < \left[\pm\sqrt{-\frac{2k_3+k_1^2\sigma^2+k_2^2}{\sigma^2}}(x+y-ct)\right] < \pi,$$

and

$$q(x,y,t) = \pm\sqrt{-\frac{2k_3+k_1^2\sigma^2+k_2^2}{\lambda}}e^{i(k_1x+k_2y+k_3t)}\sec\left[\pm\sqrt{-\frac{2k_3+k_1^2\sigma^2+k_2^2}{\sigma^2}}(x+y-ct)\right], \tag{26}$$

where

$$\left|\left[\pm\sqrt{-\frac{2k_3+k_1^2\sigma^2+k_2^2}{\sigma^2}}(x+y-ct)\right]\right| < \frac{\pi}{2},$$

and for $\frac{2k_3+k_1^2\sigma^2+k_2^2}{\sigma^2} > 0$, the following solitary solutions

$$q(x,y,t) = \pm i\sqrt{-\frac{2k_3+k_1^2\sigma^2+k_2^2}{\lambda}}e^{i(k_1x+k_2y+k_3t)}\csc h\left[\pm\sqrt{\frac{2k_3+k_1^2\sigma^2+k_2^2}{\sigma^2}}(x+y-ct)\right]; \tag{27}$$

and

$$q(x,y,t) = \pm\sqrt{-\frac{2k_3+k_1^2\sigma^2+k_2^2}{\lambda}}e^{i(k_1x+k_2y+k_3t)}\sec h\left[\pm\sqrt{\frac{2k_3+k_1^2\sigma^2+k_2^2}{\sigma^2}}(x+y-ct)\right]; \tag{28}$$

where

$$c = \sigma^2\ k_1 + k_2.$$

To find the solutions $\phi(x,y,t)$, according to (20), we have

$$V(\xi) = \frac{2\lambda}{1-\sigma^2}\int U^2(\xi)d\xi \tag{29}$$

By means of the equations (15), (16), (5) and (29) and using equation (24), we have the following perodic solutions for $\phi(x,y,t)$:

- When $\frac{2k_3+k_1^2\sigma^2+k_2^2}{\sigma^2} < 0$, we get

$$\phi(x,y,t) = \frac{-2\sigma\sqrt{\lambda}}{1-\sigma^2}\cot\left[\sqrt{-\frac{2k_3+k_1^2\sigma^2+k_2^2}{\sigma^2}}(x+y-ct\right] \tag{30}$$

and

$$\phi(x,y,t) = \frac{-2\sigma\sqrt{\lambda}}{1-\sigma^2}\tan\left[\sqrt{-\frac{2k_3+k_1^2\sigma^2+k_2^2}{\sigma^2}}(x+y-ct\right]. \tag{31}$$

- When $\frac{2k_3+k_1^2\sigma^2+k_2^2}{\sigma^2} > 0$, we get the following solitary wave solutions

$$\phi(x,y,t) = \frac{2\sigma\sqrt{-\lambda}}{1-\sigma^2}\coth\left[\sqrt{\frac{2k_3+k_1^2\sigma^2+k_2^2}{\sigma^2}}(x+y-ct\right], \tag{32}$$

and

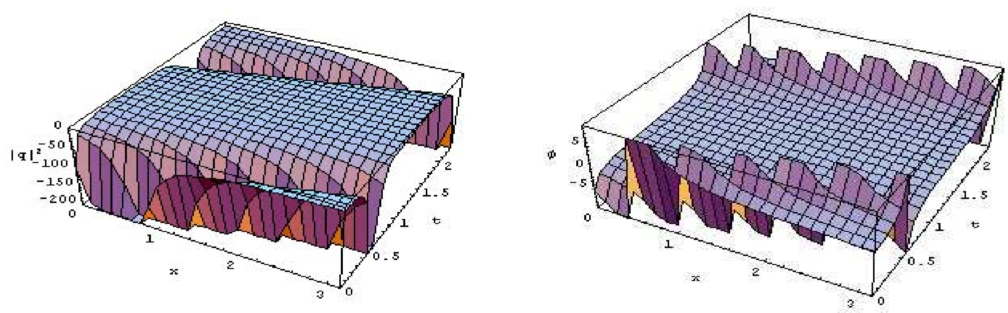
$$\phi(x,y,t) = \frac{2\sigma\sqrt{-\lambda}}{1-\sigma^2}\tanh\left[\sqrt{\frac{2k_3+k_1^2\sigma^2+k_2^2}{\sigma^2}}(x+y-ct\right] \tag{33}$$

