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Oscillation For Higher Order Sublinear Delay Difference Equation*

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Abstract

In this paper, we shall consider higher order nonlinear delay difference equation of the type

$$\Delta^m x_n + p_n \Delta^{m-1} x_n + q_n x_{n-k}^{\alpha} = 0, \quad n \geq 0$$

where $m > 2$, $\{p_n\}$ and $\{q_n\}$ are sequences of nonnegative real numbers, $0 \leq p_n < 1$ for $n \geq n_0 \geq 0$, k is a positive integer and $\alpha \in (0, 1)$ is a ratio of odd positive integers. We obtain sufficient conditions for the oscillation of all solutions of this equation.

1 Introduction

We consider the following higher order nonlinear delay difference equation:

$$\Delta^m x_n + p_n \Delta^{m-1} x_n + q_n x_{n-k}^{\alpha} = 0, \quad m > 2, \tag{1}$$

where Δ is the usual forward difference operator defined by $\Delta x_n = x_{n+1} - x_n$, k is a positive integer, $\{p_n\}$ and $\{q_n\}$ are sequences of nonnegative real numbers, $0 \leq p_n < 1$ for $n \geq n_0 \geq 0$ and $\alpha \in (0, 1)$ is a ratio of odd positive integers. If $0 < \alpha < 1$, then equation (1) is called sublinear equation, when $\alpha > 1$, it is called superlinear equation.

Recently, there are many studies concerning the behavior of the oscillatory difference equations, see [1]–[11] and the reference cited therein. In particular, in [8], Tang and Liu have investigated the oscillatory behaviour of the first order sublinear and superlinear delay difference equations of the form

$$x_{n+1} - x_n + p_n x_{n-k}^{\alpha} = 0, \tag{2}$$

where $\{p_n\}$ is a sequence of nonnegative numbers, k is a positive integer and $\alpha \in (0, \infty)$ is a quotient of odd positive integers. Tang and Liu have discussed the oscillation for

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equation (2) in the two cases where $\alpha \in (0,1)$ and $\alpha \in (1,\infty)$. For the former case, it is shown that every solution of sublinear equation (2) oscillates if and only if

$$\sum_{n=0}^{\infty} p_n = \infty.$$

For the latter case, it is shown that, if there exists a $\lambda > k^{-1}\ln\alpha$ such that

$$\liminf_{n\to\infty}\left[p_n \exp(-e^{\lambda n})\right] > 0,$$

then every solution of superlinear equation (2) oscillates. When $\alpha = 1$, (2) reduces to linear delay difference equation and the oscillatory behavior of this equation has been investigated in the literature, which we refer to [1, 5, 6, 7].

In [3], Agarwal and Grace investigated that oscillatory properties of equation (1) in the case $\alpha = 1$ and m is even.

By a solution of (1), we mean a real sequence $\{x_n\}$ which is the defined for all $n \geq -k$ and satisfies equation (1) for $n \geq 0$. A solution of (1) is said to be oscillatory of it is neither eventually positive nor eventually negative.

In this paper our aim is to obtain sufficient conditions for all solutions of equation (1) when m is even or odd.

The following lemmas are needed in the proof of our results, which Lemma 1 and Lemma 2 are obtained in [1], Lemma 3 is obtained in [11].

LEMMA 1 [Discrete Kneser's Theorem]. Let z_n be defined for $n \geq n_0$, and $z_n > 0$ with $\Delta^m z_n$ of constant sign for $n \geq n_0$ and not identically zero. Then, there exists an integer j, $0 \leq j \leq m$ with $(m+j)$ odd for $\Delta^m z_n \leq 0$, and $(m+j)$ even for $\Delta^m z_n \geq 0$ such that

(i) $j \leq m-1$ implies $(-1)^{j+i}\Delta^i z_n > 0$, for all $n \geq n_0$, $j \leq i \leq m-1$,

(ii) $j \geq 1$ implies $\Delta^i z_n > 0$, for all large $n \geq n_0$, $1 \leq i \leq j-1$.

