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# Synchronization of Strongly Coupled Neural Networks*

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**Abstract**

Two identical discrete time cellular neural networks are coupled and sharp conditions are found so that some or all neural units will eventually synchronize. In deriving these criteria, we make use of symmetry (invariance) principles, Banach contraction technique and spectral properties of several band matrices with block components.

## 1 Introduction

The fact that various parts of a biological system operate in harmony is taken to be an important indication of normal functioning of the system. One concept that is essential in describing such harmonious operations is 'synchronization'. In order to design computing machines that simulate harmonious operations, it is therefore necessary to find mathematical models that are capable of generating synchronized outputs. Synchronization can occur in a number of continuous dynamical systems. Synchronization can also occur in artificial neural network models where time and space are both assumed to be discrete. Such an example is studied in [1, 2]. For the sake of convenience, we briefly recall this network model here. Let $x_1, ..., x_n$ be $n$ ($n \geq 2$) neuron units placed at the vertices of a regular polygon. Let $x_i^{(t)}$ be the state values of the neuron unit $x_i$ in the time period $t$. During the time period $t$, if the state value $x_1^{(t)}$ is larger than $x_2^{(t)}$, information will "flow" from the unit $x_1$ to the unit $x_2$. The subsequent change of state value of $x_2$ is $x_2^{(t+1)} - x_2^{(t)}$ and under a first order approximation assumption, it is reasonable that it is proportional to the difference $x_1^{(t)} - x_2^{(t)}$, say, $\gamma\left(x_1^{(t)} - x_2^{(t)}\right)$, where $\gamma$ is a positive rate constant. Similarly, information will flow from the point $x_3$ to the point $x_2$ if $x_3^{(t)} > x_2^{(t)}$. Thus, it is reasonable that the total effect is

$$x_2^{(t+1)} - x_2^{(t)} = \gamma\left(x_1^{(t)} - x_2^{(t)}\right) + \gamma\left(x_3^{(t)} - x_2^{(t)}\right) = \gamma\left(x_1^{(t)} - 2x_2^{(t)} + x_3^{(t)}\right).$$

By similar considerations, we may then obtain the following dynamic system of equations

$$\mathbf{x}^{(t+1)} = (1-2\gamma)\mathbf{x}^{(t)} + \gamma A_n \mathbf{x}^{(t)},\ t = 0, 1, 2, ...,$$

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where $\mathbf{x}^{(t)} = \left(x_1^{(t)}, ..., x_n^{(t)}\right)^\dagger$ and $A_n$ is the circulant matrix defined by

$$A_n = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}_{n\times n} \tag{1}$$

when $n \geq 2$.

Now suppose there is another identical neural network with neuron units denoted by $y_1, ..., y_n$. Suppose further that the neuron pairs $x_i$ and $y_i$, $i = 1, 2, ..., n$, are "strongly connected" so that the change, say, $x_2^{(t+1)} - x_2^{(t)}$ is also proportional to $2\gamma\left(y_2^{(t)} - x_2^{(t)}\right)$ (note the factor 2 here), then the subsequent equation for the neuron $x_2$ is

$$x_2^{(t+1)} - x_2^{(t)} = \gamma\left(x_1^{(t)} - x_2^{(t)}\right) + \gamma\left(x_3^{(t)} - x_2^{(t)}\right) + 2\gamma\left(y_2^{(t)} - x_2^{(t)}\right).$$

The complete set of equations for the neurons $x_1, ..., x_n$ and $y_1, ..., y_n$ is of the form

$$\begin{aligned} \mathbf{x}^{(t+1)} &= (1-4\gamma_t)\mathbf{x}^{(t)} + \gamma_t A_n \mathbf{x}^{(t)} + 2\gamma_t \mathbf{y}^{(t)}, \\ \mathbf{y}^{(t+1)} &= (1-4\gamma_t)\mathbf{y}^{(t)} + \gamma_t A_n \mathbf{y}^{(t)} + 2\gamma_t \mathbf{x}^{(t)}, \end{aligned}$$

which describes the evolution process of the state values as $t$ tends to infinity. In Fig. 1, we illustrate a coupled network with 6 neuron units $x_1, ..., x_6$ and 6 neuron units $y_1, ..., y_6$ in two manners. The one on the right is a planar graph, which is more convenient for illustrating later results.

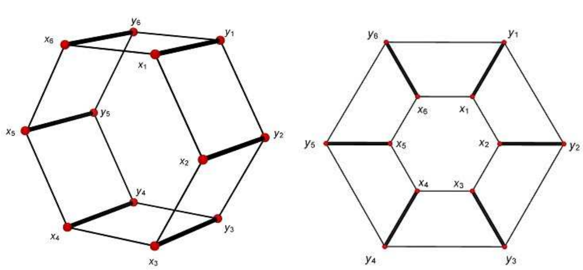


Figure 1

There are many questions of interests which we can raise regarding the above evolutionary system. An interesting one is whether some or all neuron units are ultimately in synchronization, so that, as time evolves, they differ from each other only by infinitesimal values.

In this paper, we will consider such a problem for a slightly more general nonlinear system of equations of the form

$$\begin{aligned} \mathbf{x}^{(t+1)} &= (1-4\gamma_t)\mathbf{F}\left(\mathbf{x}^{(t)}\right) + \gamma_t A_n \mathbf{F}\left(\mathbf{x}^{(t)}\right) + 2\gamma_t \mathbf{F}\left(\mathbf{y}^{(t)}\right), \\ \mathbf{y}^{(t+1)} &= (1-4\gamma_t)\mathbf{F}\left(\mathbf{y}^{(t)}\right) + \gamma_t A_n \mathbf{F}\left(\mathbf{y}^{(t)}\right) + 2\gamma_t \mathbf{F}\left(\mathbf{x}^{(t)}\right), \end{aligned} \tag{2}$$

for $t = 0, 1, 2, \ldots$, where $\{\gamma_t\}$ is a real sequence, $\mathbf{F}(\mathbf{u}) = (f(u_1), f(u_2), ..., f(u_n))^\dagger$ for $\mathbf{u} = (u_1, ..., u_n)^\dagger$ and $f : \mathbf{R} \to \mathbf{R}$ is a Lipschitz function[1] which satisfies

$$|f(x) - f(y)| \le \Gamma |x - y|, \; x, \; y \in \mathbf{R},$$

for some fixed positive constant $\Gamma$.

