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Stability And Ultimate Boundedness Of Solutions To Certain Third-Order Differential Equations*

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Abstract

We employ Lyapunov's second method to investigate uniform asymptotic stability, ultimate boundedness and uniform ultimate boundedness of solutions to certain third-order non-linear differential equations. Our results improve some well-known results in the literature.

1 Introduction

The concept of stability and boundedness of solutions cannot be overemphasized in the theory and applications of differential equations. Till now, many authors have done excellent works; see for instance Reissig *et al.,* [6] a survey book and ([1] - [5], and [7]). With respect to our observation in the relevant literature, works on uniform stability and uniform ultimate boundedness of solutions for third-order non-linear differential equation (1) using a complete Lyapunov function are scarce. The purpose of this paper is to study uniform stability, ultimate boundedness and uniform ultimate boundedness of solutions of (1) when: $p(t, x, \dot{x}, \ddot{x}) = 0$, $p(t, x, \dot{x}, \ddot{x}) = p(t)$ and $p(t, x, \dot{x}, \ddot{x}) \neq 0$ of the following third-order differential equation

$$\dddot{x} + f(x, \dot{x}, \ddot{x})\ddot{x} + g(x, \dot{x}) + h(x, \dot{x}, \ddot{x}) = p(t, x, \dot{x}, \ddot{x}), \tag{1}$$

or its equivalent system

$$\dot{x} = y, \quad \dot{y} = z, \quad \dot{z} = p(t, x, y, z) - f(x, y, z)z - g(x, y) - h(x, y, z), \tag{2}$$

where $g \in C(\mathbb{R}^2, \mathbb{R})$, $f, h \in C(\mathbb{R}^3, \mathbb{R})$, $p \in C(\mathbb{R}^+ \times \mathbb{R}^3, \mathbb{R})$, $\mathbb{R}^+ = [0, \infty)$ and $\mathbb{R} = (-\infty, \infty)$. It is supposed that the functions f, g, h and p depend only on the arguments displayed explicitly, and the dots, as elsewhere, denote differentiation with respect to t. The derivatives $\frac{\partial}{\partial x}f(x, y, z) = f_x(x, y, z)$, $\frac{\partial}{\partial x}g(x, y) = g_x(x, y)$, $\frac{\partial}{\partial x}h(x, y, z) = h_x(x, y, z)$, $\frac{\partial}{\partial y}h(x, y, z) = h_y(x, y, z)$ and $\frac{\partial}{\partial z}h(x, y, z) = h_z(x, y, z)$ exist and are continuous. Moreover, the existence and uniqueness of solutions of (1) will be assumed. We shall use Lyapunov's second (or direct) method as our tool to achieve the desired results. The results obtained in this investigation improve the existing results on the third-order non-linear differential equations in the literature.

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2 Main Results

In the case $p \equiv 0$, (2) becomes

$$\dot{x} = y, \quad \dot{y} = z, \quad \dot{z} = -f(x,y,z)z - g(x,y) - h(x,y,z), \tag{3}$$

with the following result.

THEOREM 1. In addition to the basic assumptions on f, g, h and p, suppose that $\delta_0, a, b, c, a_1, b_1$ are positive constants and that:

(i) $h(0,0,0) = 0,\ \delta_0 \leq \dfrac{h(x,y,z)}{x}$ for all $x \neq 0,\ y$ and z;

(ii) $g(0,0) = 0,\ b \leq \dfrac{g(x,y)}{y} \leq b_1$ for all x and $y \neq 0$;

(iii) $h_x(x,0,0) \leq c$ for all x;

(iv) $a \leq f(x,y,z) \leq a_1$ for all x, y, z;

(v) $g_x(x,y) \leq 0,\ yf_x(x,y,0) \leq 0$ for all x, y;

(vi) $h_y(x,y,0) \geq 0,\ h_z(x,0,z) \geq 0,\ yf_z(x,y,z) \geq 0$ for all x, y, z.

Then the zero solution of the system (3) is uniform asymptotically stable provided that $c < ab$.

REMARK 2. In the special case $f(x,y,z) = f(x,y)$, $g(x,y) = g(y)$, and $h(x,y,z) = h(x)$ the assumptions of Theorem 1 are less restrictive than those established by Ezeilo ([2], Theorem 1).

