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Two And Three Dimensional Regions From Homothetic Motions*

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Abstract

In this study, the oriented area of a region swept by a line segment under 1-parameter planar homothetic motions and the volume of the region traced by a fixed rectangle under 1-parameter spatial homothetic motions are studied. Furthermore, the polar moment of inertia of a planar region swept by a line segment under Frenet motion is given.

1 Introduction

Kinematic describes the motion of a point or a point system depending on time. If a point moves with respect to one parameter, then it traces its 1-dimensional path, orbit curve. If a line segment and a rectangle moves with respect to one parameter, then they sweep their two and three dimensional paths, respectively.

The polar moment of inertia of a closed orbit curve is studied in planar kinematics by [3, 5] and is generalized to closed projection curves by [2, 4]. The area of swept region under 1-parameter open planar motions is given by [1] and under 1-parameter open homothetic motions by [8]. They obtained the swept region by using the line segments which combine the points of open orbit curve with a fixed point chosen on the fixed plane. They also give the surface area swept by the pole ray. Urban studies the surface area swept by a fixed line segment which moves under the Frenet motion (the motion obtained by taking the Frenet frame of the curve as a moving frame) of a regular planar curve and obtains the volume of the region traced by a line segment which is fixed according to the special defined frame of a surface, [7].

In this paper, the area of the swept region obtained by planar homothetic motions and the volume of the region swept under spatial homothetic motions are studied. In section 2, the polar moment of inertia of a planar region swept by a line segment under Frenet motion is given. The method of Urban for swept area given in [7] is generalized to the 1-parameter planar homothetic motions in the third section. The result obtained in this section generalizes also the results of [1] and [8]. In the last section, the volume of the 3-dimensional region traced by a fixed rectangle under 1-parameter spatial homothetic motions is studied.

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2 Polar Moment of Inertia

In this part, we give the polar moment of inertia of the planar region swept by a line segment under the Frenet motion of a regular curve.

Let us consider a regular curve $\beta : I \rightarrow E^2$ parametrized by arc length v. If we denote the unit tangent and unit normal vectors of $\beta(v)$ with $\mathbf{t}$ and $\mathbf{n}$, respectively; we have

$$\frac{d\mathbf{t}}{dv} = \kappa(v)\mathbf{n}(v), \quad \frac{d\mathbf{n}}{dv} = -\kappa(v)\mathbf{t}(v),$$

where κ is the curvature of the curve. Let δ be a fixed line segment according to the moving Frenet frame $\{\mathbf{t}, \mathbf{n}\}$; i.e.

$$\delta = x\mathbf{t} + y\mathbf{n}, \quad x,\, y = constants.$$

The area F of the region Γ swept by the line segment δ during the Frenet motion is given by

$$F = \frac{\phi}{2}\left(x^2 + y^2\right) - Ly, \tag{1}$$

where ϕ is the total curvature and L is the total length of the base curve β, [7].

Now, let us compute the polar moment of inertia of the region Γ. For the points of Γ, we may write

$$X(u, v) = \beta(v) + u\delta(v), \quad v \in I,\, 0 \leq u \leq 1. \tag{2}$$

Then, the area element of the planar region Γ is

$$d\sigma = \left[\kappa\, u\left(x^2 + y^2\right) - y\right]\, du\, dv. \tag{3}$$

Hence, the polar moment of inertia T of Γ with respect to the origin point is obtained by

$$T = \iint_{\Gamma} ||X(u, v)||^2\, d\sigma. \tag{4}$$

Substituting (2) and (3) into (4) yields

$$\begin{aligned} T \;=\; & \oint \Big\{ \frac{1}{2}\left(x^2 + y^2\right)\kappa(v)||\beta(v)||^2 - y||\beta(v)||^2 + \frac{2}{3}\left(x^2 + y^2\right)\kappa(v)\big\langle\beta(v), \delta(v)\big\rangle \\ & -y\big\langle\beta(v), \delta(v)\big\rangle + \frac{1}{4}\left(x^2 + y^2\right)\kappa(v)||\delta(v)||^2 - \frac{1}{3}y||\delta(v)||^2 \Big\}\, dv, \end{aligned}$$

where $\langle , \rangle$ denotes scalar product and the integration is taken over I. Hence we have:

THEOREM 1. During the Frenet motion of a planar curve β, the polar moment of inertia of the region swept by a line segment with the initial point β and the end point $\beta + x\mathbf{t} + y\mathbf{n}$ is given by

$$T = \left\{\frac{\phi}{4}\left(x^2 + y^2\right) + \frac{2K_1}{3}x + \frac{2K_2 - L}{3}y + \frac{A}{2}\right\}\left(x^2 + y^2\right) - (E_1 x + E_2 y + B)y,$$

where ϕ and L denote the total curvature and total length of the curve, respectively;

$$\begin{aligned} A &= \oint \kappa||\beta||^2 dv, & B &= \oint ||\beta||^2 dv, & K_1 &= \oint \kappa\langle\beta, \mathbf{t}\rangle dv \\ K_2 &= \oint \kappa\langle\beta, \mathbf{n}\rangle dv, & E_1 &= \oint \langle\beta, \mathbf{t}\rangle dv, & E_2 &= \oint \langle\beta, \mathbf{n}\rangle dv. \end{aligned}$$

3 Area of Swept Region under Homothetic Motion

In this section, the method given by Urban, [7], for computing the area of the region swept by a line segment is generalized to the planar homothetic motions.

Let us consider 1-parameter planar homothetic motion of the moving plane E with respect to the fixed plane E' and denote the moving and fixed orthonormal frames with $\{O; \mathbf{e}_1, \mathbf{e}_2\}$ and $\{O'; \mathbf{e}'_1, \mathbf{e}'_2\}$, respectively. If we take $\vec{OO'} = \mathbf{u} = u_1\mathbf{e}_1 + u_2\mathbf{e}_2$, then we have

$$\mathbf{x}' = h\,\mathbf{x} - \mathbf{u},$$

where $h = h(t)$ is the homothetic scale and $\mathbf{x}$, $\mathbf{x}'$ are the position vectors of a fixed point $X \in E$ according to E and E', respectively.

Taking $\varphi = \varphi(t)$ as the rotation angle, i.e. the angle between the vectors $\mathbf{e}_1$ and $\mathbf{e}'_1$, yields

$$\left\{ \begin{array}{rcr} \mathbf{e}_1 & = & \mathbf{e}'_1\,\cos\varphi + \mathbf{e}'_2\,\sin\varphi \\ \mathbf{e}_2 & = & -\mathbf{e}'_1\,\sin\varphi + \mathbf{e}'_2\,\cos\varphi \end{array} \right. \tag{5}$$

and

$$\dot{\mathbf{e}}_1 = \dot{\varphi}\,\mathbf{e}_2, \;\; \dot{\mathbf{e}}_2 = -\dot{\varphi}\,\mathbf{e}_1,$$

where the dot indicates the derivative with respect to the parameter "t".

The motion is called closed if there exists $\rho > 0$ such that

$$h(t+\rho) = h(t), \quad \varphi(t+\rho) = \varphi(t) + 2\pi\nu, \quad u_i(t+\rho) = u_i(t), \quad i = 1, 2,$$

for all "t". The smallest number ρ satisfying these properties is called the period of closed motion, the integer ν is called the rotation number of the motion.