Note that if we denote the vector $\begin{pmatrix} \mathbf{x}^{(t)} \\ \mathbf{y}^{(t)} \end{pmatrix}$ by $\mathbf{z}^{(t)}$, where $\mathbf{x}^{(t)} = \left(x_1^{(t)}, ..., x_n^{(t)}\right)^\dagger$ and $\mathbf{y}^{(t)} = \left(y_1^{(t)}, ..., y_n^{(t)}\right)^\dagger$, then our system can be written in the form $\mathbf{z}^{(t+1)} = \mathbf{F}\left(t, \mathbf{z}^{(t)}\right)$. Thus, given an initial distribution $\mathbf{z}^{(0)} = \mathbf{z}$, it is easily seen that we can calculate $\mathbf{z}^{(1)}, \mathbf{z}^{(2)}, \ldots$ successively and in a unique manner from (2). Such a sequence $\left\{\mathbf{z}^{(0)}, \mathbf{z}^{(1)}, \mathbf{z}^{(2)}, \ldots\right\}$ is said to be a solution of (2).

An important property of our system (2) is its 'invariance' under 'rotations'. To be more precise, let us call the vector $(u_n, u_1, u_2, ..., u_{n-1})^\dagger$ the (forward) rotation of the vector $\mathbf{u} = (u_1, u_2, ..., u_n)\dagger$ and denote it by $\theta(\mathbf{u})$. If $\begin{pmatrix} \mathbf{x}^{(t)} \\ \mathbf{y}^{(t)} \end{pmatrix}$ is a solution of (2), then it is easily checked that $\begin{pmatrix} \theta(\mathbf{x}^{(t)}) \\ \theta(\mathbf{y}^{(t)}) \end{pmatrix}$ is also a solution since we are simply rearranging the equations in (2).

Let $\left\{\mathbf{z}^{(t)}\right\}_{t=0}^{\infty}$ be a solution sequence of (2). Given any two distinct neuron units $u, v$ in $\{x_1, \ldots, x_n, y_1, \ldots, y_n\}$, let $\left\{u^{(t)}\right\}_{t=0}^{\infty}$ and $\left\{v^{(t)}\right\}_{t=0}^{\infty}$ be the corresponding component sequences in the solution sequence $\left\{\mathbf{z}^{(t)}\right\}_{t=0}^{\infty}$. We say that the solution $\left\{\mathbf{z}^{(t)}\right\}_{t=0}^{\infty}$ is $\{u, v\}$ synchronized if

$$\lim_{t\to\infty} \left|u^{(t)} - v^{(t)}\right| = 0.$$

More generally, let $\Omega$ be a subset of $\{x_1, \ldots, x_n, y_1, \ldots, y_n\}$, if $\left\{\mathbf{z}^{(t)}\right\}_{t=0}^{\infty}$ is $\{u, v\}$ synchronized for each pair of distinct neuron units in $\Omega$, then we say that $\left\{\mathbf{z}^{(t)}\right\}_{t=0}^{\infty}$ is $\Omega$ (partially) synchronized. In case $\Omega = \{x_1, \ldots, x_n, y_1, \ldots, y_n\}$, then it is natural to say that $\left\{\mathbf{z}^{(t)}\right\}_{t=0}^{\infty}$ is (fully) synchronized. In Fig. 12, a fully synchronized neural network ($n = 5$) can be found in which any two (distinct) units are connected with a dash line to show synchronization.

Two neuron units from $\{x_1, .., x_n\}$ or from $\{y_1, ..., y_n\}$ are said to be of the *same type*, while a neuron unit $u$ from $\{x_1, ..., x_n\}$ and $v$ from $\{y_1, ..., y_n\}$ are said to be of *different type*. In this paper, we are interested in finding various synchronization phenomena that involves neuron units of the same or different types.

[1]For example, the tent map $g$ defined by $g(x) = 2x$ for $0 \le x \le 1/2$, $2(1-x)$ for $1/2 \le x \le 1$, and 0 elsewhere is a Lipschitz function with Lipschitz constant 2.

To motivate the main results that follow, let us consider the case where $n = 2$. For the sake of convenience, we use $\rho(W)$ to denote the *spectral radius* of a square matrix $W$.

When $n = 2$, our system is

$$\begin{cases} x_1^{(t+1)} = (1-4\gamma_t) f\left(x_1^{(t)}\right) + 2\gamma_t f\left(x_2^{(t)}\right) + 2\gamma_t f\left(y_1^{(t)}\right), \\ x_2^{(t+1)} = 2\gamma_t f\left(x_1^{(t)}\right) + (1-4\gamma_t) f\left(x_2^{(t)}\right) + 2\gamma_t f\left(y_2^{(t)}\right), \\ y_1^{(t+1)} = 2\gamma_t f\left(x_1^{(t)}\right) + (1-4\gamma_t) f\left(y_1^{(t)}\right) + 2\gamma_t f\left(y_2^{(t)}\right), \\ y_2^{(t+1)} = 2\gamma_t f\left(x_2^{(t)}\right) + 2\gamma_t f\left(y_1^{(t)}\right) + (1-4\gamma_t) f\left(y_2^{(t)}\right). \end{cases} \tag{3}$$

We assert that if

$$\limsup_{t\to\infty} |1-4\gamma_t| < \frac{1}{\Gamma}, \tag{4}$$

then every solution of (3) is $\{x_2, y_1\}$ synchronized. Indeed, note that

$$x_2^{(t+1)} - y_1^{(t+1)} = (1-4\gamma_t)\left(f\left(x_2^{(t)}\right) - f\left(y_1^{(t)}\right)\right). \tag{5}$$

If (4) holds, then there exist constant $d \in (0,1)$ and positive integer $T$ such that (cf. Banach contraction principle)

$$\Gamma\,|1-4\gamma_t| < d < 1,\ \ t \geq T,$$

so that by (5),

$$\left|x_2^{(t+1)} - y_1^{(t+1)}\right| = |1-4\gamma_t| \left|f\left(x_2^{(t)}\right) - f\left(y_1^{(t)}\right)\right| \leq \Gamma\,|1-4\gamma_t| \left|x_2^{(t)} - y_1^{(t)}\right|,$$

and hence

$$\left|x_2^{(T+n)} - y_1^{(T+n)}\right| < d\left|x_2^{(T+n-1)} - y_1^{(T+n-1)}\right| < \cdots < d^n\left|x_2^{(T)} - y_1^{(T)}\right|$$

for $n = 1, 2, \ldots$. If we now let $n \to \infty$, we see that

$$\lim_{t\to\infty}\left|x_2^{(t)} - y_1^{(t)}\right| = 0.$$

This shows that $\{x_2, y_1\}$ are synchronized. By considering the absolute difference $\left|x_1^{(t)} - y_2^{(t)}\right|$, we may proceed in a similar manner to show that $\lim_{t\to\infty}\left|x_1^{(t)} - y_2^{(t)}\right| = 0$. That is, every solution of (3) is $\{x_1, y_2\}$ synchronized.