LEMMA 2. Let z_n be defined for $n \geq n_0$, and $z_n > 0$ with $\Delta^m z_n \leq 0$ for $n \geq n_0$ and not identically zero. Then, there exists a large integer $n_1 \geq n_0$ such that

$$z_n \geq \frac{1}{(m-1)!}(n-n_1)^{m-1}\Delta^{m-1} z_{2^{m-j-1}n} \quad n \geq n_1,$$

where j is defined in Lemma 1. Further, if z_n is increasing, then

$$z_n \geq \frac{1}{(m-1)!}\left(\frac{n}{2^{m-1}}\right)^{m-1}\Delta^{m-1} z_n, \quad n \geq 2^{m-1}n_1.$$

LEMMA 3. Assume that for large n,

$$(p_n, p_{n+1}, \ldots, p_{n+k-1}) \neq 0.$$

Then

$$x_{n+1} - x_n + p_n x_{n-k}^{\alpha} = 0, \quad n = 0, 1, 2, \ldots$$

has an eventually positive solution if and only if the corresponding inequality

$$x_{n+1} - x_n + p_n x_{n-k}^{\alpha} \leq 0, \quad n = 0, 1, 2, \ldots$$

has an eventually positive solution.

2 Oscillation of Equation (1)

THEOREM 1. Let $\Delta p_n \leq 0$, for $n \geq n_0 \geq 0$, $\alpha \in (0,1)$, $\liminf_{n\to\infty} q_n > 0$, m is an even positive integer, $k > 1$ and

$$C_n = \min\{Q_n, R_n\} > p_{n-k}, \quad \text{for all large } n,$$

where

$$Q_n = \left(\frac{\theta}{(m-2)!}\right)^{\alpha} \left(\sum_{j=n-k}^{n-1} (j-k)^{\alpha(m-2)} q_j\right), \quad \theta = \frac{1}{2^{(m-2)^2}}$$

and

$$R_n = \sum_{j=n-k}^{n-1} q_j \left(\sum_{s=j-k}^{n-1} \frac{1}{(m-3)!} (s+m+k-j-3)^{(m-3)}\right)^{\alpha}.$$

If the difference equation

$$\Delta z_n + c_n z_{n-k}^{\alpha} = 0, \tag{3}$$

where

$$c_n = \min\left\{C_n - p_{n-k},\ \left(\frac{1}{(m-1)!}\frac{1}{2^{(m-1)^2}}\right)^{\alpha} (n-k)^{\alpha(m-1)} q_n\right\} \tag{4}$$

is oscillatory, then (1) is oscillatory.

PROOF. Let $\{x_n\}$ be a nonoscillatory solution of (1), say $x_n > 0$ for $n \geq n_0 \geq 1$. First, we claim that $\{\Delta^{m-1} x_n\}$ is eventually of one sign. To this end, we assume that $\{\Delta^{m-1} x_n\}$ is oscillatory. There exists $N \geq n_0 + k$ such that $\Delta^{m-1} x_N < 0$. Let $n = N$ in (1) and the multiply the resulting equation by $\Delta^{m-1} x_N$, to obtain

$$\Delta^m x_N \Delta^{m-1} x_N = -p_N (\Delta^{m-1} x_N)^2 - q_N x_{N-k}^{\alpha} \Delta^{m-1} x_N \geq -p_N (\Delta^{m-1} x_N)^2$$

or

$$\Delta^{m-1} x_{N+1} \Delta^{m-1} x_N \geq (1 - p_N)(\Delta^{m-1} x_N)^2 > 0,$$

which implies that

$$\Delta^{m-1} x_{N+1} < 0.$$

By induction, we obtain $\Delta^{m-1} x_n < 0$ for $n \geq N$, contradicting the assumption that $\{\Delta^{m-1} x_n\}$ is oscillatory.

Next, suppose there exists $N^* \geq n_0 + k$ such that $\Delta^{m-1} x_{N^*} = 0$. Let $n = N^*$ in (1) leads to

$$\Delta^m x_{N^*} = -q_{N^*} x_{N^*-k}^{\alpha} \leq 0,$$

which implies that

$$\Delta^{m-1} x_{N^*+1} \leq \Delta^{m-1} x_{N^*} = 0.$$

As in the above case, we have seen that this contradicts the assumption that $\{\Delta^{m-1} x_n\}$ is oscillatory.

Now, we consider the following two cases either $\Delta^{m-1} x_n < 0$ or $\Delta^{m-1} x_n > 0$ eventually.