For the rest of this article, δ_i $(i = 0,1,\cdots,13)$, D_i $(i = 0,1,2,\cdots,10)$, D_i^* $(i = 0,1)$, E_1 and D stand for positive constants. Their meaning is preserved throughout this paper.

The proof of this and subsequent results depend on the fundamental properties of the continuously differentiable function $V = V(x(t),y(t),z(t))$ defined as:

$$\begin{aligned} 2V = {} & 2(\alpha + a)\int_0^x h(\xi,0,0)d\xi + 4\int_0^y g(x,\tau)d\tau + 2\beta xz + 2(\alpha + a)yz \\ & + 2(\alpha + a)\int_0^y \tau f(x,\tau,0)d\tau + 4h(x,0,0)y + b\beta x^2 + \beta y^2 + 2z^2 + 2a\beta xy \end{aligned} \tag{4}$$

where α and β are positive constants chosen so that

$$\frac{c}{b} < \alpha < a \tag{5a}$$

and

$$0 < \beta < \min\left[(ab-c)a^{-1};\ \frac{(ab-c)\delta_0}{[\frac{g(x,y)}{y} - b]^2};\ \frac{\frac{1}{2}(a-\alpha)\delta_0}{[f(x,y,z) - a]^2}\right], \tag{5b}$$

$a \neq 0, \frac{g(x,y)}{y} - b \neq 0\ y \neq 0, f(x,y,z) - a \neq 0.$

These are discussed in the following lemmas.

LEMMA 3. Subject to the hypotheses (i)-(iv) of Theorem 1, $V(0,0,0) = 0$ and there exist constants $D_0 = D_0(\alpha, \beta, \delta_0, a, b, c) > 0$, $D_1 = D_1(\alpha, \beta, \delta_0, a, a_1, b, b_1, c) > 0$ such that

$$D_0(x^2 + y^2 + z^2) \leq V(x, y, z) \leq D_1(x^2 + y^2 + z^2),$$

and that

$$V(x, y, z) \to +\infty \text{ as } x^2 + y^2 + z^2 \to \infty.$$

PROOF. From (4), it is clear that $V(0,0,0) = 0$. Indeed we can rearrange the terms in (4) to obtain

$$2V = V_1 + V_2 + V_3 \tag{6}$$

where

$$\begin{aligned} \frac{1}{2}V_1 &= \frac{1}{b}\int_0^x \Big[(\alpha + a)b - 2h_\xi(\xi, 0, 0)\Big]h(\xi, 0, 0)d\xi + 2\int_0^y \Big[\frac{g(x,\tau)}{\tau} - b\Big]\tau d\tau \\ &\quad + \frac{1}{b}\Big[by + h(x, 0, 0)\Big]^2, \end{aligned}$$

$$V_2 = b\beta x^2 + (a^2 + \beta)y^2 + z^2 + 2a\beta xy + 2\beta xz + 2ayz$$

and

$$V_3 = 2\int_0^y \Big[(\alpha + a)f(x, \tau, 0) - (\alpha^2 + a^2)\Big]\tau d\tau + (\alpha y + z)^2.$$

We note that $\frac{1}{2}V_1$ is obviously positive definite. This follows from the conditions $h(0,0,0) = 0$, $\frac{h(x,0,0)}{x} \geq \delta_0$ $(x \neq 0)$, $h_x(x, 0, 0) \leq c$ for all x, $\frac{g(x,y)}{y} \geq b$ for all $x, y \neq 0$ and (5a). Hence, for all $x \neq 0$, $y \neq 0$, we have

$$V_1 \geq [(\alpha + a)b - 2c]\delta_0 b^{-1}x^2 + (by + \delta_0 x)^2 > 0.$$

Also, V_2 regarded as quadratic form in x, y and z can be rearranged in the form XAX^T, where $X = \begin{pmatrix} x & y & z \end{pmatrix}$, X^T the transpose of X and

$$A = \begin{pmatrix} b\beta & a\beta & \beta \\ a\beta & a^2 + \beta & a \\ \beta & a & 1 \end{pmatrix}.$$

