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Periodic Solutions Of A Three-Species Food Chain Model*

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Abstract

A fairly realistic three-species food chain model based on the Leslie-Gower scheme is investigated, and some sufficient conditions for the existence of positive periodic solutions of the model are obtained.

1 Introduction

Letellier and Aziz-Alaoui in paper [3] introduced the following fairly realistic three-species food chain model based on the Leslie-Gower scheme:

$$\begin{cases} x'(t) &= a_1x(t) - b_1x^2(t) - \frac{\omega_0 x(t)y(t)}{x(t)+d_0}, \\ y'(t) &= -a_2y(t) + \frac{\omega_1 x(t)y(t)}{x(t)+d_1} - \frac{\omega_2 y(t)z(t)}{y(t)+d_2}, \\ z'(t) &= c_0z^2(t) - \frac{\omega_3 z^2(t)}{y(t)+d_3}, \end{cases} \tag{1}$$

where a_1 is the rate of the self-growth for prey x, a_2 measures the rate at which y will die out when no x remains; $\omega_0, ..., \omega_3$ are the maximum value which per capita rate can attain; d_0 and d_1 signify the extent to which environment provides protection to the prey x; b_1 measures the strength of competition among individuals of the species x; d_2 is the value of y at which per capita removal rate of y becomes $\omega_2/2$; d_3 represents the residual loss in z population due to severe scarcity of its favorite food y; c_0 describes the growth rate of the generalist predator z by sexual reproduction, the number of males and females being assumed to be equal. For further ecological sense of system (1), we refer to [3] and the references cited therein.

The variation of the environment plays an important role in many biological and ecological systems. In particular, the effects of a periodically varying environment are important for evolutionary theory as the selective forces on systems in a fluctuating environment differ from those in a stable environment. Thus, the assumption of periodicity of the parameters is one way to describe the periodicity of the environment.

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In this note, we consider the following three-species food chain model based on the Leslie-Gower scheme:

$$\begin{cases} x'(t) &= a_1(t)x(t) - b_1(t)x^2(t) - \frac{\omega_0(t)x(t)y(t)}{x(t)+d_0}, \\ y'(t) &= -a_2(t)y(t) + \frac{\omega_1(t)x(t)y(t)}{x(t)+d_1} - \frac{\omega_2(t)y(t)z(t)}{y(t)+d_2}, \\ z'(t) &= c_0(t)z^2(t) - \frac{\omega_3(t)z^2(t)}{y(t)+d_3}, \end{cases} \tag{2}$$

where $a_1(t),\ b_1(t),\ a_2(t),\ c_0(t)$ and $\omega_i(t),\ i = 0,\ 1,\ 2,\ 3$ are positive continuous ω-periodic functions.

To our best knowledge, system (2) has not been investigated so far. Therefore, it is meaningful and interesting to study the existence of positive periodic solutions of (2), and this is the aim of this note. We approach the result by using coincidence degree theory. For the works on the existence of periodic solutions of differential equation or difference equations by using coincidence degree theory, we refer to [2, 4, 5] and references cited therein.

LEMMA 1.1 ([1]). Let X and Z be two Banach spaces, L a Fredholm mapping of index zero and $N : \overline{\Omega} \to Z$ an L-compact operator on $\overline{\Omega}$ with Ω open bounded in X. Furthermore, the following statements are true:

(a) For each $\lambda \in (0,\ 1)$, every solution x of $Lx = \lambda Nx$ is such that $x \overline{\in} \partial\Omega \cap Dom\, L$.

(b) $QNx \neq 0$ for each $x \in Ker\, L \cap \partial\Omega$.

(c) The Brouwer degree $deg\, \{JQN, \Omega \cap Ker L, 0\} \neq 0$.

Then the operator equation $Lx = Nx$ has at least one solution lying in $Dom\, L \cap \overline{\Omega}$.

To facilitate the discussion below, we adopt the notations:

$$\overline{g} = \frac{1}{\omega}\int_0^{\omega} g(t)dt, \quad g^l = \min_{t\in[0,\omega]} g(t), \quad g^M = \max_{t\in[0,\omega]} g(t),$$

where g is a continuous ω-periodic function.