CASE A. Suppose that $\Delta^{m-1}x_n < 0$ for $n \geq n_1 \geq n_0$. By Lemma 1, there exists an $n_2 \geq n_1$ such that for $n \geq n_2$

$$\Delta^{m-2}x_n > 0, \quad \text{and either (i) } \Delta x_n > 0 \text{ or (ii) } \Delta x_n < 0.$$

(i) Suppose that $\Delta x_n > 0$ for $n \geq n_2$. By applying Lemma 2, there exists $n_3 \geq 2^{m-1}n_2$ such that

$$x_n \geq \frac{1}{(m-2)!}\left(\frac{n}{2^{m-2}}\right)^{m-2}\Delta^{m-2}x_n, \quad \text{for } n \geq n_3.$$

There exists $n_4 \geq n_3$ such that

$$x_{n-k}^{\alpha} \geq \left(\frac{1}{(m-2)!}\frac{1}{2^{(m-2)^2}}\right)^{\alpha}(n-k)^{\alpha(m-2)}\left(\Delta^{m-2}x_{n-k}\right)^{\alpha}, \quad \text{for } n \geq n_4. \tag{5}$$

Using (5) in (1) yields

$$\Delta^2 z_n + p_n\Delta z_n + \left(\frac{\theta}{(m-2)!}\right)^{\alpha}(n-k)^{\alpha(m-2)}q_n z_{n-k}^{\alpha} \leq 0, \quad \text{for } n \geq n_4, \tag{6}$$

where $z_n = \Delta^{m-2}x_n$, $n \geq n_4$ and $\theta = 1/2^{(m-2)^2}$. Summing up both sides of (6) from $n-k$ to $n-1$, we have

$$\Delta z_n - \Delta z_{n-k} + \sum_{j=n-k}^{n-1} p_j\Delta z_j + \left(\frac{\theta}{(m-2)!}\right)^{\alpha}\sum_{j=n-k}^{n-1}(j-k)^{\alpha(m-2)}q_j z_{j-k}^{\alpha} \leq 0 \tag{7}$$

or

$$\begin{aligned} 0 \geq \Delta z_n &+ \left[p_n z_n - p_{n-k}z_{n-k} - \sum_{j=n-k+1}^{n-1} z_j\Delta p_{j-1}\right] \\ &+ \left(\frac{\theta}{(m-2)!}\right)^{\alpha}\left(\sum_{j=n-k}^{n-1}(j-k)^{\alpha(m-2)}q_j\right)z_{n-k}^{\alpha}. \end{aligned}$$

Since $\Delta z_n < 0$, $z_n > 0$, $\lim_{n\to\infty} z_n = c \in [0,\infty)$ exists. We claim that $c = 0$. Otherwise, $\lim_{n\to\infty} z_n = c \in (0,\infty)$. So, if we take limit on both sides of (7) we get the contradiction $\infty \leq 0$. Thus, we must have $c = 0$ and $z_n < z_n^{\alpha} < 1$ for sufficiently large n. And we have

$$\begin{aligned} 0 \geq \Delta z_n &+ \left[p_n z_n - p_{n-k}z_{n-k}^{\alpha} - \sum_{j=n-k+1}^{n-1} z_j\Delta p_{j-1}\right] \\ &+ \left(\frac{\theta}{(m-2)!}\right)^{\alpha}\left(\sum_{j=n-k}^{n-1}(j-k)^{\alpha(m-2)}q_j\right)z_{n-k}^{\alpha}. \end{aligned}$$

Since $\Delta p_n \leq 0$, we have

$$\Delta z_n + \left[\left(\frac{\theta}{(m-2)!} \right)^{\alpha} \left(\sum_{j=n-k}^{n-1} (j-k)^{(m-2)\alpha} q_j \right) - p_{n-k} \right] z_{n-k}^{\alpha} \leq 0, \tag{8}$$

for some $n_5 \geq n_4$ and hence by (4), we have

$$\Delta z_n + c_n z_{n-k}^{\alpha} \leq 0, \quad \text{for } n \geq n_5.$$

Therefore, by Lemma 3, (3) has an eventually positive solution, which is a contradiction.

(ii) Suppose that $\Delta x_n < 0$ for $n \geq n_2$. By applying Lemma 3, we must have $j = 0$ and one can easily see that

$$(-1)^i \Delta^i x_n > 0, \quad \text{for } i = 0, 1, \ldots, m-1 \text{ and } n \geq n_3 \geq n_2. \tag{9}$$