Let us consider a line segment ℓ which is fixed according to the moving coordinate frame, i.e.

$$\ell(t) = \lambda\,\mathbf{e}_1(t) + \mu\,\mathbf{e}_2(t), \quad \lambda,\, \mu = constants$$

and let $X = (x_1, x_2)$ be the initial point of this line segment. Then, for the region swept by ℓ during the 1-parameter planar homothetic motion, we may write

$$\psi(s,t) = \mathbf{x}'(t) + sh(t)\,\ell(t), \quad s \in [0,1], \quad t \in [t_1, t_2]. \tag{6}$$

Thus, we get

$$\psi_t = \dot{\mathbf{x}}'(t) + s\,\dot{h}(t)\ell(t) + s\,h(t)\,\dot{\ell}(t), \quad \psi_s = \lambda\,h(t)\mathbf{e}_1(t) + \mu\,h(t)\mathbf{e}_2(t). \tag{7}$$

Since $X \in E$ is fixed point in E, then $\dot{\mathbf{x}}'(t)$ corresponds to the sliding velocity vector of X which was given by [6]:

$$\dot{\mathbf{x}}'(t) = \Big\{\dot{h}\,x_1 - \dot{u}_1 + \dot{\varphi}\,(u_2 - h\,x_2)\Big\}\,\mathbf{e}_1 + \Big\{\dot{h}\,x_2 - \dot{u}_2 + \dot{\varphi}\,(h\,x_1 - u_1)\Big\}\,\mathbf{e}_2. \tag{8}$$

Furthermore, the area F_X of the region swept by the line segment ℓ is given by

$$F_X = \int_{t_1}^{t_2}\!\!\int_0^1 \det\{\psi_s, \psi_t\}\,ds\,dt. \tag{9}$$

If we substitute (7) and (8) into (9), the area swept by the line segment ℓ is obtained as

$$F_X = \frac{C}{2}\left(\lambda^2+\mu^2\right) + (Cx_1 + Hx_2 + D)\,\lambda + (-Hx_1 + C\,x_2 + E)\,\mu, \tag{10}$$

where

$$\begin{aligned} C &= \int_{t_1}^{t_2} h^2(t)\,\dot{\varphi}(t)\,dt, \qquad D = \int_{t_1}^{t_2} \Big\{ -h(t)\dot{u}_2(t) - h(t)u_1(t)\,\dot{\varphi}(t)\Big\}\,dt, \\ E &= \int_{t_1}^{t_2} \Big\{h(t)\dot{u}_1(t) - h(t)u_2(t)\,\dot{\varphi}(t)\Big\}\,dt, H = \int_{t_1}^{t_2} h(t)\dot{h}(t)dt = \frac{1}{2}\Big(h^2(t_2) - h^2(t_1)\Big). \end{aligned}$$

We may give the following theorem:

THEOREM 2. The end points of the line segments which have the same swept area under the 1-parameter planar homothetic motions lie on the circle with center

$$M = \left(-\frac{C\,x_1 + H\,x_2 + D}{C}, -\frac{-Hx_1 + C\,x_2 + E}{C}\right)$$

on the moving plane, where (x_1, x_2) is the initial point of these segments on E.

COROLLARY 1. In the case of closed planar homothetic motion, since $H = 0$ and $C = 2h^2(t_0)\pi\nu,\ t_0 \in [0, \rho]$ [see 6], the swept area of a line segment is given by

$$F_X = h^2(t_0)\pi\nu\left(\lambda^2+\mu^2\right) + \Big(2h^2(t_0)\pi\nu x_1 + D\Big)\,\lambda + \Big(2h^2(t_0)\pi\nu x_2 + E\Big)\,\mu.$$

COROLLARY 2. Let X and Y be two fixed points on E and Z be another point on the line segment XY, i.e. $z_i = \xi_1 x_i + \xi_2 y_i$. Then, the swept areas of the parallel line segments with initial points X, Y, Z have the relation

$$F_Z = \xi_1 F_X + \xi_2 F_Y.$$

4 Volume of Region Swept under Homothetic Motion

In this part of our study, we obtain the volume formula of the region swept by a fixed rectangle under the 1-parameter homothetic motions in Euclidean 3-space.