2 Existence of Positive Periodic Solutions

By the first equation in (2) we can get

$$x(t) = x(0) \exp\left\{\int_0^t \left[a_1(s) - b_1(s)x(s) - \frac{\omega_0(s)y(s)}{x(s)+d_0}\right] ds\right\}. \tag{3}$$

$y(t),\ z(t)$ can also be obtained by the same way. Given any initial condition $x(0)$, $y(0),\ z(0) \in (0, \infty)$, the exponential form of (3) assures that the forward trajectory $(x(t), y(t), z(t))^T$ of (2) remains positive for $t \in [0, \infty)$. So for biological reasons, we only consider solutions $(x(t), y(t), z(t))^T$ with positive initial conditions.

THEOREM 2.1. Suppose $\overline{\omega_1} > \overline{a_2}$, $\overline{\omega_2} > \overline{\omega_1}$ and $d_2 > d_3$. Then system (2) has at least one ω-periodic positive solution.

PROOF. Let $x(t) = \exp\{J_1(t)\},\ y(t) = \exp\{J_2(t)\},\ z(t) = \exp\{J_3(t)\}$. Then system (2) is equivalent to

$$J'(t) = \Theta(t), \tag{4}$$

where $J'(t) = (J_1'(t), J_2'(t), J_3'(t))^T$ and $\Theta(t) = (g_1(t), g_1(t), g_1(t))^T$ with

$$\begin{cases} g_1(t) &= & a_1(t) - b_1(t)\exp\{J_1(t)\} - \frac{\omega_0(t)\exp\{J_2(t)\}}{\exp\{J_1(t)\}+d_0}, \\ g_2(t) &= & -a_2(t) + \frac{\omega_1(t)\exp\{J_1(t)\}}{\exp\{J_1(t)\}+d_1} - \frac{\omega_2(t)\exp\{J_3(t)\}}{\exp\{J_2(t)\}+d_2}, \\ g_3(t) &= & c_0(t)\exp\{J_3(t)\} - \frac{\omega_3(t)\exp\{J_3(t)\}}{\exp\{J_2(t)\}+d_3}. \end{cases}$$

It is easy to see that $(J_1(t), J_2(t), J_3(t))^T$ is an ω-periodic solution of (4) if and only if $(x(t), y(t), z(t))^T$ is an ω-periodic positive solution of system (2). Hence, it suffices to show the existence of ω-periodic solutions of system (4).

For $\lambda \in (0, 1)$, consider the following system:

$$J'(t) = \lambda\Theta(t). \tag{5}$$

Suppose that $(J_1(t), J_2(t), J_3(t))^T$ is an ω-periodic solution of (5) for some $\lambda \in (0, 1)$. Integrating (5) over $[0, \omega]$, we obtain

$$\int_0^\omega b_1(t)\exp\{J_1(t)\}dt + \int_0^\omega \frac{\omega_0(t)\exp\{J_2(t)\}}{\exp\{J_1(t)\}+d_0}dt = \omega\overline{a_1}, \tag{6}$$

$$\int_0^\omega \frac{\omega_1(t)\exp\{J_1(t)\}}{\exp\{J_1(t)\}+d_1}dt = \omega\overline{a_2} + \int_0^\omega \frac{\omega_2(t)\exp\{J_3(t)\}}{\exp\{J_2(t)\}+d_2}dt, \tag{7}$$

$$\int_0^\omega c_0(t)\exp\{J_3(t)\}dt = \int_0^\omega \frac{\omega_3(t)\exp\{J_3(t)\}}{\exp\{J_2(t)\}+d_3}dt. \tag{8}$$

By virtue of (5)–(8),

$$\int_0^\omega |J_1'(t)|dt \le \int_0^\omega |g_1(t)|dt \le 2\omega\overline{a_1} := R_1, \tag{9}$$

$$\int_0^\omega |J_2'(t)|dt \le \int_0^\omega |g_2(t)|dt \le 2\omega\overline{a_2} + 2\int_0^\omega \frac{\omega_2(t)\exp\{J_3(t)\}}{\exp\{J_2(t)\}+d_2}dt, \tag{10}$$

$$\int_0^\omega |J_3'(t)|dt \le \int_0^\omega |g_3(t)|dt \le 2\int_0^\omega c_0(t)\exp\{J_3(t)\}dt. \tag{11}$$

Choose $\xi_i,\ \eta_i \in [0, \omega]$, $i = 1,\ 2,\ 3$, such that

$$J_i(\xi_i) = \min_{t\in[0,\omega]} J_i(t),\ J_i(\eta_i) = \max_{t\in[0,\omega]} J_i(t).$$