Similarly, note that

$$\begin{pmatrix} x_1^{(t+1)} - x_2^{(t+1)} \\ y_1^{(t+1)} - y_2^{(t+1)} \end{pmatrix} = \begin{pmatrix} 1-6\gamma_t & 2\gamma_t \\ 2\gamma_t & 1-6\gamma_t \end{pmatrix} \begin{pmatrix} f\left(x_1^{(t)}\right) - f\left(x_2^{(t)}\right) \\ f\left(y_1^{(t)}\right) - f\left(y_2^{(t)}\right) \end{pmatrix}. \tag{6}$$

If

$$\limsup_{t\to\infty} \rho \begin{pmatrix} 1-6\gamma_t & 2\gamma_t \\ 2\gamma_t & 1-6\gamma_t \end{pmatrix} < \frac{1}{\Gamma}, \tag{7}$$

there exist constant $d \in (0,1)$ and positive integer $T$ such that

$$\begin{aligned}
\left\| \begin{pmatrix} x_1^{(t+1)} - x_2^{(t+1)} \\ y_1^{(t+1)} - y_2^{(t+1)} \end{pmatrix} \right\|_2 &\le \left\| \begin{pmatrix} 1-6\gamma_t & 2\gamma_t \\ 2\gamma_t & 1-6\gamma_t \end{pmatrix} \right\|_2 \left\| \begin{pmatrix} f\left(x_1^{(t)}\right) - f\left(x_2^{(t)}\right) \\ f\left(y_1^{(t)}\right) - f\left(y_2^{(t)}\right) \end{pmatrix} \right\|_2 \\
&\le \Gamma \rho \begin{pmatrix} 1-6\gamma_t & 2\gamma_t \\ 2\gamma_t & 1-6\gamma_t \end{pmatrix} \left\| \begin{pmatrix} x_1^{(t)} - x_2^{(t)} \\ y_1^{(t)} - y_2^{(t)} \end{pmatrix} \right\|_2 \\
&\le d \left\| \begin{pmatrix} x_1^{(t)} - x_2^{(t)} \\ y_1^{(t)} - y_2^{(t)} \end{pmatrix} \right\|_2, \ t \ge T.
\end{aligned}$$

so that

$$\left\| \begin{pmatrix} x_1^{(T+n)} - x_2^{(T+n)} \\ y_1^{(T+n)} - y_2^{(T+n)} \end{pmatrix} \right\|_2 < \cdots < d^n \left\| \begin{pmatrix} x_1^{(T)} - x_2^{(T)} \\ y_1^{(T)} - y_2^{(T)} \end{pmatrix} \right\|_2$$

for $n = 1, 2, \ldots$. As $n \to \infty$, we see that

$$\lim_{t\to\infty} \begin{pmatrix} x_1^{(t)} - x_2^{(t)} \\ y_1^{(t)} - y_2^{(t)} \end{pmatrix} = 0.$$

That is, every solution of (3) is $\{x_1, x_2\}$ and $\{y_1, y_2\}$ synchronized. Similarly, we may show that every solution of (2) is $\{x_1, y_1\}$ and $\{x_2, y_2\}$ synchronized.

We remark that when $\gamma_t = \gamma$ for all $t$ and $\Gamma = 1$, condition (4) holds if, and only if, $|1-4\gamma| < 1$ or $0 < \gamma < 1/2$. Note that the eigenvalues of

$$\begin{pmatrix} 1-6\gamma & 2\gamma \\ 2\gamma & 1-6\gamma \end{pmatrix}$$

are $1-8\gamma$ and $1-4\gamma$. Thus, condition (7) becomes

$$\max\{|1-8\gamma|, |1-4\gamma|\} < 1,$$

or $0 < \gamma < 1/4$. Note that this condition is sharp. Indeed, when $\gamma = 1/4$ and $f$ is the identity function, (6) becomes

$$\begin{pmatrix} x_1^{(t+1)} - x_2^{(t+1)} \\ y_1^{(t+1)} - y_2^{(t+1)} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1^{(t)} - x_2^{(t)} \\ y_1^{(t)} - y_2^{(t)} \end{pmatrix}.$$

If we let

$$x_1^{(0)} = -1, \ y_1^{(0)} = 1, \ x_2^{(0)} = y_2^{(0)} = 0,$$

then since

$$\left(\frac{1}{2}\right)^t \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}^t \begin{pmatrix} -1 \\ 1 \end{pmatrix} = (-1)^t \begin{pmatrix} -1 \\ 1 \end{pmatrix},$$

we see that neither $\left\{\left|x_1^{(t)} - x_2^{(t)}\right|\right\}_{t=0}^{\infty}$ nor $\left\{\left|y_1^{(t)} - y_2^{(t)}\right|\right\}_{t=0}^{\infty}$ converge to zero.

The condition (4) is also sharp since when $f$ is the identity function,

$$x_2^{(t+1)} - y_1^{(t+1)} = -\left(x_2^{(t)} - y_1^{(t)}\right) = (-1)^{t+2},$$

we see that $\left\{\left|x_2^{(t)} - y_1^{(t)}\right|\right\}_{t=0}^{\infty}$ does not converge to zero. Thus every solution of (2), where $n = 2$, is synchronized when $0 < \gamma < 1/4$ and this condition is sharp.

To facilitate later discussions, we need to introduce some simple notations. The $n$ by $n$ identity matrix is denoted by $I_n$. We let $J$, $U$ and $V$ be respectively the matrices

$$J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ U = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} \text{ and } V = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}. \tag{8}$$

Note that the matrix $J$ has eigenvalues $-1$ and $1$ with the corresponding eigenvectors $\widehat{u} = (-1,1)^\dagger$ and $\widehat{v} = (1,1)^\dagger$ respectively. Note further that $\widehat{u}$ and $\widehat{v}$ are linearly independent. The matrix $U$ has eigenvalues 0 and 1 with corresponding eigenvectors $(0,1)^\dagger$ and $\widehat{u}$ respectively. The matrix $V$ has eigenvalues 0 and $-1$ with eigenvectors $(1,0)^\dagger$ and $\widehat{u}$ respectively. The matrix $U^\dagger$ has eigenvalues 0 and 1 with corresponding eigenvectors $\widehat{v}$ and $(1,0)^\dagger$ respectively. The matrix $V^\dagger$ has eigenvalues 0 and $-1$ with corresponding eigenvectors $\widehat{v}$ and $(0,1)^\dagger$ respectively.