By discrete Taylor's formula [1, p. 26], x_g can be expressed as

$$x_g = \sum_{i=0}^{m-3} \frac{(z+i-1-g)^{(i)}}{i!} (-1)^i \Delta^i x_z + \frac{1}{(m-3)!} \sum_{s=g}^{z-1} (s+m-g-3)^{(m-3)} \Delta^{m-2} x_s, \tag{10}$$

for all $g \in N(n_3, z) = \{n_3, n_3+1, \quad , z\}$, where $z \in N(n_3) = \{n_3, n_3+1, \ldots\}$. Using (9) in (10) and letting $z - 1 = n - k$, we have

$$x_g \geq \frac{1}{(m-3)!} \sum_{s=g}^{n-k} (s+m-g-3)^{(m-3)} \Delta^{m-2} x_s, \quad n \geq n_3.$$

Letting $g = j - k$, and using the fact that $\Delta^{m-2} x_n$ is decreasing, we have

$$x_{j-k}^{\alpha} \geq \left(\frac{1}{(m-3)!} \right)^{\alpha} \left(\sum_{s=j-k}^{n-k} (s+m-j+k-3)^{(m-3)} \right)^{\alpha} \left(\Delta^{m-2} x_{n-k} \right)^{\alpha}. \tag{11}$$

Summing both sides of (1) from $n - k$ to $n - 1$, one can easily see that

$$\Delta^{m-1} x_n - p_{n-k} \Delta^{m-2} x_{n-k} + \sum_{j=n-k}^{n-1} q_j x_{j-k}^{\alpha} \leq 0, \tag{12}$$

using (11) in (12), we have

$$\Delta z_n - p_{n-k} z_{n-k}^{\alpha} + \left[\sum_{j=n-k}^{n-1} q_j \left(\frac{1}{(m-3)!} \right)^{\alpha} \left(\sum_{s=j-k}^{n-k} (s+m-j+k-3)^{(m-3)} \right)^{\alpha} \right] z_{n-k}^{\alpha} \leq 0,$$

and

$$\Delta z_n + \left[\sum_{j=n-k}^{n-1} q_j \left(\frac{1}{(m-3)!} \right)^{\alpha} \left(\sum_{s=j-k}^{n-k} (s+m-j+k-3)^{(m-3)} \right)^{\alpha} - p_{n-k} \right] z_{n-k}^{\alpha} \leq 0,$$

where $z_n = \Delta^{m-2}x_n$, $n \geq n_4 \geq n_3$. Next, by (4), we see that

$$\Delta z_n + c_n z_{n-k}^{\alpha} \leq 0, \quad \text{for } n \geq n_4.$$

Therefore, by Lemma 3, (3) has an eventually positive solution, which is a contradiction.

CASE B. Suppose that $\Delta^{m-1}x_n > 0$ for $n \geq n_1$. From (1) it follows that

$$\Delta^m x_n + q_n x_{n-k}^{\alpha} \leq 0, \quad \text{for } n \geq n_1. \tag{13}$$

By applying Lemma 2, there exists $n_2 \geq 2^{m-1}n_1$ such that

$$x_n \geq \frac{1}{(m-1)!}\left(\frac{n}{2^{m-1}}\right)^{m-1}\Delta^{m-1}x_n, \quad \text{for } n \geq n_2.$$

There exists $n_3 \geq n_2$ such that

$$x_{n-k} \geq \frac{1}{(m-1)!}\left(\frac{1}{2^{m-1}}\right)^{m-1}(n-k)^{m-1}\Delta^{m-1}x_{n-k} \quad \text{for } n \geq n_3.$$

There exists $n_4 \geq n_3$ such that

$$x_{n-k}^{\alpha} \geq \left(\frac{1}{(m-1)!}\frac{1}{2^{(m-1)^2}}\right)^{\alpha}(n-k)^{\alpha(m-1)}\left(\Delta^{m-1}x_{n-k}\right)^{\alpha}, \quad \text{for } n \geq n_4. \tag{14}$$

Using (14) in (13), we obtain

$$\Delta z_n + \left(\frac{1}{(m-1)!}\frac{1}{2^{(m-1)^2}}\right)^{\alpha}(n-k)^{(m-1)\alpha}q_n z_{n-k}^{\alpha} \leq 0, \quad \text{for } n \geq n_4,$$

where $z_n = \Delta^{m-1}x_n$, $n \geq n_4$. By (4), we see that

$$\Delta z_n + c_n z_{n-k}^{\alpha} \leq 0, \quad \text{for } n \geq n_4.$$

The rest of the proof is similar to that of the above case.