A one-parameter homothetic (equiform) motion of a rigid body in 3-dimensional Euclidean space is given analytically by

$$\mathbf{x}' = hA\mathbf{x} + C \tag{11}$$

in which $\mathbf{x}'$ and $\mathbf{x}$ are the position vectors, represented by column matrices, of a point X in the fixed space R' and the moving space R respectively; A is an orthogonal 3×3-matrix, C a translation vector and h is the homothetic scale of the motion. Also, h, A and C are continuously differentiable functions of a real parameter t.

Let $\{O;\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3\}$ and $\{O';\mathbf{r}'_1,\mathbf{r}'_2,\mathbf{r}'_3\}$ be two right-handed sets of orthonormal vectors that are rigidly linked to the moving space R and fixed space R', respectively and let the derivative equations be

$$\dot{\mathbf{r}}_i = \omega_k \mathbf{r}_j - \omega_j \mathbf{r}_k, \quad (i,j,k = 1,2,3;2,3,1;3,1,2), \tag{12}$$

where ω_i are the functions of the parameter t.

Let X be a fixed point in R with

$$\vec{OX} = \mathbf{x} = x_1\mathbf{r}_1 + x_2\mathbf{r}_2 + x_3\mathbf{r}_3.$$

If we denote

$$\vec{OO'} = \mathbf{u} = u_1\mathbf{r}_1 + u_2\mathbf{r}_2 + u_3\mathbf{r}_3,$$

for the position vector of X in R' we may write

$$\mathbf{x}' = h\mathbf{x} - \mathbf{u}. \tag{13}$$

Let

$$d_1 = \lambda_1\mathbf{r}_1 + \lambda_2\mathbf{r}_2 + \lambda_3\mathbf{r}_3, \quad \lambda_i = \text{constants}$$

and

$$d_2 = \mu_1\mathbf{r}_1 + \mu_2\mathbf{r}_2 + \mu_3\mathbf{r}_3, \quad \mu_i = \text{constants}$$

be two orthogonal line segments (beginning from X) with lengths a and b, respectively. Then, we have

$$\sum_{i=1}^{3}\lambda_i^2 = a^2, \quad \sum_{i=1}^{3}\mu_i^2 = b^2, \quad \sum_{i=1}^{3}\lambda_i\mu_i = 0. \tag{14}$$

Now, let us consider the rectangle defined by the line segments d_1 and d_2. We want to obtain the volume of the region G in R' swept by this rectangle under the 1-parameter homothetic motion. We may write

$$G(u,v,t) = \mathbf{x}'(t) + h(t)\Big(u\,d_1(t) + v\,d_2(t)\Big), \quad u,v \in [0,1],\ t \in [t_1,t_2]. \tag{15}$$

Thus, the volume of G is given by

$$V = \int_{t_1}^{t_2}\int_0^1\int_0^1 \det\{G_u, G_v, G_t\}\,du\,dv\,dt. \tag{16}$$

Since

$$\begin{aligned}
G_u &= h(t)\sum_{i=1}^{3}\lambda_i\mathbf{r}_i(t), \quad G_v = h(t)\sum_{i=1}^{3}\mu_i\mathbf{r}_i(t)\\
G_t &= \sum_{i=1}^{3}\Big[\dot{h}x_i - \dot{u}_i + \dot{h}\,(\lambda_i\,u + \mu_i\,v) + \omega_j\Big(hx_k - u_k + h(\lambda_k u + \mu_k v)\Big)\\
&\qquad -\omega_k\Big(hx_j - u_j + h(\lambda_j u + \mu_j v)\Big)\Big]\mathbf{r}_i,
\end{aligned}$$

we find

$$\begin{aligned}
&\det\{G_u, G_v, G_t\}\\
= &\sum_{i=1}^{3} h^2 \Bigg\{ A_i \left(\dot{h}x_i - \dot{u}_i + \dot{h}(\lambda_i\, u + \mu_i\, v)\right)\\
+ &\left[A_k\Big(hx_j - u_j + h(\lambda_j u + \mu_j v)\Big) - A_j\Big(hx_k - u_k + h(\lambda_k u + \mu_k v)\Big)\right]\omega_i \Bigg\},
\end{aligned}$$

where $A_i = \lambda_j\mu_k - \lambda_k\mu_j, \quad i,j,k = 1,2,3(cyclic)$.