## 2 The Case where $n = 3$

From (2), we see that

$$\begin{pmatrix} x_1^{(t+1)} - y_1^{(t+1)} \\ x_2^{(t+1)} - y_2^{(t+1)} \\ x_3^{(t+1)} - y_3^{(t+1)} \end{pmatrix} = \begin{pmatrix} 1-6\gamma_t & \gamma_t & \gamma_t \\ \gamma_t & 1-6\gamma_t & \gamma_t \\ \gamma_t & \gamma_t & 1-6\gamma_t \end{pmatrix} \begin{pmatrix} f\left(x_1^{(t)}\right) - f\left(y_1^{(t)}\right) \\ f\left(x_2^{(t)}\right) - f\left(y_2^{(t)}\right) \\ f\left(x_3^{(t)}\right) - f\left(y_3^{(t)}\right) \end{pmatrix}, \tag{9}$$

and

$$\begin{pmatrix} x_1^{(t+1)} - x_2^{(t+1)} \\ y_1^{(t+1)} - y_2^{(t+1)} \end{pmatrix} = \begin{pmatrix} 1-5\gamma_t & 2\gamma_t \\ 2\gamma_t & 1-5\gamma_t \end{pmatrix} \begin{pmatrix} f\left(x_1^{(t)}\right) - f\left(x_2^{(t)}\right) \\ f\left(y_1^{(t)}\right) - f\left(y_2^{(t)}\right) \end{pmatrix} \tag{10}$$

as well as

$$\begin{pmatrix} x_1^{(t+1)} - y_2^{(t+1)} \\ y_1^{(t+1)} - x_2^{(t+1)} \\ x_3^{(t+1)} - y_3^{(t+1)} \end{pmatrix} = \begin{pmatrix} 1-4\gamma_t & \gamma_t & \gamma_t \\ \gamma_t & 1-4\gamma_t & -\gamma_t \\ \gamma_t & -\gamma_t & 1-6\gamma_t \end{pmatrix} \begin{pmatrix} f\left(x_1^{(t)}\right) - f\left(y_2^{(t)}\right) \\ f\left(y_1^{(t)}\right) - f\left(x_2^{(t)}\right) \\ f\left(x_3^{(t)}\right) - f\left(y_3^{(t)}\right) \end{pmatrix}, \tag{11}$$

and by the rotational invariance of (2), we also have

$$\begin{pmatrix} x_2^{(t+1)} - x_3^{(t+1)} \\ y_2^{(t+1)} - y_3^{(t+1)} \end{pmatrix} = \begin{pmatrix} 1-5\gamma_t & 2\gamma_t \\ 2\gamma_t & 1-5\gamma_t \end{pmatrix} \begin{pmatrix} f\left(x_2^{(t)}\right) - f\left(x_3^{(t)}\right) \\ f\left(y_2^{(t)}\right) - f\left(y_3^{(t)}\right) \end{pmatrix}, \tag{12}$$

$$\begin{pmatrix} x_1^{(t+1)} - x_3^{(t+1)} \\ y_1^{(t+1)} - y_3^{(t+1)} \end{pmatrix} = \begin{pmatrix} 1-5\gamma_t & 2\gamma_t \\ 2\gamma_t & 1-5\gamma_t \end{pmatrix} \begin{pmatrix} f\left(x_1^{(t)}\right) - f\left(x_3^{(t)}\right) \\ f\left(y_1^{(t)}\right) - f\left(y_3^{(t)}\right) \end{pmatrix}, \tag{13}$$

and

$$\begin{pmatrix} x_1^{(t+1)} - y_3^{(t+1)} \\ y_1^{(t+1)} - x_3^{(t+1)} \\ x_2^{(t+1)} - y_2^{(t+1)} \end{pmatrix} = \begin{pmatrix} 1-4\gamma_t & \gamma_t & \gamma_t \\ \gamma_t & 1-4\gamma_t & -\gamma_t \\ \gamma_t & -\gamma_t & 1-6\gamma_t \end{pmatrix} \begin{pmatrix} f\left(x_1^{(t)}\right) - f\left(y_3^{(t)}\right) \\ f\left(y_1^{(t)}\right) - f\left(x_3^{(t)}\right) \\ f\left(x_2^{(t)}\right) - f\left(y_2^{(t)}\right) \end{pmatrix},$$

$$\begin{pmatrix} x_2^{(t+1)} - y_3^{(t+1)} \\ y_2^{(t+1)} - x_3^{(t+1)} \\ x_1^{(t+1)} - y_1^{(t+1)} \end{pmatrix} = \begin{pmatrix} 1-4\gamma_t & \gamma_t & \gamma_t \\ \gamma_t & 1-4\gamma_t & -\gamma_t \\ \gamma_t & -\gamma_t & 1-6\gamma_t \end{pmatrix} \begin{pmatrix} f\left(x_2^{(t)}\right) - f\left(y_3^{(t)}\right) \\ f\left(y_2^{(t)}\right) - f\left(x_3^{(t)}\right) \\ f\left(x_1^{(t)}\right) - f\left(y_1^{(t)}\right) \end{pmatrix}.$$

For the same reason as used in the case where $n = 2$, we may now see that if

$$\limsup_{t\to\infty} \rho \begin{pmatrix} 1-5\gamma_t & 2\gamma_t \\ 2\gamma_t & 1-5\gamma_t \end{pmatrix} < \frac{1}{\Gamma}, \tag{14}$$

then every solution of (2) is $\{x_1, x_2\}$, $\{x_2, x_3\}$, $\{x_1, x_3\}$, $\{y_1, y_2\}$, $\{y_2, y_3\}$, and $\{y_1, y_3\}$ synchronized. Moreover, every solution of (2) is $\{x_1, x_2, x_3\}$ and $\{y_1, y_2, y_3\}$ synchronized. If

$$\limsup_{t\to\infty} \rho \begin{pmatrix} 1-6\gamma_t & \gamma_t & \gamma_t \\ \gamma_t & 1-6\gamma_t & \gamma_t \\ \gamma_t & \gamma_t & 1-6\gamma_t \end{pmatrix} < \frac{1}{\Gamma}, \tag{15}$$

then every solution of (2) is $\{x_1, y_1\}$, $\{x_2, y_2\}$ and $\{x_3, y_3\}$ synchronized; and that if

$$\limsup_{t\to\infty} \rho \begin{pmatrix} 1-4\gamma_t & \gamma_t & \gamma_t \\ \gamma_t & 1-4\gamma_t & -\gamma_t \\ \gamma_t & -\gamma_t & 1-6\gamma_t \end{pmatrix} < \frac{1}{\Gamma}, \tag{16}$$

then every solution of (2) is $\{x_1, y_2\}$, $\{x_2, y_1\}$, $\{x_3, y_3\}$, $\{x_2, y_3\}$, $\{x_3, y_2\}$, $\{x_1, y_1\}$, $\{x_1, y_3\}$, $\{x_3, y_1\}$ and $\{x_2, y_2\}$ synchronized.

