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Convexity Properties And Inequalities For A Generalized Gamma Function*

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Abstract

For the Γ_p-function, defined by Euler, are given some properties related to convexity and log-convexity. Also, some properties of p analogue of the ψ function have been established. The p-analogue of some inequalities from [6] and [7] have been proved. As an application, when $p \to \infty$, we obtain all results of [6].

1 Introduction

In this section we will present definitions used in this paper. The Euler gamma function $\Gamma(x)$ is defined for $x > 0$ by

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt.$$

The digamma (or psi) function is defined for positive real numbers x as the logarithmic derivative of Euler's gamma function, that is $\psi(x) = \frac{d}{dx} \ln \Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)}$. The following integral and series representations are valid (see [1]):

$$\psi(x) = -\gamma + \int_0^\infty \frac{e^{-t} - e^{-xt}}{1 - e^{-t}} dt = -\gamma - \frac{1}{x} + \sum_{n \geq 1} \frac{x}{n(n+x)}, \tag{1}$$

where $\gamma = 0.57721...$ denotes Euler's constant.

Euler, gave another equivalent definition for the $\Gamma(x)$ (see [2],[5])

$$\Gamma_p(x) = \frac{p! p^x}{x(x+1) \cdots (x+p)} = \frac{p^x}{x(1+\frac{x}{1}) \cdots (1+\frac{x}{p})}, x > 0 \tag{2}$$

where p is positive integer, and

$$\Gamma(x) = \lim_{p \to \infty} \Gamma_p(x). \tag{3}$$

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We define the p-analogue of the psi function as the logarithmic derivative of the Γ_p function, that is

$$\psi_p(x) = \frac{d}{dx} \ln \Gamma_p(x) = \frac{\Gamma_p'(x)}{\Gamma_p(x)}. \tag{4}$$

DEFINITION 1.1. The function f is called log-convex if for all $\alpha, \beta > 0$ such that $\alpha + \beta = 1$ and for all $x, y > 0$ the following inequality holds

$$\log f(\alpha x + \beta y) \leq \alpha \log f(x) + \beta \log f(y)$$

or equivalently

$$f(\alpha x + \beta y) \leq (f(x))^\alpha \cdot (f(y))^\beta.$$

DEFINITION 1.2. Let $f : I \subseteq (0, \infty) \longrightarrow (0, \infty)$ be a continuous function. Then f is called geometrically convex on I if there exists $n \geq 2$ such that one of the following two inequalities holds:

$$f(\sqrt{(x_1 x_2)}) \leq \sqrt{f(x_1) f(x_2)} \tag{5}$$

$$f\Big(\prod_{i=1}^{n} x_i^{\lambda_i}\Big) \leq \prod_{i=1}^{n} (f(x_i))^{\lambda_i} \tag{6}$$

where $x_1, \ldots, x_n \in I; \lambda_1, \ldots, \lambda_n > 0$ with $\sum_{i=1}^{n} \lambda_i = 1$. If inequalities (5) and (6) are reversed, then f is called geometrically concave function on I.

In the next section, we derive several convexity and log-convexity properties related the Γ_p.

2 Some Properties of Γ_p

We begin with recurrent relations for Γ_p and ψ_p.

LEMMA 2.1. Let Γ_p be defined as in (2). Then

$$\Gamma_p(x+n) = p^n \cdot \frac{\prod_{i=0}^{n-1}(x+i)}{\prod_{i=1}^{n}(x+p+i)} \Gamma_p(x), x+n > 0. \tag{7}$$

PROOF. Using (2) one finds that:

$$\frac{\Gamma_p(x+n)}{\Gamma_p(x+n-1)} = \frac{x+n-1}{p^{-1}(x+n+p)}.$$

Hence

$$\Gamma_p(x+n) = \frac{p(x+n-1)}{(x+n+p)} \cdot \Gamma_p(x+n-1).$$

In a similar way, we have:

$$\Gamma_p(x+n-1) = \frac{p(x+n-2)}{(x+(n-1)+p)} \cdot \Gamma_p(x+n-2)$$

It means

$$\Gamma_p(x+n) = \frac{p^2(x+n-1)(x+n-2)}{(x+n+p)(x+(n-1)+p)} \cdot \Gamma_p(x+n-2).$$

Continuing in this way we obtain:

$$\Gamma_p(x+n) = \frac{p^n(x+n-1)(x+n-2)\cdot\ldots\cdot x}{(x+n+p)(x+p+n-1)\cdot\ldots\cdot(x+p+1)} \cdot \Gamma_p(x),$$

completing the proof.