THEOREM 2. Let $\Delta p_n \leq 0$, for $n \geq n_0 \geq 0$, $\alpha \in (0,1)$, $\liminf_{n\to\infty} q_n > 0$, m is an odd positive integer, $k > 1$ and $Q_n > p_{n-k}$, for all large n, where

$$Q_n = \left(\frac{\theta}{(m-2)!}\right)^{\alpha}\left(\sum_{j=n-k}^{n-1}(j-k)^{\alpha(m-2)}q_j\right), \ \theta = \frac{1}{2^{(m-2)^2}}.$$

If the difference equation

$$\Delta z_n + c_n z_{n-k}^{\alpha} = 0, \tag{15}$$

where

$$c_n = \min\left\{Q_n - p_{n-k}, \ \left(\frac{1}{(m-1)!}\frac{1}{2^{(m-1)^2}}\right)^{\alpha}(n-k)^{\alpha(m-1)}q_n, \ \left(\frac{(m-2)^{(m-2)}}{(m-2)!}\right)^{\alpha}q_n\right\} \tag{16}$$

is oscillatory, then (1) is oscillatory.

PROOF. Let $\{x_n\}$ be a nonoscillatory solution of (1), say $x_n > 0$ for $n \geq n_0 \geq 1$. As in Theorem 1, we see that $\left\{\Delta^{m-1} x_n\right\}$ is eventually of one sign. To this end, we consider the two as in Theorem 1.

CASE A. Suppose that $\Delta^{m-1} x_n < 0$ for $n \geq n_1 \geq n_0$. By Lemma 1, there exists an $n_2 \geq n_1$ such that for $n \geq n_2$

$$\Delta^{m-2} x_n > 0, \quad \text{and} \quad \Delta x_n > 0.$$

Since $\Delta x_n > 0$ for $n \geq n_2$. By applying Lemma 2, there exists $n_3 \geq 2^{m-1} n_2$ such that

$$x_n \geq \frac{1}{(m-2)!} \left(\frac{n}{2^{m-2}}\right)^{m-2} \Delta^{m-2} x_n, \quad \text{for } n \geq n_3.$$

There exists $n_4 \geq n_3$ such that

$$x_{n-k}^{\alpha} \geq \left(\frac{1}{(m-2)!} \frac{1}{2^{(m-2)^2}}\right)^{\alpha} (n-k)^{\alpha(m-2)} \left(\Delta^{m-2} x_{n-k}\right)^{\alpha}, \quad \text{for } n \geq n_4. \tag{17}$$

Using (17) in (1) yields

$$\Delta^2 z_n + p_n \Delta z_n + \left(\frac{1}{(m-2)!} \frac{1}{2^{(m-2)^2}}\right)^{\alpha} (n-k)^{\alpha(m-2)} q_n z_{n-k}^{\alpha} \leq 0, \quad \text{for } n \geq n_4, \tag{18}$$

where $z_n = \Delta^{m-2} x_n$, $n \geq n_4$ and $\theta = 1/2^{(m-2)^2}$. Summing up both sides of (18) from $n-k$ to $n-1$, we have

$$\Delta z_n - \Delta z_{n-k} + \sum_{j=n-k}^{n-1} p_j \Delta z_j + \left(\frac{\theta}{(m-2)!}\right)^{\alpha} \sum_{j=n-k}^{n-1} (j-k)^{\alpha(m-2)} q_j z_{j-k}^{\alpha} \leq 0$$

and using the fact that $\Delta^{m-2} x_n$ is decreasing, we have

$$\begin{aligned} 0 \geq \Delta z_n &+ \left[p_n z_n - p_{n-k} z_{n-k}^{\alpha} - \sum_{j=n-k+1}^{n-1} z_j \Delta p_{j-1}\right] \\ &+ \left(\frac{\theta}{(m-2)!}\right)^{\alpha} \left(\sum_{j=n-k}^{n-1} (j-k)^{\alpha(m-2)} q_j\right) z_{n-k}^{\alpha}. \end{aligned}$$

Since $\Delta p_n \leq 0$, we have

$$\Delta z_n + \left[\left(\frac{\theta}{(m-2)!}\right)^{\alpha} \left(\sum_{j=n-k}^{n-1} (j-k)^{\alpha(m-2)} q_j\right) - p_{n-k}\right] z_{n-k}^{\alpha} \leq 0, \tag{19}$$

for some $n_5 \geq n_4$ and hence by (16), we have

$$\Delta z_n + c_n z_{n-k}^{\alpha} \leq 0, \quad \text{for } n \geq n_5.$$

Therefore, by Lemma 3, (15) has an eventually positive solution, which is a contradiction.