Substituting the last equation into (16) and using (14) yield

$$\begin{aligned}
V &= \sum_{i=1}^{3}\int_{t_1}^{t_2} \Bigg\{ A_i\left(\dot{h}x_i - \dot{u}_i\right) + \Big[A_k(hx_j - u_j) - A_j(hx_k - u_k)\Big]\omega_i\\
&\quad + \frac{1}{2}\left[\Big(\lambda_i b^2 - \mu_i a^2\Big)h^3\omega_i + A_i(\lambda_i + \mu_i)h^2\dot{h}\right]\Bigg\}dt. \qquad (17)
\end{aligned}$$

So, we may give the following theorem:

THEOREM 3. Let us consider the 1-parameter spatial homothetic motion in Euclidean 3-space. The volume of the region swept by a fixed rectangle (with side lengths a and b) during the homothetic motion is given by the formula

$$\begin{aligned}
V &= \frac{1}{3}\Big(h^3(t_2) - h^3(t_1)\Big)\sum_{i=1}^{3} A_i x_i + \sum_{i=1}^{3} A_i B_i + \sum_{i=1}^{3}(A_k x_j - A_j x_k)C_i\\
&\quad + \frac{1}{2}\left\{\sum_{i=1}^{3}\left(\lambda_i b^2 - \mu_i a^2\right)C_i + \frac{1}{3}\Big(h^3(t_2) - h^3(t_1)\Big)\sum_{i=1}^{3} A_i(\lambda_i + \mu_i)\right\},
\end{aligned}$$

where (x_1, x_2, x_3) denotes the coordinate of a vertex of the rectangle; λ_i, μ_i are the fixed directions of the sides of rectangle and

$$B_i = \int_{t_1}^{t_2}(u_j\omega_k - u_k\omega_j - \dot{u}_i)h^2 dt, \quad C_i = \int_{t_1}^{t_2} h^3\omega_i dt.$$

5 Examples

5.1 The area of the swept region

Let us consider the 1-parameter planar homothetic motion with

$$\mathbf{e}_1(t) = (-\sin t, \cos t),\ \mathbf{e}_2(t) = (-\cos t, -\sin t),\ h(t) = \cos\left(\frac{t}{4}\right)$$

and let the point O of the moving plane moves along the curve $\alpha(t) = (1 + \cos t, \sin t)$. During such a motion, let us compute the area of the swept region by the fixed line segment $\ell(t) = -\mathbf{e}_1(t) - \mathbf{e}_2(t)$ at the point $X = (0, -1)$ on the moving plane (see Figure 1). Then, we have

$$\dot{\varphi}(t) = 1,\ u_1(t) = \sin t,\ u_2(t) = 1 + \cos t,\ \lambda = -1,\ \mu = -1.$$

If we restrict the motion to the time interval $[0, 2\pi]$, we obtain

$$C = \int_0^{2\pi} \cos^2\left(\frac{t}{4}\right) dt = \pi, \quad D = 0, \quad E = -\int_0^{2\pi} \cos\left(\frac{t}{4}\right) dt = -4, \quad H = -\frac{1}{2}.$$

Thus, we get the area of the swept region from (10) as $F_X = 2\pi + \frac{7}{2} \simeq 9.7832$.