REMARK 2.2. When $p \to \infty$, we obtain the well known relation

$$\Gamma(x) = \frac{\Gamma(x+n)}{x(x+1)\cdot\ldots\cdot(x+n-1)},\ \ x+n>0.$$

LEMMA 2.3. a) The function ψ_p defined by (4) has the following series representation

$$\psi_p(x) = \ln p - \sum_{k=0}^{p} \frac{1}{x+k}. \tag{8}$$

b) The function ψ_p is increasing on $(0,\infty)$.

c) The function ψ_p' is strictly completely monotonic on $(0,\infty)$.

PROOF. a) By (2) we have:

$$\begin{aligned}
\psi_p(x) &= \frac{d}{dx}(\ln\Gamma_p(x)) \\
&= \frac{d}{dx}\Big(x\ln p - \Big(\ln x + \ln(1+x) + \ln\Big(1+\frac{x}{2} + \ldots + \ln\Big(1+\frac{x}{p}\Big)\Big)\Big)\Big) \\
&= \ln p - \Big(\frac{1}{x} + \frac{1}{1+x} + \frac{1}{1+\frac{x}{2}}\cdot\frac{1}{2} + \ldots + \frac{1}{1+\frac{x}{p}}\cdot\frac{1}{p}\Big) \\
&= \ln p - \sum_{k=0}^{p} \frac{1}{x+k}.
\end{aligned}$$

b) Let $0 < x < y$. Using (8) we obtain

$$\psi_p(x) - \psi_p(y) = -\sum_{k=0}^{p}\frac{1}{x+k} + \sum_{k=0}^{p}\frac{1}{y+k} = \sum_{k=0}^{p}\frac{(x-y)}{(x+k)(y+k)} < 0.$$

c) Deriving n times the relation (8) one finds that:

$$\psi_p^{(n)}(x) = \sum_{k=0}^{p}\frac{(-1)^{n-1}\cdot n!}{(x+k)^{n+1}}, \tag{9}$$

hence $(-1)^n(\psi_p'(x))^{(n)} > 0$ for $x > 0, n \ge 0$.

REMARK 2.4. We note that $\lim_{p\to\infty}\psi_p^{(n)}(x) = \psi^{(n)}(x)$.

By (8) one has the following:

COROLLARY 2.5.

$$\psi_p(x+1) = \frac{1}{x} - \frac{1}{x+p+1} + \psi_p(x).$$

COROLLARY 2.6. The function $\log \Gamma_p(x)$ is convex for $x > 0$.

PROOF. Taking $n = 2$ in (9) we have

$$\psi_p'(x) = \sum_{k=0}^{p} \frac{1}{(x+k)^2}. \tag{10}$$

So, for $x > 0$, $\psi_p'(x) > 0$ hence ψ_p is a monotonous function on the positive axis and therefore the function $\log \Gamma_p(x)$ is convex for $x > 0$.

LEMMA 2.7. Let ψ_p be as in (8). Then

$$\lim_{p\to\infty} \psi_p(x) = \psi(x). \tag{11}$$

PROOF. By (8) we have:

$$\begin{aligned}
\lim_{p\to\infty} \psi_p(x) &= \lim_{p\to\infty} \ln p - \lim_{p\to\infty} \Big(\frac{1}{x} + \sum_{k=1}^{p} \frac{1}{x+k}\Big) \\
&= \lim_{p\to\infty} \Big(\ln p - 1 - \frac{1}{2} - \ldots - \frac{1}{p}\Big) - \frac{1}{x} - \lim_{p\to\infty} \Big(\sum_{k=1}^{p} \frac{1}{x+k} - \sum_{k=1}^{p} \frac{1}{k}\Big) \\
&= -\gamma - \frac{1}{x} + \sum_{k=1}^{\infty} \frac{x}{k(k+x)} \\
&= \psi(x).
\end{aligned}$$

THEOREM 2.8. The function

$$\Gamma_p(x) = \frac{p^x}{x(1+\frac{x}{1})\ldots(1+\frac{x}{p})}, x > 0$$

is log-convex.