From (6) we have $\omega\overline{a_1} \ge \omega\overline{b_1}\exp\{J_1(\xi_1)\}$. That is

$$J_1(\xi_1) \le \ln(\overline{a_1}/\overline{b_1}) := \rho_1. \tag{12}$$

From (9) and (12), we obtain for any $t \in [0, \infty)$,

$$J_1(t) \le J_1(\xi_1) + \int_0^\omega |J_1'(t)|dt \le \rho_1 + R_1 := \delta_1. \tag{13}$$

From (7), we have

$$\frac{\omega\overline{\omega_1}}{1 + d_1/\exp\{J_1(\eta_1)\}} \ge \int_0^\omega \frac{\omega_1(t)}{1 + d_1/\exp\{J_1(t)\}}dt \ge \omega\overline{a_2}.$$

By the hypothesis $\overline{\omega_1} > \overline{a_2}$ of this theorem, we have

$$J_1(\eta_1) \ge \ln(\overline{a_2}d_1/(\overline{\omega_1} - \overline{a_2})) := \rho_2. \tag{14}$$

Combining (9) and (14), we obtain for any $t \in [0, \infty)$,

$$J_1(t) \ge J_1(\eta_1) - \int_0^\omega |J_1'(t)|dt \ge \rho_2 - R_1 := \delta_2, \tag{15}$$

which, together with (13), yields

$$|J_1(t)| \le \max\{|\delta_1|, |\delta_2|\} := B_1. \tag{16}$$

From (7) and (16), we have

$$\int_0^\omega \frac{\omega_2(t)\exp\{J_3(t)\}}{\exp\{J_2(t)\} + d_2}dt + \omega\overline{a_2} = \int_0^\omega \frac{\omega_1(t)}{1 + d_1/\exp\{J_1(t)\}}dt \le \frac{\omega\overline{\omega_1}}{1 + d_1/\exp\{B_1\}}, \tag{17}$$

which, together with (10), leads us to

$$\int_0^\omega |J_2'(t)|dt \le \frac{2\omega\overline{\omega_1}\exp\{B_1\}}{d_1 + \exp\{B_1\}} := R_2. \tag{18}$$

From (6) and (10), we have

$$\omega\overline{a_1} \ge \int_0^\omega \frac{\omega_0(t)\exp\{J_2(t)\}}{\exp\{J_1(t)\} + d_0}dt \ge \frac{\omega\overline{\omega_0}\exp\{J_2(\xi_2)\}}{d_0 + \exp\{B_1\}},$$

which implies

$$J_2(\xi_2) \le \ln[\overline{a_1}(d_0 + \exp\{B_1\})/\overline{\omega_0}] := \rho_3. \tag{19}$$

By virtue of (18) and (19), we obtain, for any $t \in [0, \infty)$,

$$J_2(t) \le J_2(\xi_2) + \int_0^\omega |J_2'(t)|dt \le \rho_3 + R_2 := \delta_3. \tag{20}$$

From (7), we have

$$\int_0^\omega \frac{\omega_1(t)}{1 + d_1/\exp\{\delta_2\}} \ge \int_0^\omega \frac{\omega_2(t)}{\exp\{J_2(\eta_2)\} + d_2}.$$

By the hypothesis $\overline{\omega_2} > \overline{\omega_1}$ of this theorem, we have

$$J_2(\eta_2) \geq \ln[(\overline{\omega_2}(\exp\{\delta_2\} + d_1) - d_2\overline{\omega_1}\exp\{\delta_2\})/\overline{\omega_1}\exp\{\delta_2\}] := \rho_4. \tag{21}$$

Combining (18) with (21), we obtain for any $t \in [0, \infty)$,

$$J_2(t) \geq J_2(\eta_2) - \int_0^\omega |J_2'(t)|dt \geq \rho_4 - R_2 := \delta_4. \tag{22}$$

It follows from (20) and (22) that

$$|J_2(t)| \leq \max\{|\delta_3|, |\delta_4|\} := B_2. \tag{23}$$

From (7), we have

$$\frac{\omega\overline{\omega_1}}{1 + d_1/\exp\{B_1\}} \geq \int_0^\omega \frac{\omega_1(t)\exp\{J_1(t)\}}{\exp\{J_1(t)\} + d_1}dt - \omega\overline{a_2} \geq \int_0^\omega \frac{\omega_2^l \exp\{J_3(t)\}}{\exp\{B_2\} + d_2}dt.$$