We remark that when $\gamma_t = \gamma$ for all $t$ and $\Gamma = 1$, the condition (14) becomes

$$\max\{|1-7\gamma|, |1-3\gamma|\} < 1,$$

or $0 < \gamma < 2/7$. When $\gamma = 2/7$ and $f$ is the identity function, (10) becomes

$$\begin{pmatrix} x_1^{(t+1)} - x_2^{(t+1)} \\ y_1^{(t+1)} - y_2^{(t+1)} \end{pmatrix} = \frac{1}{7} \begin{pmatrix} -3 & 4 \\ 4 & -3 \end{pmatrix} \begin{pmatrix} x_1^{(t)} - x_2^{(t)} \\ y_1^{(t)} - y_2^{(t)} \end{pmatrix}.$$

If we let

$$x_1^{(0)} = y_2^{(0)} = 1,\ x_2^{(0)} = y_1^{(0)} = 0,$$

then since

$$\left(\frac{1}{7}\right)^t \begin{pmatrix} -3 & 4 \\ 4 & -3 \end{pmatrix}^t \begin{pmatrix} 1 \\ -1 \end{pmatrix} = (-1)^t \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

we see that there is a solution of (2) which is not $\{x_1, x_2\}$ nor $\{y_1, y_2\}$ synchronized.

Next, the matrix

$$\begin{pmatrix} 1-6\gamma_t & \gamma_t & \gamma_t \\ \gamma_t & 1-6\gamma_t & \gamma_t \\ \gamma_t & \gamma_t & 1-6\gamma_t \end{pmatrix}$$

has the eigenvalues $1-7\gamma$, $1-7\gamma$ and $1-4\gamma$. The condition (15) becomes

$$\max\{|1-7\gamma|, |1-4\gamma|\} < 1,$$

or $0 < \gamma < 2/7$. When $\gamma = 2/7$ and $f$ is the identity function, (9) becomes

$$\begin{pmatrix} x_1^{(t+1)} - y_1^{(t+1)} \\ x_2^{(t+1)} - y_2^{(t+1)} \\ x_3^{(t+1)} - y_3^{(t+1)} \end{pmatrix} = \frac{1}{7}\begin{pmatrix} -5 & 2 & 2 \\ 2 & -5 & 2 \\ 2 & 2 & -5 \end{pmatrix}\begin{pmatrix} x_1^{(t)} - y_1^{(t)} \\ x_2^{(t)} - y_2^{(t)} \\ x_3^{(t)} - y_3^{(t)} \end{pmatrix}.$$

If we let

$$x_1^{(0)} = x_2^{(0)} = x_3^{(0)} = 1,\ y_1^{(0)} = y_2^{(0)} = y_3^{(0)} = 0,$$

then since

$$\left(\frac{1}{7}\right)^t \begin{pmatrix} -5 & 2 & 2 \\ 2 & -5 & 2 \\ 2 & 2 & -5 \end{pmatrix}^t \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \left(\frac{-1}{7}\right)^t \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},$$

we see that $\left\{\left|x_i^{(t)} - y_i^{(t)}\right|\right\}_{t=0}^{\infty}$ does not converge to zero for $i = 1, 2, 3$.

Finally, the matrix

$$\begin{pmatrix} 1-4\gamma_t & \gamma_t & \gamma_t \\ \gamma_t & 1-4\gamma_t & -\gamma_t \\ \gamma_t & -\gamma_t & 1-6\gamma_t \end{pmatrix}$$

has the eigenvalues $1-7\gamma$, $1-4\gamma$ and $1-3\gamma$. By similar reasonings, we see that the condition (16) is equivalent to $0 < \gamma < 2/7$ which is also sharp. When $\gamma = 2/7$ and $f$ is the identity function, (11) becomes

$$\begin{pmatrix} x_1^{(t+1)} - y_2^{(t+1)} \\ y_1^{(t+1)} - x_2^{(t+1)} \\ x_3^{(t+1)} - y_3^{(t+1)} \end{pmatrix} = \frac{1}{7}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & -2 \\ 2 & -2 & -5 \end{pmatrix}\begin{pmatrix} x_1^{(t)} - y_2^{(t)} \\ y_1^{(t)} - x_2^{(t)} \\ x_3^{(t)} - y_3^{(t)} \end{pmatrix}.$$

Since

$$\left(\frac{1}{7}\right)^t \begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & -2 \\ 2 & -2 & -5 \end{pmatrix}^t \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \left(\frac{-1}{7}\right)^t \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix},$$

we see that there is a solution of (2) which is not $\{x_1, y_2\}$, $\{x_2, y_1\}$ nor $\{x_3, y_3\}$ synchronized. Thus every solution of (2), where $n = 3$, is synchronized when $0 < \gamma < 2/7$ and the condition is sharp.

# 3 The Case where $n = 4$

In view of our previous discussions for the cases $n = 2$ and $n = 3$, it is reasonable to proceed to the general case. However, there are enough difference in the case $n = 4$ from the general case to warrant a sketch of the various synchronization phenomena.

- If
$$\limsup_{t\to\infty} \rho\left((1-6\gamma_t)I_4 + \gamma_t \begin{pmatrix} J & J \\ J & J \end{pmatrix}\right) < 1/\Gamma, \tag{17}$$
then every solution of (2) is $\{x_i, y_i\}$ synchronized for $i = 1, 2, 3, 4$. Indeed, from (2), we see that
$$\begin{pmatrix} x_1^{(t+1)} - y_1^{(t+1)} \\ x_2^{(t+1)} - y_2^{(t+1)} \\ x_3^{(t+1)} - y_3^{(t+1)} \\ x_4^{(t+1)} - y_4^{(t+1)} \end{pmatrix} = \left((1-6\gamma_t)I_4 + \gamma_t \begin{pmatrix} J & J \\ J & J \end{pmatrix}\right) \begin{pmatrix} f\left(x_1^{(t)}\right) - f\left(y_1^{(t)}\right) \\ f\left(x_2^{(t)}\right) - f\left(y_2^{(t)}\right) \\ f\left(x_3^{(t)}\right) - f\left(y_3^{(t)}\right) \\ f\left(x_4^{(t)}\right) - f\left(y_4^{(t)}\right) \end{pmatrix}. \tag{18}$$
Then by the Banach contraction technique described previously, the condition
$$\limsup_{t\to\infty} \rho\left((1-6\gamma_t)I_4 + \gamma_t \begin{pmatrix} J & J \\ J & J \end{pmatrix}\right) < 1/\Gamma$$
will imply
$$\lim_{t\to\infty} \left| x_i^{(t+1)} - y_i^{(t+1)} \right| = 0 \text{ for } i = 1, 2, 3, 4.$$

- If
$$\limsup_{t\to\infty} \rho\left((1-4\gamma_t)I_2 + 2\gamma_t J\right) < 1/\Gamma, \tag{19}$$
then every solution of (2) is $\{x_1, x_3\}$ and $\{y_1, y_3\}$ synchronized. Indeed, from (2), we may show that
$$\begin{pmatrix} x_1^{(t+1)} - x_3^{(t+1)} \\ y_1^{(t+1)} - y_3^{(t+1)} \end{pmatrix} = \left((1-4\gamma_t)I_2 + 2\gamma_t J\right) \begin{pmatrix} f\left(x_1^{(t)}\right) - f\left(x_3^{(t)}\right) \\ f\left(y_1^{(t)}\right) - f\left(y_3^{(t)}\right) \end{pmatrix}. \tag{20}$$
Then the same argument which we used before leads to the desired conclusion. Furthermore, by the rotation invariance of our system (2), we may also conclude that every solution is $\{x_2, x_4\}$ and $\{y_2, y_4\}$ synchronized.