Since condition (5b) implies that $b > \beta$ it is clear that the principal minors are positive and therefore A is positive definite. Moreover, since $f(x, y, 0) \geq a$ for all x, y and by (5a), V_3 is positive definite. Thus V is positive semidefinite. We can therefore, find a constant $\delta_1 = \delta_1(\alpha, \beta, \delta_0, a, b, c,) > 0$, such that

$$V \geq \delta_1(x^2 + y^2 + z^2). \tag{7}$$

Furthermore, since $h(0,0,0) = 0$ and $h_x(x, 0, 0) \leq c$ for all x, we have

$$h(x, 0, 0) \leq cx \tag{8}$$

for all $x \neq 0$. By using Schwartz inequality and (8), we have

$$|V_1| \leq D_0^*(x^2 + y^2), \quad D_0^* = D_0^*(\alpha, a, b_1, c) > 0,$$

and

$$|V_2 + V_3| \leq D_1^*(x^2 + y^2 + z^2), \quad D_1^* = D_1^*(\alpha, \beta, a, a_1, b) > 0.$$

Hence, there exists a constant $\delta_2 > 0$ such that

$$V \leq \delta_2(x^2 + y^2 + z^2), \tag{9}$$

where $\delta_2 = \max(D_0^*; D_1^*)$. Moreover, from (7) $V(x, y, z) = 0$ if $x^2 + y^2 + z^2 = 0$, $V(x, y, z) > 0$ if $x^2 + y^2 + z^2 \neq 0$, it follows that $V(x, y, z) \to +\infty$ as $x^2 + y^2 + z^2 \to \infty$. This completes the proof of Lemma 3.

LEMMA 4. Under the hypotheses of Theorem 1, there exists a positive constant D_2 depending only on $\alpha, \beta, \delta_0, a, b, c$ such that along a solution of (3)

$$\dot{V} = \frac{d}{dt}V(x, y, z) \leq -D_2(x^2 + y^2 + z^2).$$

PROOF. Along any solution $(x(t), y(t), z(t))$ of (3), we have

$$\begin{aligned}\dot{V}_{(3)} =& 2y\int_0^y g_x(x, \tau)d\tau + (\alpha + a)y\int_0^y \tau f_x(x, \tau, 0)d\tau + \beta(ay^2 + 2yz)\\ &- \beta x h(x, y, z) - [(\alpha + a)yg(x, y) - 2h_x(x, 0, 0)y^2] - W_1 - W_2\\ &- [2f(x, y, z) - (\alpha + a)]z^2 - \beta[f(x, y, z) - a]xz - \beta[g(x, y) - by]x,\end{aligned} \tag{10}$$

where

$$W_1 := (\alpha + a)yz[f(x, y, z) - f(x, y, 0)]$$

and

$$W_2 := [(\alpha + a)y + 2z][h(x, y, z) - h(x, 0, 0)].$$

In view of hypothesis (v) of Theorem 1, the first two terms in $\dot{V}_{(3)}$ satisfies

$$2y\int_0^y g_x(x, \tau)d\tau + (\alpha + a)y\int_0^y \tau f_x(x, \tau, 0)d\tau \leq 0. \tag{11}$$

Furthermore, from hypothesis (vi) of Theorem 1, we obtain

$$W_1 = (\alpha + a)yz^2\left[\frac{f(x, y, z) - f(x, y, 0)}{z}\right] = (\alpha + a)z^2 y f_z(x, y, \theta_1 z) \geq 0, \tag{12}$$

where $0 \leq \theta_1 \leq 1$ and $W_1 = 0$ when $z = 0$. Similarly

$$W_2 = 2z^2 h_z(x, 0, \theta_2 z) + (\alpha + a)y^2 h_y(x, \theta_3 y, 0) \geq 0, \tag{13}$$

where $0 \leq \theta_i \leq 1$, $(i = 2, 3)$ and $W_2 = 0$ when $y = z = 0$. By hypotheses (i) - (iv) of Theorem 1, we note the following:

$$\beta x h(x, y, z) \geq \beta\delta_0 x^2, \ x \neq 0, \tag{14}$$

$$(\alpha+a)\frac{g(x,y)}{y}-2h_x(x,0,0)\geq(\alpha+a)b-2c,\quad y\neq 0, \tag{15}$$

and

$$2f(x,y,z)-(\alpha+a)\geq a-\alpha. \tag{16}$$

On gathering estimates (11) - (16) into (10), we obtain

$$\dot{V}_{(3)}\leq-\frac{1}{2}\beta\delta_0x^2-[(\alpha+a)b-2c]y^2-(a-\alpha)z^2-W_3-W_4, \tag{17}$$

where

$$W_3:=\frac{1}{4\delta_0}\beta\left[\delta_0^2x^2+4\delta_0\left[\frac{g(x,y)}{y}-b\right]xy\right]$$

and

$$W_4:=\frac{\beta}{4\delta_0}\left[\delta_0^2x^2+4\delta_0[f(x,y,z)-a]xz\right].$$

On completing the square, we have

$$W_3\geq-\frac{\beta}{\delta_0}\left[\frac{g(x,y)}{y}-b\right]^2y^2 \text{ and } W_4\geq-\frac{\beta}{\delta_0}\left[f(x,y,z)-a\right]^2z^2,$$

since $\left[\delta_0x+2\left[\frac{g(x,y)}{y}-b\right]y\right]^2\geq 0$ for all x,y and $\left[\delta_0x+2[f(x,y,z)-a]z\right]^2\geq 0$ for all x,y,z respectively. Estimates W_3 and W_4 into (17), yields

$$\begin{aligned}\dot{V}_{(3)}\leq&-\frac{1}{2}\beta\delta_0x^2-(\alpha b-c)y^2-\frac{1}{2}(a-\alpha)z^2-\left[(ab-c)-\frac{\beta}{\delta_0}\left[\frac{g(x,y)}{y}-b\right]^2\right]y^2\\&-\left[\frac{1}{2}(a-\alpha)-\frac{\beta}{\delta_0}\left[f(x,y,z)-a\right]^2\right]z^2.\end{aligned} \tag{18}$$

In view of (5b), there exists a positive constant $\delta_3=\delta_3(\alpha,\beta,\delta_0,a,b,c)$ such that

$$\dot{V}_{(3)}\leq-\delta_3(x^2+y^2+z^2) \tag{19}$$

for all x,y,z. This completes the proof of Lemma 4.

PROOF OF THEOREM 1. We shall use the usual limit point argument as contained in [8] to show that when Lemma 3 and Lemma 4 hold, then $V(t)\equiv V(x(t),y(t),z(t))\to 0$ as $t\to\infty$. From Lemma 3, we find that $V(x,y,z)=0$ if and if only if $x^2+y^2+z^2=0$, $V(x,y,z)>0$ if and if only if $x^2+y^2+z^2\neq 0$, and $V(x,y,z)\to+\infty$ if and if only if $x^2+y^2+z^2\to\infty$. The remaining of this proof follows the strategy indicated in [1]. This completes the proof of Theorem 1.

In the case $p=p(t)$, (2) becomes

$$\dot{x}=y,\quad \dot{y}=z,\quad \dot{z}=p(t)-f(x,y,z)z-g(x,y)-h(x,y,z), \tag{20}$$

with the following result.

THEOREM 5. Suppose that δ_0,a,b,c,P_0 are positive constants and that:

(i) conditions (i), (iii), (v) and (vi) of Theorem 1 hold and $c < ab$;

(ii) $\dfrac{g(x,y)}{y} \geq b \ (y \neq 0)$;

(iii) $f(x,y,z) \geq a$, for all x, y, z;

(iv) $|p(t)| \leq P_0$ where $P_0 > 0$ is a finite constant.

Then solution $(x(t), y(t), z(t))$ of (20) ultimately satisfies

$$|x(t)| \leq D, \ |y(t)| \leq D, \ |z(t)| \leq D \tag{21}$$

for all $t \geq 0$, where D is a constant depending only on $\alpha, \beta, \delta_0, a, b, c, P_0$.