That is

$$\int_0^\omega \exp\{J_3(t)\}dt \leq \frac{\omega\omega_1\exp\{B_1\}(\exp\{B_2\} + d_2)}{\omega_2^l(\exp\{B_1\} + d_1)},$$

which, together with (11), yields

$$\int_0^\omega |J_3'(t)|dt \leq \frac{2c_0\omega\overline{\omega_1}\exp\{B_1\}(\exp\{B_2\} + d_2)}{\omega_2^l(\exp\{B_1\} + d_1)} := R_3. \tag{24}$$

On the other hand, (7) and (8) produce

$$\frac{\omega\overline{\omega_1}\exp\{B_1\}}{\exp\{-B_1\} + d_1} \geq \int_0^\omega \frac{\omega_1(t)\exp\{J_1(t)\}}{\exp\{J_1(t)\} + d_1}dt - \omega\overline{a_2} \geq \frac{\omega\omega_2^l\exp\{J_3(\xi_3)\}}{\exp\{B_2\} + d_2}.$$

and

$$\int_0^\omega c_0(t)\exp\{J_3(\eta_3)\}dt \geq \int_0^\omega \frac{\omega_3(t)}{\exp\{B_1\} + d_3}.$$

According to the above two inequalities, we get

$$J_3(\xi_3) \leq \ln[\frac{\overline{\omega_1}\exp\{2B_1\}(\exp\{B_2\} + d_2)}{\omega_2^l(1 + d_1\exp\{B_1\})}] := \rho_5 \tag{25}$$

and

$$J_3(\eta_3) \geq \ln[\frac{\overline{\omega_3}}{\overline{c_0}(\exp\{B_1\} + d_3)}] := \rho_6. \tag{26}$$

It follows from (24)–(26) that

$$J_3(t) \leq J_3(\zeta_3) + \int_0^\omega |J_3'(t)|dt \leq \rho_5 + R_3 := \delta_5. \tag{27}$$

$$J_3(t) \geq J_3(\eta_3) - \int_0^\omega |J_3'(t)|dt \geq \rho_6 - R_3 := \delta_6. \tag{28}$$

Combining (27) with (28), we have

$$|J_3(t)| \leq \max\{|\delta_5|, |\delta_6|\} := B_3.$$

Clearly, B_1, B_2, B_3 are independent of the choice of $\lambda \in (0,1)$. Take $B = B_1 + B_2 + B_3 + B_0$, where B_0 is taken sufficiently large such that each solution $(J_1^*, J_2^*, J_3^*)^T$ of the following system:

$$\begin{aligned} \overline{a_1} - \overline{b_1}\exp\{J_1\} - \frac{\overline{\omega_0}\exp\{J_2\}}{\exp\{J_1\} + d_0} \quad &: \; = h_1 = 0, \\ -\overline{a_2} + \frac{\overline{\omega_1}\exp\{J_1\}}{\exp\{J_1\} + d_1} - \frac{\overline{\omega_2}\exp\{J_3\}}{\exp\{J_2\} + d_2} \quad &: \; = h_2 = 0, \\ \overline{c_0}\exp\{J_3\} - \frac{\overline{\omega_3}\exp\{J_3\}}{\exp\{J_2\} + d_3} \quad &: \; = h_3 = 0. \end{aligned} \tag{29}$$

satisfies $\|(J_1^*, J_2^*, J_3^*)^T\| \leq B$, provided that system (29) has a solution and that

$$\max\{|\rho_1|, |\rho_2|\} + \max\{|\rho_3|, |\rho_4|\} + \max\{|\rho_5|, |\rho_6|\} < B.$$

We now take

$$X = Z = \{J = (J_1, J_2, J_3)^T \in C(R, R^3) \mid J_i(t+\omega) = x(t), i = 1,2,3\},$$
$$\Omega = \{(J_1, J_2, J_3)^T \in X : \|(J_1, J_2, J_3)^T\| < B\},$$
$$\|(x(t), y(t), z(t))^T\| = \max_{t\in[0,\omega]} x(t) + \max_{t\in[0,\omega]} y(t) + \max_{t\in[0,\omega]} z(t).$$