- If
$$\lim_{t\to\infty} \rho\left((1-4\gamma_t)I_2 + \gamma_t \begin{pmatrix} 2J & I_2 - J \\ I_2 - J & 2J \end{pmatrix}\right) < 1/\Gamma, \tag{21}$$

then every solution of (2) is $\{x_1, y_3\}$, $\{x_3, y_1\}$, $\{x_2, y_4\}$ and $\{x_4, y_2\}$ synchronized. Indeed, from (2), we may show that

$$\begin{pmatrix} x_1^{(t+1)} - y_3^{(t+1)} \\ y_1^{(t+1)} - x_3^{(t+1)} \\ x_2^{(t+1)} - y_4^{(t+1)} \\ y_2^{(t+1)} - x_4^{(t+1)} \end{pmatrix} = \left((1-4\gamma_t)I_2 + \gamma_t \begin{pmatrix} 2J & I_2 - J \\ I_2 - J & 2J \end{pmatrix}\right) \times \begin{pmatrix} f\left(x_1^{(t)}\right) - f\left(y_3^{(t)}\right) \\ f\left(y_1^{(t)}\right) - f\left(x_3^{(t)}\right) \\ f\left(x_2^{(t)}\right) - f\left(y_4^{(t)}\right) \\ f\left(y_2^{(t)}\right) - f\left(x_4^{(t)}\right) \end{pmatrix}. \tag{22}$$

- If

$$\lim_{t\to\infty} \rho\left((1-5\gamma_t)I_2 + \gamma_t \begin{pmatrix} 2J & I_2 \\ I_2 & 2J \end{pmatrix}\right) < 1/\Gamma, \tag{23}$$

then every solution of (2) is $\{x_1, x_2\}$, $\{x_4, x_3\}$, $\{y_1, y_2\}$ and $\{y_4, y_3\}$ synchronized. Indeed, from (2), we may show that

$$\begin{pmatrix} x_1^{(t+1)} - x_2^{(t+1)} \\ y_1^{(t+1)} - y_2^{(t+1)} \\ x_4^{(t+1)} - x_3^{(t+1)} \\ y_4^{(t+1)} - y_3^{(t+1)} \end{pmatrix} = \left((1-5\gamma_t)I_2 + \gamma_t \begin{pmatrix} 2J & I_2 \\ I_2 & 2J \end{pmatrix}\right) \begin{pmatrix} f\left(x_1^{(t)}\right) - f\left(x_2^{(t)}\right) \\ f\left(y_1^{(t)}\right) - f\left(y_2^{(t)}\right) \\ f\left(x_4^{(t)}\right) - f\left(x_3^{(t)}\right) \\ f\left(y_4^{(t)}\right) - f\left(y_3^{(t)}\right) \end{pmatrix}. \tag{24}$$

Then the same argument which we used before leads to the desired conclusion. Furthermore, by the rotation invariance of our system (2), we may also conclude that every solution is $\{x_1, x_4\}$, $\{x_2, x_3\}$, $\{y_1, y_4\}$ and $\{y_2, y_3\}$ synchronized.

- If

$$\lim_{t\to\infty} \rho\left((1-4\gamma_t)I_2 + \gamma_t \begin{pmatrix} J & I_2 \\ I_2 & J \end{pmatrix}\right) < 1/\Gamma, \tag{25}$$

then every solution of (2) is $\{x_1, y_2\}$, $\{x_2, y_1\}$, $\{x_4, y_3\}$ and $\{x_3, y_4\}$ synchronized. Indeed, from (2), we may show that

$$\begin{pmatrix} x_1^{(t+1)} - y_2^{(t+1)} \\ y_1^{(t+1)} - x_2^{(t+1)} \\ x_4^{(t+1)} - y_3^{(t+1)} \\ y_4^{(t+1)} - x_3^{(t+1)} \end{pmatrix} = \left((1-4\gamma_t)I_2 + \gamma_t \begin{pmatrix} J & I_2 \\ I_2 & J \end{pmatrix}\right) \begin{pmatrix} f\left(x_1^{(t)}\right) - f\left(y_2^{(t)}\right) \\ f\left(y_1^{(t)}\right) - f\left(x_2^{(t)}\right) \\ f\left(x_4^{(t)}\right) - f\left(y_3^{(t)}\right) \\ f\left(y_4^{(t)}\right) - f\left(x_3^{(t)}\right) \end{pmatrix}. \tag{26}$$

Then the same argument which we used before leads to the desired conclusion. Furthermore, by the rotation invariance of our system (2), we may also conclude that every solution is $\{x_1, y_4\}$, $\{x_4, y_1\}$, $\{x_2, y_3\}$ and $\{x_3, y_2\}$ synchronized.

The conditions derived above are sharp. To see this, we assume that $\gamma_t = \gamma$ for all $t$, $f$ is the identity function and $\Gamma = 1$.

- Then the matrix
$$(1-6\gamma)I_4 + \gamma_t \begin{pmatrix} J & J \\ J & J \end{pmatrix}$$
has eigenvalues $1-8\gamma$, $1-6\gamma$, $1-6\gamma$ and $1-4\gamma$. The condition (17) becomes
$$\max\{|1-8\gamma|, |1-6\gamma|, |1-4\gamma|\} < 1$$
or $0 < \gamma < 1/4$. When $\gamma = 1/4$, (18) becomes
$$\begin{pmatrix} x_1^{(t+1)} - y_1^{(t+1)} \\ x_2^{(t+1)} - y_2^{(t+1)} \\ x_3^{(t+1)} - y_3^{(t+1)} \\ x_4^{(t+1)} - y_4^{(t+1)} \end{pmatrix} = \left( \frac{-1}{2} I_4 + \frac{1}{4} \begin{pmatrix} J & J \\ J & J \end{pmatrix} \right) \begin{pmatrix} x_1^{(t)} - y_1^{(t)} \\ x_2^{(t)} - y_2^{(t)} \\ x_3^{(t)} - y_3^{(t)} \\ x_4^{(t)} - y_4^{(t)} \end{pmatrix}.$$
If we let
$$x_2^{(0)} = x_4^{(0)} = y_1^{(0)} = y_3^{(0)} = 1,\ x_1^{(0)} = x_3^{(0)} = y_2^{(0)} = y_4^{(0)} = 0,$$
then since
$$\left( \frac{-1}{2} I_4 + \frac{1}{4} A_4 \right)^t \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix} = (-1)^t \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix},$$
we see that $\left\{ \left| x_i^{(t)} - y_i^{(t)} \right| \right\}_{t=0}^{\infty}$ does not converge to zero for $i = 1, 2, 3, 4$.