REMARK 6. Whenever $f(x,y,z) = p(t)$, $g(x,y) = q(t)g(y)$ and $h(x,y,z) = h(x)$, the hypotheses of Theorem 5 coincide with those of Swick ([7], Theorem 5). Thus Theorem 5 extends that of [7].

LEMMA 7. Subject to the hypotheses of Theorem 5, there are finite constants $D_3 > 0$ and $D_4 > 0$ dependent only on $\alpha, \beta, \delta_0, a, b, c, P_0$ such that any solution $(x(t), y(t), z(t))$ of (20) satisfies

$$\dot{V} \equiv \frac{d}{dt} V(x(t), y(t), z(t)) \leq -D_3 \tag{22}$$

provided that $x^2 + y^2 + z^2 \geq D_4$.

PROOF. Let $(x(t), y(t), z(t))$ be any solution of (20). Since $p(t) \neq 0$, a direct differentiation of (4) with respect to independent variable t yields

$$\dot{V}_{(20)}(t) = \dot{V}_{(3)}(t) + [\beta x + (\alpha + a)y + 2z]p(t).$$

By (19), Schwartz inequality and condition (iv) of Theorem 5 the last equation becomes

$$\dot{V}_{(20)}(t) \leq -\delta_3(x^2 + y^2 + z^2) + \delta_4(x^2 + y^2 + z^2)^{1/2}, \tag{23}$$

where $\delta_4 = 3^{1/2} P_0 \max[\beta; (\alpha + a); 2]$. Choose $(x^2 + y^2 + z^2)^{1/2} \geq \delta_5 = 2\delta_3^{-1}\delta_4$, the inequality in (23) becomes

$$\dot{V}_{(20)}(t) \leq -\frac{1}{2}\delta_3(x^2 + y^2 + z^2).$$

It follows that

$$\dot{V}_{(20)}(t) \leq -\delta_6 \tag{24}$$

provided that $x^2 + y^2 + z^2 \geq \delta_7 = 2\delta_3^{-1}\delta_6$. This completes the proof of Lemma 7.

PROOF OF THEOREM 5. Let $(x(t), y(t), z(t))$ be any solution of (20). Then there is a $t_0 \geq 0$ such that

$$x^2(t_0) + y^2(t_0) + z^2(t_0) < D_4$$

where D_4 is the constant in Lemma 7; for otherwise

$$x^2(t) + y^2(t) + z^2(t) \geq D_4, \quad t \geq 0,$$

and by (22)

$$\dot{V} \leq -D_3 < 0, \quad t \geq 0$$

so that $V(t) \to -\infty$ as $t \to \infty$ which contradicts (7). To prove (21) we shall show that if

$$x^2(t) + y^2(t) + z^2(t) < D_5 \text{ for t = T}, \tag{25}$$

where $D_5 \geq D_4$ is a finite constant, then there is a constant $D_6 > 0$ depending on $\alpha, \beta, \delta_0, a, b, c, P_0$ and D_5 such that

$$x^2(t) + y^2(t) + z^2(t) \leq D_6 \text{ for t} \geq \text{T}. \tag{26}$$

The remaining of this proof follows the strategy indicated in [5]. This completes the prove of Theorem 5.

THEOREM 8. Suppose that $\delta_0, a, a_1, b, b_1, c, \epsilon, E_1$ are positive constants and that:

(i) hypotheses (i)-(vi) of Theorem 1 hold and $c < ab$;

(ii) there are non-negative continuous functions $p_1(t)$ and $p_2(t)$ such that

$$|p(t, x, y, z)| \leq p_1(t) + p_2(t)(|x| + |y| + |z|), \tag{27}$$

for all $t \geq 0$, and $|x| + |y| + |z| \geq D_8$, $(D_8 > 0)$, where

$$p_1(t) \leq E_1 < \infty \tag{28a}$$

$E_1 \geq 0$, and there exists $\epsilon > 0$ satisfying

$$0 \leq p_2(t) \leq \epsilon. \tag{28b}$$

Then the solution $(x(t), y(t), z(t))$ of (2) is uniformly ultimately bounded.