PROOF. We have to prove that for all $\alpha, \beta > 0, \alpha + \beta = 1, x, y > 0$

$$\log \Gamma_p(\alpha x + \beta y) \le \alpha \log \Gamma_p(x) + \beta \log \Gamma_p(y) \tag{12}$$

which is equivalent to

$$\Gamma_p(\alpha x + \beta y) \le (\Gamma_p(x))^{\alpha} \cdot (\Gamma_p(y))^{\beta}. \tag{13}$$

By Young's inequality (see [3]) we have:

$$x^{\alpha} \cdot y^{\beta} \le \alpha x + \beta y. \tag{14}$$

From (14) we obtain:

$$\Big(1+\frac{x}{k}\Big)^{\alpha}\cdot\Big(1+\frac{y}{k}\Big)^{\beta}\le\alpha\Big(1+\frac{x}{k}\Big)+\beta\Big(1+\frac{y}{k}\Big)=1+\frac{\alpha x+\beta y}{k}\tag{15}$$

for all $k\ge 1, k\in\mathbf{N}$.

Multiplying (15) for $k=1,2,\dots,p$ one obtains

$$\Big(1+\frac{x}{1}\Big)^{\alpha}\dots\Big(1+\frac{x}{p}\Big)^{\alpha}\cdot\Big(1+\frac{y}{1}\Big)^{\beta}\dots\Big(1+\frac{y}{p}\Big)^{\beta}\le\Big(1+\frac{\alpha x+\beta y}{1}\Big)\dots\Big(1+\frac{\alpha x+\beta y}{p}\Big).$$

Now, taking the reciprocal values and multiplying by $p^{\alpha x+\beta y}$ one obtains (13) and thus the proof is completed.

For the proof of the following result see [5].

PROPOSITION 2.9. Let f be a log-convex function on $(0,\infty)$. Then the function F_a given by $F_a(x)=a^x f(x)$ is convex for any $a>0$.

From Proposition 2.9 and Theorem 2.8 immediately follows the following corollary.

COROLLARY 2.10. The functions F_a, G_a given by

$$F_a(x)=a^x\Gamma_p(x), x>0; G_a(x)=x^a\Gamma_p(x), x>0,$$

respectively, are convex.

Another easily established property related to ψ_p is the following proposition.

PROPOSITION 2.11. The function $x\longmapsto x\psi_p(x), x>0$ is strictly convex.

PROOF. We have

$$(x\psi_p(x))'=\psi_p(x)+x\psi_p'(x)$$

$$(x\psi_p(x))''=2\psi_p'(x)+x\psi_p''(x).$$

Using (9) we obtain

$$(x\psi_p(x))''=2\sum_{k=0}^{p}\frac{1}{(x+k)^2}-2\sum_{k=0}^{p}\frac{x}{(x+k)^3}=2\sum_{k=0}^{p}\frac{k}{(x+k)^3}>0.$$

Next we will prove a result on geometric convexity related to Γ_p that will be used in the next section.

For the proof of the following Lemma see [4].

LEMMA 2.12. Let $(a,b)\subset(0,\infty)$ and $f:(a,b)\longrightarrow(0,\infty)$ be a differentiable function. Then f is geometrically convex if and only if the function $\frac{xf'(x)}{f(x)}$ is nondecreasing.

THEOREM 2.13. The function $f(x)=e^x\cdot\Gamma_p(x)$ is geometrically convex.