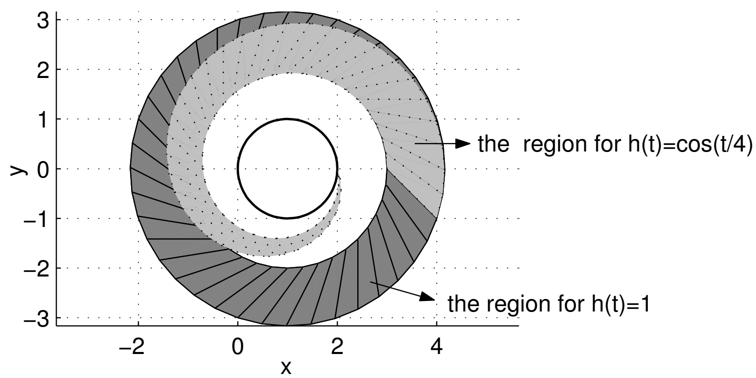

Figure 1: The swept regions by the line segment $-e_1 - e_2$ at the point $(0, -1)$ under the homothetic motions

5.2 The volume of the swept region

Now, let us compute the volume of the three dimensional region (see Figure 2) swept by the rectangle defined by the fixed line segments $d_1(t) = \mathbf{r}_3(t)$ and $d_2(t) = -\mathbf{r}_1(t) - \mathbf{r}_2(t)$ at the fixed point $X = (0, -1, 1)$ of the moving space under the 1-parameter homothetic motion with

$$\mathbf{r}_1(t) = (-\sin t, \cos t, 0),\ \mathbf{r}_2(t) = (-\cos t, -\sin t, 0),\ \mathbf{r}_3(t) = (0, 0, 1),\ h(t) = \frac{1}{3}\cos(t+1)$$

while the origin point O of moving space moves along the curve $\beta(t) = (1+\cos t, \sin t, 0)$. For this motion, we have

$$\left\{\begin{array}{lllllllllllll} \omega_1(t) & = & 0, & u_1(t) & = & \sin t, & \lambda_1 & = & 0, & \mu_1 & = & -1, & a = 1, \\ \omega_2(t) & = & 0, & u_2(t) & = & 1+\cos t, & \lambda_2 & = & 0, & \mu_2 & = & -1, & b = \sqrt{2}, \\ \omega_3(t) & = & 1, & u_3(t) & = & 0, & \lambda_3 & = & 1, & \mu_3 & = & 0. & \end{array}\right.$$

As in the first example, restricting the motion to $[0, 2\pi]$ yields

$$\left\{\begin{array}{lllllllll} A_1 & = & 1, & B_1 & = & \frac{\pi}{9}, & C_1 & = & 0, \\ A_2 & = & -1, & B_2 & = & 0, & C_2 & = & 0, \\ A_3 & = & 0, & B_3 & = & 0, & C_3 & = & 0. \end{array}\right.$$

Hence, the volume of the region swept by the rectangle during the homothetic motion is $V = \frac{\pi}{9} \simeq 0.3491$.

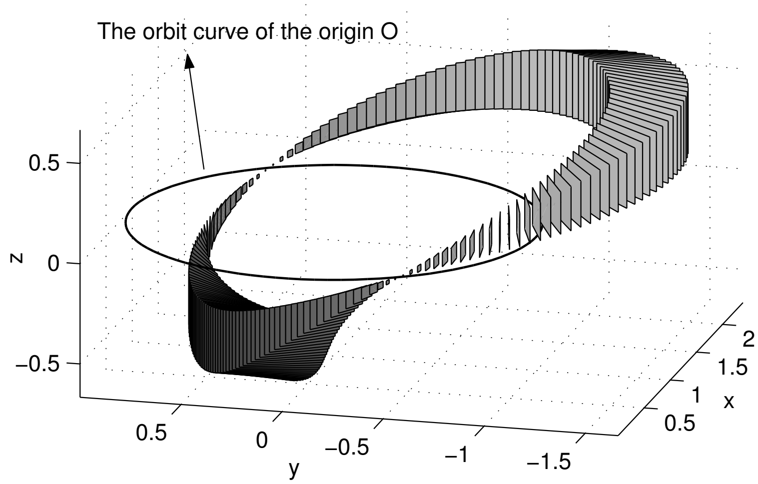

Figure 2: The swept region by the rectangle under $h(t) = \frac{1}{3}\cos(t+1)$

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