CASE B. Suppose that $\Delta^{m-1}x_n > 0$ for $n \geq n_1$. From (1) it follows that

$$\Delta^m x_n + q_n x_{n-k}^{\alpha} \leq 0, \quad \text{for } n \geq n_1. \tag{20}$$

By Lemma 1, there exists an $n_2 \geq n_1$ such that $\Delta^{m-2}x_n < 0$ for $n \geq n_2$. Therefore either $\Delta x_n > 0$ or $\Delta x_n < 0$ for $n \geq n_2$. We consider these cases separately as follows:

(i) Suppose that $\Delta x_n > 0$ for $n \geq n_2$. By applying Lemma 2, there exists $n_3 \geq 2^{m-1}n_2$ such that

$$x_n \geq \frac{1}{(m-1)!}\left(\frac{n}{2^{m-1}}\right)^{m-1}\Delta^{m-1}x_n, \quad \text{for } n \geq n_3.$$

There exists $n_4 \geq n_3$ such that

$$x_{n-k}^{\alpha} \geq \left(\frac{1}{(m-1)!}\frac{1}{2^{(m-1)^2}}\right)^{\alpha}(n-k)^{\alpha(m-1)}\left(\Delta^{m-1}x_{n-k}\right)^{\alpha}, \quad \text{for } n \geq n_4. \tag{21}$$

Using (21) in (1) yields

$$\Delta z_n + \left(\frac{1}{(m-1)!}\frac{1}{2^{(m-1)^2}}\right)^{\alpha}(n-k)^{\alpha(m-1)}q_n z_{n-k}^{\alpha} \leq 0, \quad \text{for } n \geq n_4, \tag{22}$$

where $z_n = \Delta^{m-1}x_n$, $n \geq n_4$. By (16), we have

$$\Delta z_n + c_n z_{n-k}^{\alpha} \leq 0, \quad \text{for } n \geq n_5.$$

Therefore, by Lemma 3, (15) has an eventually positive solution, which is a contradiction.

(ii) Suppose that $\Delta x_n < 0$ for $n \geq n_2$. By applying Lemma 1, we must have $j = 0$ and one can easily see that

$$(-1)^i \Delta^i x_n > 0, \quad \text{for } i = 0, 1, \ldots, m-1 \text{ and } n \geq n_3 \geq n_2. \tag{23}$$

By discrete Taylor's formula [1, p. 26], x_g can be expressed as

$$x_g = \sum_{i=0}^{m-2}\frac{(z+i-1-g)^{(i)}}{i!}(-1)^i\Delta^i x_z + \frac{1}{(m-2)!}\sum_{s=g}^{z-1}(s+m-g-2)^{(m-2)}\Delta^{m-1}x_s, \tag{24}$$

for all $g \in N(n_3, z) = \{n_3, n_3+1, \ldots, z\}$, where $z \in N(n_3) = \{n_3, n_3+1, \ldots\}$. Using (23) in (24) and letting $z - 1 = n - k$, we have

$$x_g \geq \frac{1}{(m-2)!}\sum_{s=g}^{n-k}(s+m-g-2)^{(m-2)}\Delta^{m-1}x_s, \quad n \geq n_3.$$

Letting $g = j - k$, and using the fact that $\Delta^{m-1}x_n$ is decreasing, we have

$$x_{j-k}^{\alpha} \geq \left(\frac{1}{(m-2)!}\right)^{\alpha}\left(\sum_{s=j-k}^{n-k}(s+m+k-j-2)^{(m-2)}\right)^{\alpha}\left(\Delta^{m-1}x_{n-k}\right)^{\alpha}. \tag{25}$$

Using (25) in (20), we obtain

$$\Delta z_n + q_n \left(\frac{1}{(m-2)!} \right)^{\alpha} \left(\sum_{s=n-k}^{n-k} (s+m+k-n-2)^{(m-2)} \right)^{\alpha} z_{n-k}^{\alpha} \le 0, \quad \text{for } n \ge n_4$$

and

$$\Delta z_n + q_n \left(\frac{(m-2)^{(m-2)}}{(m-2)!} \right)^{\alpha} z_{n-k}^{\alpha} \le 0, \quad \text{for } n \ge n_4 \tag{26}$$

where $z_n = \Delta^{m-1} x_n$, $n \ge n_4 \ge n_3$. Next, by (16), we see that

$$\Delta z_n + c_n z_{n-k}^{\alpha} \le 0, \text{for } n \ge n_4.$$

Therefore, by Lemma 3, (15) has an eventually positive solution, which is a contradiction.

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