- Next, the condition (19) becomes $\max\{|1-6\gamma|, |1-2\gamma|\} < 1$ or $0 < \gamma < 1/3$. When $\gamma = 1/3$, (20) becomes
$$\begin{pmatrix} x_1^{(t+1)} - x_3^{(t+1)} \\ y_1^{(t+1)} - y_3^{(t+1)} \end{pmatrix} = \left( \frac{-1}{3} I_2 + \frac{2}{3} J \right) \begin{pmatrix} x_1^{(t)} - x_3^{(t)} \\ y_1^{(t)} - y_3^{(t)} \end{pmatrix}.$$
If we let
$$x_3^{(0)} = y_1^{(0)} = 1,\ x_1^{(0)} = y_3^{(0)} = 0,$$
then since
$$\left( \frac{-1}{3} I_2 + \frac{2}{3} J \right)^t \begin{pmatrix} -1 \\ 1 \end{pmatrix} = (-1)^t \begin{pmatrix} -1 \\ 1 \end{pmatrix},$$
we see that there is a solution of (2) which is not $\{x_1 x_3\}$ nor $\{y_1, y_3\}$ synchronized.

- Next, the matrix
$$(1-4\gamma_t)I_2 + \gamma_t \begin{pmatrix} 2J & I_2 - J \\ I_2 - J & 2J \end{pmatrix}$$

has eigenvalues $1-8\gamma$, $1-4\gamma$ and $1-2\gamma$, $1-2\gamma$. The condition (21) becomes

$$\max\{|1-8\gamma|, |1-4\gamma|, |1-2\gamma|\} < 1$$

or $0<\gamma<1/4$. When $\gamma = 1/4$, (22) becomes

$$\begin{pmatrix} x_1^{(t+1)} - y_3^{(t+1)} \\ y_1^{(t+1)} - x_3^{(t+1)} \\ x_2^{(t+1)} - y_4^{(t+1)} \\ y_2^{(t+1)} - x_4^{(t+1)} \end{pmatrix} = \left( (1-4\gamma_t) I_2 + \gamma_t \begin{pmatrix} 2J & I_2 - J \\ I_2 - J & 2J \end{pmatrix} \right) \begin{pmatrix} x_1^{(t)} - y_3^{(t)} \\ y_1^{(t)} - x_3^{(t)} \\ x_2^{(t)} - y_4^{(t)} \\ y_2^{(t)} - x_4^{(t)} \end{pmatrix}.$$

If we let

$$x_1^{(0)} = x_3^{(0)} = y_2^{(0)} = y_4^{(0)} = 1,\ x_2^{(0)} = x_4^{(0)} = y_1^{(0)} = y_3^{(0)} = 0,$$

then since

$$\left(\frac{1}{4}\right)^t \begin{pmatrix} 2J & I_2 - J \\ I_2 - J & 2J \end{pmatrix}^t \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} = (-1)^t \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix},$$

we see that there is a solution of (2) which is not $\{x_1, y_3\}$, $\{x_3, y_1\}$, $\{x_2, y_4\}$ nor $\{x_4, y_2\}$ synchronized.

- Next, the matrix

$$(1-5\gamma_t) I_2 + \gamma_t \begin{pmatrix} 2J & I_2 \\ I_2 & 2J \end{pmatrix}$$

has eigenvalues $1-8\gamma$, $1-6\gamma$, $1-4\gamma$ and $1-2\gamma$. The condition (23) then becomes

$$\max\{|1-8\gamma|, |1-6\gamma|, |1-4\gamma|, |1-2\gamma|\} < 1$$

or $0<\gamma<1/4$. When $\gamma = 1/4$, then (24) becomes

$$\begin{pmatrix} x_1^{(t+1)} - x_2^{(t+1)} \\ y_1^{(t+1)} - y_2^{(t+1)} \\ x_4^{(t+1)} - x_3^{(t+1)} \\ y_4^{(t+1)} - y_3^{(t+1)} \end{pmatrix} = \left( (1-5\gamma_t) I_2 + \gamma_t \begin{pmatrix} 2J & I_2 \\ I_2 & 2J \end{pmatrix} \right) \begin{pmatrix} x_1^{(t)} - x_2^{(t)} \\ y_1^{(t)} - y_2^{(t)} \\ x_4^{(t)} - x_3^{(t)} \\ y_4^{(t)} - y_3^{(t)} \end{pmatrix}.$$

If we let

$$x_1^{(0)} = x_3^{(0)} = y_2^{(0)} = y_4^{(0)} = 1,\ x_2^{(0)} = x_4^{(0)} = y_1^{(0)} = y_3^{(0)} = 0,$$

then since

$$\left( \frac{-1}{4} I_2 + \frac{1}{4} \begin{pmatrix} 2J & I_2 \\ I_2 & 2J \end{pmatrix} \right)^t \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} = (-1)^t \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix},$$

we see that there is a solution of (2) which is not $\{x_1, x_2\}$, $\{x_4, x_3\}$, $\{y_1, y_2\}$ nor $\{y_4, y_3\}$ synchronized.

- Finally, the matrix
$$(1-4\gamma_t)I_2+\gamma_t\begin{pmatrix} J & I_2 \\ I_2 & J \end{pmatrix}$$
has eigenvalues $1-6\gamma$, $1-4\gamma$, $1-4\gamma$ and $1-2\gamma$. The condition (25) then becomes
$$\max\{|1-6\gamma|,|1-4\gamma|,|1-2\gamma|\}<1$$
or $0<\gamma<1/3$. When $\gamma=1/3$, then (26) becomes
$$\begin{pmatrix} x_1^{(t+1)}-y_2^{(t+1)} \\ y_1^{(t+1)}-x_2^{(t+1)} \\ x_4^{(t+1)}-y_3^{(t+1)} \\ y_4^{(t+1)}-x_3^{(t+1)} \end{pmatrix}=\left((1-4\gamma_t)I_2+\gamma_t\begin{pmatrix} J & I_2 \\ I_2 & J \end{pmatrix}\right)\begin{pmatrix} x_1^{(t)}-y_2^{(t)} \\ y_1^{(t)}-x_2^{(t)} \\ x_4^{(t)}-y_3^{(t)} \\ y_4^{(t)}-x_3^{(t)} \end{pmatrix}.$$
If we let
$$x_1^{(0)}=x_2^{(0)}=y_3^{(0)}=y_4^{(0)}=1,\ x_3^{(0)}=x_4^{(0)}=y_1^{(0)}=y_2^{(0)}=0,$$
then since
$$\left(\frac{-1}{3}I_2+\frac{1}{3}\begin{pmatrix} J & I_2 \\ I_2 & J \end{pmatrix}\right)^t\begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}=(-1)^t\begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix},$$
we see that there is a solution of (2) which is not $\{x_1,y_4\}$, $\{x_4,y_1\}$, $\{x_2,y_3\}$ nor $\{x_3,y_2\}$ synchronized.