where

$$c = \sigma^2\ k_1 + k_2.$$

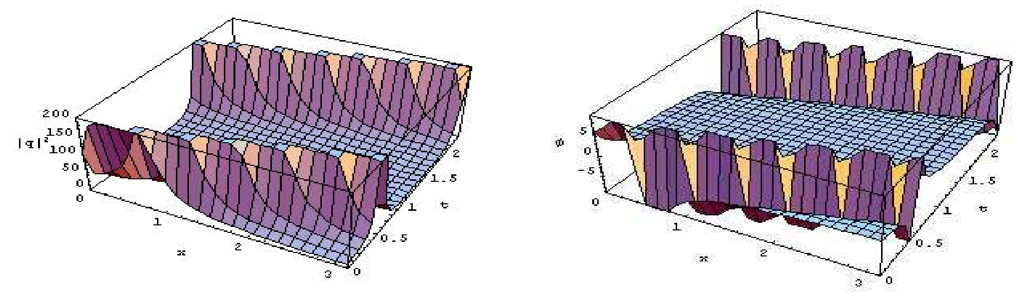
4 Illustrations

We now plot a few solutions found in our previous discussions.



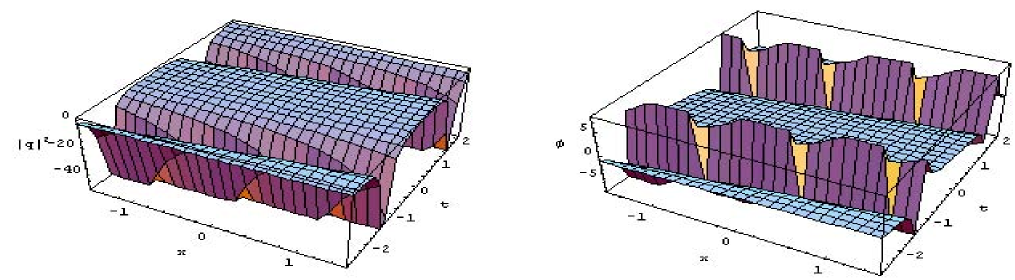
(a) (b)

Figure 1: $q(x,y,t)$ in (25) and $\phi(x,y,t)$ in (30) where $y = 0.1$, $k_1 = 0.3$, $k_2 = 0.5$, $k_3 = 1.5$, $\sigma = I$, $\lambda = 1$.



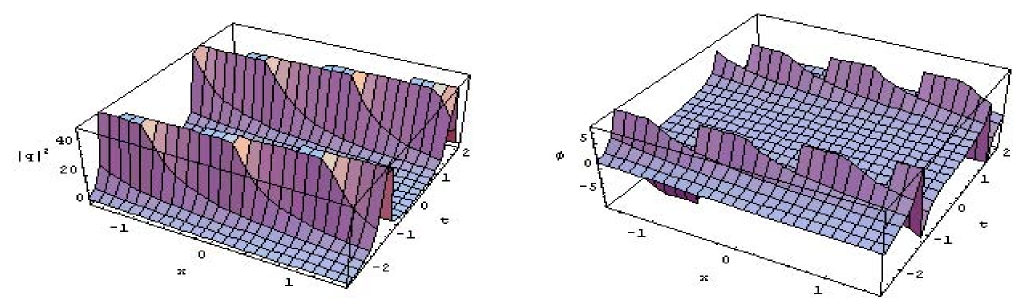
(a) (b)

Figure 2: $q(x, y, t)$ in (25) and $\phi(x, y, t)$ in (30) where $y = 0.1$, $k_1 = 0.3$, $k_2 = 0.5$, $k_3 = 1.5$, $\sigma = I$, $\lambda = -1$.



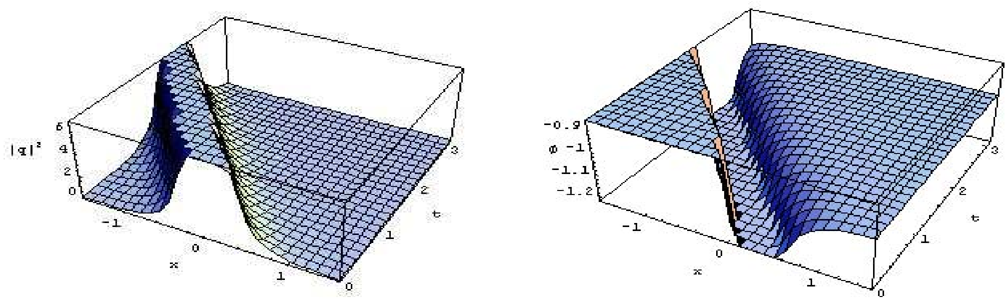
(a) (b)

Figure 3: $q(x, y, t)$ in (26) and $\phi(x, y, t)$ in (31) where $y = 0.3$, $k_1 = 1.3$, $k_2 = 1.5$, $k_3 = 0.3$, $\sigma = I$, $\lambda = 1$.