PROOF. Let $f(x)=e^x\cdot\Gamma_p(x)$. Then $\ln f(x)=x+\ln\Gamma_p(x)$. Hence

$$\frac{f'(x)}{f(x)}=1+\psi_p(x).\tag{16}$$

So, $x\frac{f'(x)}{f(x)} = x + x\psi_p(x)$. Let $\theta(x) = x + x\psi_p(x)$. Then we have

$$\theta'(x) = 1 + \psi_p(x) + x\psi_p'(x).$$

Using (8) and (9) one obtains

$$\begin{aligned}\theta'(x) &= 1 + \ln p - \sum_{k=0}^{p} \frac{1}{x+k} + x\sum_{k=0}^{p} \frac{1}{(x+k)^2} \\ &= 1 + \ln p + \sum_{k=0}^{p} \Big(\frac{x}{(x+k)^2} - \frac{1}{(x+k)}\Big) \\ &= 1 + \ln p - \sum_{k=1}^{p} \frac{k}{(x+k)^2}.\end{aligned}$$

Let $v(x) = 1 + \ln p - \sum_{k=1}^{p} \frac{k}{(x+k)^2}$. One can easily show that for $x > 0$ the function v is nondecreasing. Hence, $v(x) > v(0)$. On the other side

$$v(0) = 1 - \Big(\sum_{k=1}^{p} \frac{1}{k} - \ln p\Big) \geq 0.$$

Hence $\theta'(x) > 0$ so θ is nondecreasing.

REMARK 2.14. Using similar approach, one can show that the function $f(x) = \frac{e^x \cdot \Gamma_p(x)}{x^a}$, $a \neq 0$, is geometrically convex.

REMARK 2.15. In [7], it is proved that the function $f(x) = \frac{e^x \Gamma(x)}{x^x}$ is geometrically convex.

In relation to the function $f_1(x) = \frac{e^x \cdot \Gamma_p(x)}{x^x}$ one can show that it is geometrically convex in the neighborhood of zero, and it is not geometrically convex for $x > p$, while for the rest the proof could not be established.

3 Inequalities and Applications

In this section we prove some inequalities related to Γ_p function. Some applications of Γ_p are presented at the end of the section.

LEMMA 3.1. Let $x > 1$. Then

$$\gamma + \ln p + \psi_(x) - \psi_p(x) > 0.$$

PROOF. Using the series representations of the functions ψ and ψ_p we obtain:

$$\gamma + \ln p + \psi(x) - \psi_p(x) = (x-1)\sum_{k=0}^{\infty} \frac{1}{(1+k)(x+k)} + \sum_{k=0}^{p} \frac{1}{(x+k)} > 0.$$

Using previous Lemma we have:

LEMMA 3.2. Let a be a positive real number such that $a+x>1$. Then

$$\gamma + \ln p + \psi(x+a) - \psi_p(x+a) > 0.$$

THEOREM 3.3. Let f be a function defined by

$$f(x) = \frac{e^{\gamma x}\Gamma(x+a)}{p^{-x}\Gamma_p(x+a)}, x \in (0,1) \tag{17}$$

where a, b are real numbers such that $a+x>1$. If $\psi(x+a)>0$ or $\psi_p(x+a)>0$ then the function f is increasing for $x \in (0,1)$ and the following double inequality holds

$$\frac{\Gamma(a)}{p^x \cdot e^{\gamma x}\Gamma_p(a)} < \frac{\Gamma(x+a)}{\Gamma_p(x+a)} < p^{1-x} \cdot e^{\gamma(1-x)} \cdot \frac{\Gamma(1+a)}{\Gamma_p(1+a)}. \tag{18}$$

PROOF. Let g be a function defined by $g(x) = \ln f(x)$ for $x \in (0,1)$. Then

$$g(x) = \gamma x + \ln\Gamma(x+a) + x\ln p - \ln\Gamma_p(x+a).$$

Then

$$g'(x) = \gamma + \ln p + \psi(x+a) - \psi_p(x+a).$$

By Lemma 18 we have $g'(x)>0$. It means that g is increasing on $(0,1)$. This implies that f is increasing on $(0,1)$ so we have $f(0)<f(x)<f(1)$ and the result follows.

For the proof of the following Lemma see [4].

LEMMA 3.4. Let $(a,b) \subset (0,\infty)$ and $f:(a,b) \longrightarrow (0,\infty)$ be a differentiable function. Then f is geometrically convex if and only if the inequality

$$\frac{f(x)}{f(y)} \geq \left(\frac{x}{y}\right)^{\frac{yf'(y)}{f(y)}} \tag{19}$$

holds for any $x, y \in (a,b)$.