REMARK 9. If $h(x, y, z) = h(x) = \phi_3(x)$ system (2) reduces to the case studied by Hara [4] and Ezeilo [3]. The hypotheses on (2) are considerably weaker than those of [3] and [4]. Furthermore, assumption (ii) of Theorem 8 generalizes the situation given by Ezeilo [3], Hara [4] and Omeike [5]. Finally, the hypothesis on $f(x, y, x)$ in [5] is too restrictive compare with (iv) of Theorem 1. Thus Theorem 8 extends [3], [4] and [5].

LEMMA 10. Subject to the hypotheses of Theorem 8, there exist positive constants D_9 and D_{10} dependent only on $\alpha, \beta, \delta_0, \epsilon, a, b, c, E_1$ such that for any solution $(x(t), y(t), z(t))$ of (2),

$$\dot{V} \leq -D_9$$

provided that $x^2 + y^2 + z^2 \geq D_{10}$.

PROOF. Let $(x(t), y(t), z(t))$ be any solution of (2). Since $p(t, x, y, z) \neq 0$, an elementary calculation from (2) and (4) yields

$$\dot{V}_{(2)}(t) = \dot{V}_{(3)}(t) + [\beta x + (\alpha + a)y + 2z]p(t, x, y, z).$$

Estimates (19), (27), (28a), (28b) and the fact that $(|x|+|y|+|z|)^2 \leq 3(x^2+y^2+z^2)$, the last equation yields

$$\dot{V}_{(2)}(t) \leq -(\delta_3 - \delta_8\epsilon)(x^2+y^2+z^2) + \delta_9(x^2+y^2+z^2)^{1/2},$$

where $\delta_8 = 3\max(\beta; \alpha+a; 2)$ and $\delta_9 = 3^{1/2}E_1\max(\beta; \alpha+a; 2)$. If we choose ϵ so small such that $\delta_3 > \delta_8\epsilon$, then there exists a positive constants δ_{10} such that

$$\dot{V}_{(2)}(t) \leq -\delta_{10}(x^2+y^2+z^2) + \delta_9(x^2+y^2+z^2)^{1/2}. \tag{29}$$

Choose $(x^2+y^2+z^2)^{1/2} \geq \delta_{11} = 2\delta_9\delta_{10}^{-1}$, so that estimate (29) becomes

$$\dot{V}_{(2)}(t) \leq -\frac{1}{2}\delta_{10}(x^2+y^2+z^2).$$

We see at once that

$$\dot{V}_{(2)}(t) \leq -\delta_{12} \tag{30}$$

provided that $x^2+y^2+z^2 \geq \delta_{13} = 2\delta_{10}^{-1}\delta_{12}$. This completes the proof of Lemma 10.

Finally, the proof of Theorem 8 follows from Lemma 3 and Lemma 10, see ([8], p. 38).

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References

[1] A. T. Ademola, M. O. Ogundiran, P. O. Arawomo and O. A. Adesina, Stability results for the solutions of a certain third-order nonlinear differential equation, Math. Sci. Res. J., 12(6)(2008), 124–134.

[2] J. O. C. Ezeilo, On the stability of solutions of some third-order differential equations, J. London Math. Soc., 43(1968), 161–167.

[3] J. O. C. Ezeilo, A generalization of a boundedness theorem for the equation $\dddot{x} + a\ddot{x} + \phi_2(\dot{x}) + \phi_3(x) = \psi(t, x, \dot{x}, \ddot{x})$, Atti. Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur., 50(13)(1971), 424–431.

[4] T. Hara, On the uniform ultimate boundedness of solutions of certain third-order differential equations, J. Math. Anal. Appl. 80(2)(1981), 533–544.

[5] M. O. Omeike, New result in the ultimate boundedness of solutions of a third-order nonlinear ordinary differential equation, J. Inequal. Pure and Appl. Math., 9(1)(2008), Art. 15, 8 pp.

[6] R. Reissig, G. Sansone and R. Conti, Nonlinear Differential Equations of Higher Order, Noordhoff International Publishing, Leyeden, 1974.

[7] K. E. Swick, On the boundedness and stability of solutions for some non-autonomous differential equations of the third order, J. London Math. Soc., 44(1969), 347–359.

[8] T. Yoshizawa, Stability Theory by Liapunov's Second Method, The Mathematical Society of Japan, 1966.