Then X and Z are Banach spaces when they are endowed with the norm $\|\cdot\|$, and Ω is open bounded in X. Set $L : DomL \cap X \to Z$,

$$LJ(t) = J'(t),$$

where $J(t) = (J_1(t), J_2(t), J_3(t))^T$, $Dom\, L = \{J(t) \in C'(R,\ R^3)\}$, and $N : X \to Z$,

$$NJ(t) = \Theta(t).$$

Define two projectors P and Q as

$$PJ(t) = QJ(t) = \frac{1}{\omega}\int_0^{\omega} J(t)dt, J \in X.$$

Clearly, $Ker\, L = \{J | J \in R^3\}$, $Im\, L = \{J \in X | \int_0^{\omega} J_I(t)dt = 0, i = 1,2,3\}$ is closed in X and dim $Ker\, L = codim\, Im\, L = 3$. Therefore, L is a Fredholm mapping of index 0. Furthermore the generalized inverse Kp of L, has the form$K_p : Im\, L \to Dom\, L \cap Ker\, P$,

$$KpJ(t) = \int_0^t J(t)dt - \frac{1}{\omega}\int_0^{\omega}\int_0^{\eta} J(t)dtd\eta.$$

Evidently, QN and $K_P(I-Q)N$ are continuous and $QN(\overline{\Omega})$, $K_P(I-Q)N(\overline{\Omega})$ are relatively compact for open bounded set $\Omega \in X$. Therefore, N is L-compact on $\overline{\Omega}$ for open bounded set $\Omega \in X$.

Corresponding to (4) we have the operator equation:

$$LJ(t) = \lambda NJ(t), \tag{30}$$

which is system (2) when $\lambda = 1$. According to the prior estimate of periodic solution for (30), we have proved that for each $\lambda \in (0,1)$, $LJ(t) \neq \lambda NJ(t), J \in \partial\Omega \cap DomL$. This proves that condition (a) of Lemma 1.1 is satisfied.

When $J \in \partial\Omega \cap Ker\,L = \partial\Omega \cap R^3$, J is a constant vector in R^3 with $\|J\| = B$. If system (29) has a solution, then $QNJ(t) = h(t) \neq 0$, where $h(t) = (h_1(t), h_2(t), h_3(t))^T$. If system (29) does not have a solution, then naturally $QNJ(t) = h(t) \neq 0$. This proves that Condition (b) is satisfied.

To complete the proof, we will prove that condition (c) is satisfied. We define mapping $\Phi : Dom\,L \times [0,1] \to Z$ by

$$\Phi J(t) = \begin{bmatrix} \overline{a_1} - \overline{b_1}\exp\{J_1\} \\ \frac{\overline{\omega_1}\exp\{J_1\}}{\exp\{J_1\}+d_1} - \frac{\overline{\omega_2}\exp\{J_3\}}{\exp\{J_2\}+d_2} \\ \overline{c_0}\exp\{J_3\} - \frac{\overline{\omega_3}\exp\{J_3\}}{\exp\{J_2\}+d_3} \end{bmatrix} + \mu \begin{bmatrix} -\frac{\overline{\omega_0}\exp\{J_2\}}{\exp\{J_1\}+d_0} \\ -\overline{a_2} \\ 0 \end{bmatrix},$$

where $\mu \in [0,1]$ is a parameter. It is not difficult to show that when $J \in \partial\Omega \cap KerL, \Phi J(t) \neq 0$.Thus, by the property of topological degree and by taking J the identity mapping, a direct calculation shows that

$$\deg\{JQN, \Omega \cap KerL, 0\} = \deg\{\Phi(J_1, J_2, J_3, 0),\ \Omega \cap KerL, 0\} \neq 0.$$

By now, we have shown that Ω verifies all requirements of Lemma 1.1. It follows that system (4) has at least one solution in $DomL \cap \overline{\Omega}$. So system (2) has at least one ω-periodic strictly positive solution. The proof is complete.

3 Example

The following three-species food chain model based on the Leslie-Gower scheme

$$\begin{aligned} x'(t) &= (1/3+\sin t)x(t) - (1/8+\sin t)x^2(t) - \frac{x(t)y(t)}{x(t)+1}, \\ y'(t) &= -(1/4+\cos t)y(t) + \frac{(1+\cos t)x(t)y(t)}{x(t)+1/2} - \frac{2y(t)z(t)}{y(t)+1/4}, \\ z'(t) &= 1/6z^2(t) - \frac{(1/5+\sin t)z^2(t)}{y(t)+1/5}, \end{aligned} \tag{31}$$

has at least one 2π–periodic solutions.

Indeed, direct calculations yield

$$\overline{\omega_1} = 1 > 1/4 = \overline{a_2},\ \overline{\omega_2} = 2 > 1 = \overline{\omega_1},\ d_2 = 1/4 > 1/5 = d_3.$$

Hence, all the conditions required in Theorem 2.1 are satisfied. By Theorem 2.1, (31) has at least one 2π–periodic solutions.

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