The following result is the analogue of the Theorem 1.2 from [7].

THEOREM 3.5. For $x>0, y>0$ the double inequality holds

$$\left(\frac{x}{y}\right)^{y(1+\psi_p(y))} \cdot e^{y-x} \leq \frac{\Gamma_p(x)}{\Gamma_p(y)} \leq \left(\frac{x}{y}\right)^{x(1+\psi_p(x))} \cdot e^{y-x}. \tag{20}$$

PROOF. Combination of Theorem 2.13, Lemma 3.4 and relation (16) leads to:

$$\frac{e^x\Gamma_p(x)}{e^y\Gamma_p(y)} \geq \left(\frac{x}{y}\right)^{y(1+\psi_p(y))}$$

and

$$\frac{e^y\Gamma_p(y)}{e^x\Gamma_p(x)} \geq \left(\frac{y}{x}\right)^{x(1+\psi_p(x))}.$$

Hence the inequality (20) is established.

In the following, we give the Γ_p analogue of results from [6]. Since the proofs are almost similar, we omit them.

LEMMA 3.6. Let a, b, c, d, e be real numbers such that $a + bx > 0,\ d + ex > 0$ and $a + bx \leq d + ex$. Then

$$\psi_p(a + bx) - \psi_p(d + ex) \leq 0. \tag{21}$$

LEMMA 3.7. Let a, b, c, d, e, f be real numbers such that $a + bx > 0, d + ex > 0, a + bx \leq d + ex$ and $ef \geq bc > 0$. If (i) $\psi_p(a + bx) > 0$, or (ii) $\psi_p(d + ex) > 0$, then

$$bc\psi_p(a + bx) - ef\psi_p(d + ex) \leq 0. \tag{22}$$

LEMMA 3.8. Let a, b, c, d, e, f be real numbers such that $a + bx > 0, d + ex > 0, a + bx \leq d + ex$ and $bc \geq ef > 0$. If (i) $\psi_p(d + ex) < 0$, or (ii) $\psi_p(a + bx) < 0$, then

$$bc\psi_p(a + bx) - ef\psi_p(d + ex) \leq 0. \tag{23}$$

THEOREM 3.9. Let f_1 be a function defined by

$$f_1(x) = \frac{\Gamma_p(a + bx)^c}{\Gamma_p(d + ex)^f}, \quad x \geq 0 \tag{24}$$

where a, b, c, d, e, f are real numbers such that: $a + bx > 0, d + ex > 0, a + bx \leq d + ex, ef \geq bc > 0$. If $\psi_p(a + bx) > 0$ or $\psi_p(d + ex) > 0$ then the function f_1 is decreasing for $x \geq 0$ and for $x \in [0, 1]$ the following double inequality holds:

$$\frac{\Gamma_p(a + b)^c}{\Gamma_p(d + e)^f} \leq \frac{\Gamma_p(a + bx)^c}{\Gamma_p(d + ex)^f} \leq \frac{\Gamma_p(a)^c}{\Gamma_p(d)^f}. \tag{25}$$

In a similar way, using Lemma 3.8, it is easy to prove the following Theorem.

THEOREM 3.10. Let f_1 be a function defined by

$$f_1(x) = \frac{\Gamma_p(a + bx)^c}{\Gamma_p(d + ex)^f}, \quad x \geq 0, \tag{26}$$

where a, b, c, d, e, f are real numbers such that: $a + bx > 0, d + ex > 0, a + bx \leq d + ex, bc \geq ef > 0$. If $\psi_p(d + ex) < 0$ or $\psi_p(a + bx) < 0$ then the function f_1 is decreasing for $x \geq 0$ and for $x \in [0, 1]$ the inequality (25) holds.

At the end we provide some applications related to the Γ_p function.

REMARK 3.11. Using (2) and (3) and the fact that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ we obtain the following representation for π

$$\sqrt{\pi} = \lim_{p\to\infty} \frac{\sqrt{p}}{\frac{1}{2}\left(1 + \frac{1}{2}\right)\left(1 + \frac{1}{4}\right)\cdots\left(1 + \frac{1}{2p}\right)}.$$

REMARK 3.12. Using (3) in equations (25) and (26) we obtain all the results of [6].

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