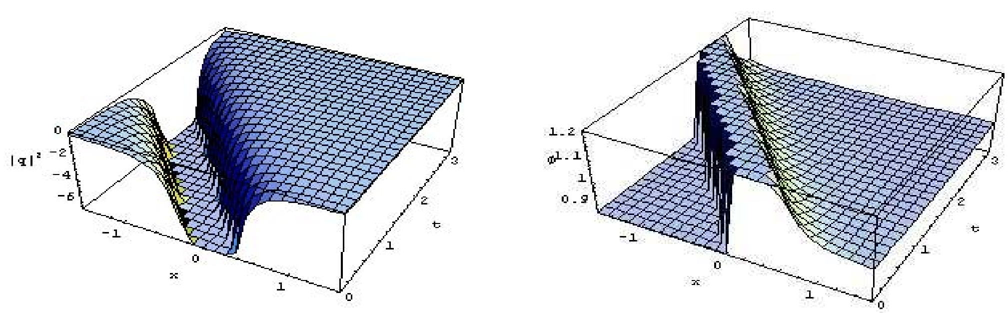
(a) (b)

Figure 4: $q(x, y, t)$ in (26) and $\phi(x, y, t)$ in (31) where $y = 0.3$, $k_1 = 1.3$, $k_2 = 1.5$, $k_3 = 0.3$, $\sigma = I$, $\lambda = -1$.



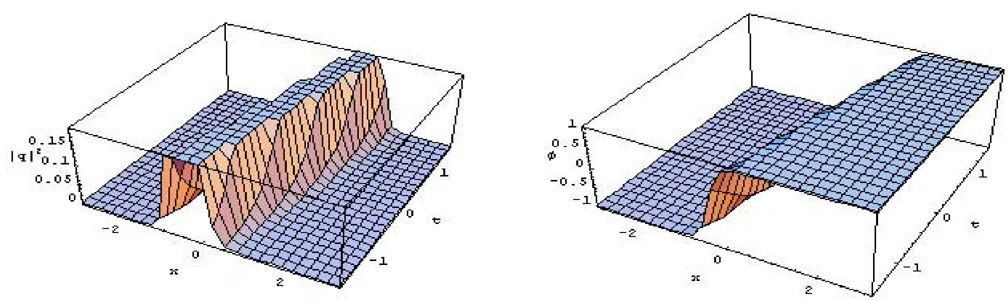
(a) (b)

Figure 5: $q(x, y, t)$ in (27) and $\phi(x, y, t)$ in (32) where $y = -0.1$, $k_1 = 1.5$, $k_2 = 0.3$, $k_3 = -0.6$, $\sigma = I$, $\lambda = 1$.



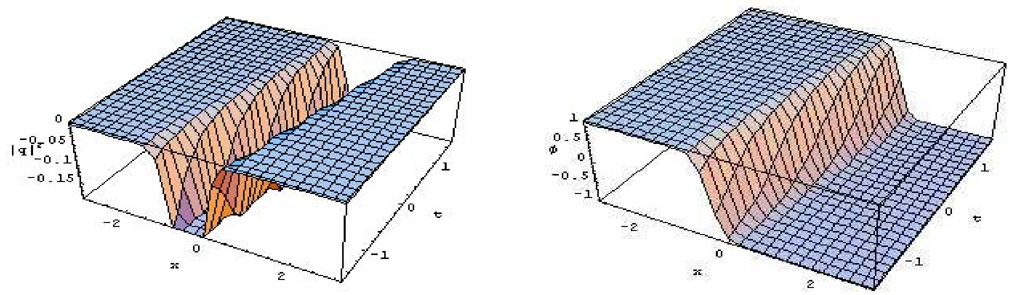
(a) (b)

Figure 6: $q(x, y, t)$ in (27) and $\phi(x, y, t)$ in (32) where $y = -0.1$, $k_1 = 1.5$, $k_2 = 0.3$, $k_3 = -0.6$, $\sigma = I$, $\lambda = -1$.



(a) (b)

Figure 7: $q(x, y, t)$ in (28) and $\phi(x, y, t)$ in (33) where $y = 0.6$, $k_1 = -2$, $k_2 = 0.7$, $k_3 = 0.5$, $\sigma = I$, $\lambda = 1$.



(a) (b)

Figure 8: $q(x, y, t)$ in (28) and $\phi(x, y, t)$ in (33) where $y = 0.6$, $k_1 = -2$, $k_2 = 0.7$, $k_3 = 0.5$, $\sigma = I$, $\lambda = -1$.

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