Hence we concluded that every solution of (2), where $n=4$, is synchronized when $0<\gamma<1/4$ and the condition is sharp.

# 4 Synchronization Criteria for $n\geq 5$

The same principles used in the previous discussions can be used again for the case where $n\geq 2$, although there are some other details.

THEOREM 4.1. Suppose $n\geq 5$. Let
$$\widehat{A}_t=(1-6\gamma_t)I_n+\gamma_tA_n \tag{27}$$
where $A_n$ is defined by (1). If $\limsup_{t\to\infty}\rho\left(\widehat{A}_t\right)<1/\Gamma$, then every solution of (2) is $\{x_i,y_i\}$ synchronized for $i=1,2,\ldots,n$.

Indeed, from (2), we see that
$$x_i^{(t+1)}-y_i^{(t+1)}=\sum_{k=1}^{n}\left(\widehat{A}_t\right)_{ik}\left(f\left(x_k^{(t)}\right)-f\left(y_k^{(t)}\right)\right)\text{ for }i=1,2,\ldots,n. \tag{28}$$

Then by the same arguments used in the derivations in the last section, the condition $\limsup_{t\to\infty} \rho\left(\widehat{A}_t\right) < 1/\Gamma$ will imply

$$\lim_{t\to\infty} \left|x_i^{(t+1)} - y_i^{(t+1)}\right| = 0 \text{ for } i = 1, 2, \ldots, n.$$

THEOREM 4.2. Suppose $n = 2m$, where $m \geq 3$. For $t \geq 0$, let

$$\widehat{B}_t = (1 - 4\gamma_t) I_{2m-2} + \gamma_t \begin{pmatrix} 2J & I_2 & & 0 \\ I_2 & 2J & \ddots & \\ & \ddots & \ddots & I_2 \\ 0 & & I_2 & 2J \end{pmatrix}_{(2m-2)\times(2m-2)}. \tag{29}$$

If $\limsup_{t\to\infty} \rho\left(\widehat{B}_t\right) < 1/\Gamma$, then every solution of (2) is $\{x_1, x_3, \ldots, x_{2m-1}\}$, $\{x_2, x_4, \ldots, x_{2m}\}$, $\{y_1, y_3, \ldots, y_{2m-1}\}$ and $\{y_2, y_4, \ldots, y_{2m}\}$ synchronized.

Indeed, from (2) we have

$$\begin{pmatrix} x_1^{(t+1)} - x_3^{(t+1)} \\ y_1^{(t+1)} - y_3^{(t+1)} \\ x_{2m}^{(t+1)} - x_4^{(t+1)} \\ y_{2m}^{(t+1)} - y_4^{(t+1)} \\ \cdots \\ x_{m+3}^{(t+1)} - x_{m+1}^{(t+1)} \\ y_{m+3}^{(t+1)} - y_{m+1}^{(t+1)} \end{pmatrix} = \widehat{B}_t \begin{pmatrix} f\left(x_1^{(t)}\right) - f\left(x_3^{(t)}\right) \\ f\left(y_1^{(t)}\right) - f\left(y_3^{(t)}\right) \\ f\left(x_{2m}^{(t)}\right) - f\left(x_4^{(t)}\right) \\ f\left(y_{2m}^{(t)}\right) - f\left(y_4^{(t)}\right) \\ \cdots \\ f\left(x_{m+3}^{(t)}\right) - f\left(x_{m+1}^{(t)}\right) \\ f\left(y_{m+3}^{(t)}\right) - f\left(y_{m+1}^{(t)}\right) \end{pmatrix}.$$

Then by reasonings we have explained before, we see that

$$\lim_{t\to\infty} \left(\left|x_1^{(t+1)} - x_3^{(t+1)}\right|, \left|x_{2m}^{(t+1)} - x_4^{(t+1)}\right|, \ldots, \left|x_{m+3}^{(t+1)} - x_{m+1}^{(t+1)}\right|\right) = 0 \tag{30}$$

and

$$\lim_{t\to\infty} \left(\left|y_1^{(t+1)} - y_3^{(t+1)}\right|, \left|y_{2m}^{(t+1)} - y_4^{(t+1)}\right|, \ldots, \left|y_{m+3}^{(t+1)} - y_{m+1}^{(t+1)}\right|\right) = 0. \tag{31}$$

By the rotational invariance of (2), we see further that every solution of (2) is $\{x_2, x_4\}$, $\{y_2, y_4\}$, $\{x_1, x_5\}$, $\{y_1, y_5\}$, $\{x_{2m+6-j}, x_j\}$, and $\{y_{2m+6-j}, y_j\}$ synchronized for $j = 6, 7, \ldots, m+2$ as well. To complete our explanation, let us consider a neural network in which $n = 6$. To illustrate (30) and (31), we draw dash lines connecting $x_1$ and $x_3$, $x_4$ and $x_6$, $y_1$ and $y_3$, as well as $y_4$ and $y_6$ to show synchronization. Then we add new dash lines to connect $x_2$ and $x_4$, $y_2$ and $y_4$, etc. These are illustrated in Fig. 2. From this Figure, we may easily see that every solution is $\{x_1, x_3, x_5\}$, $\{y_1, y_3, y_5\}$, $\{x_2, x_4, x_6\}$ and $\{y_2, y_4, y_6\}$ synchronized (See Fig. 3). The same reasoning shows that every solution of the general system (2) is $\{x_1, x_3, \ldots, x_{2m-1}\}$, $\{x_2, x_4, \ldots, x_{2m}\}$, $\{y_1, y_3, \ldots, y_{2m-1}\}$, and $\{y_2, y_4, \ldots, y_{2m}\}$ synchronized[2].

[2] The formal approach is to consider a graph with vertices $\{x_1, ..., x_n, y_1, .., y_n\}$ and synchronized vertices as edges. Then the proof reduces to finding the connected components which is an easy matter.

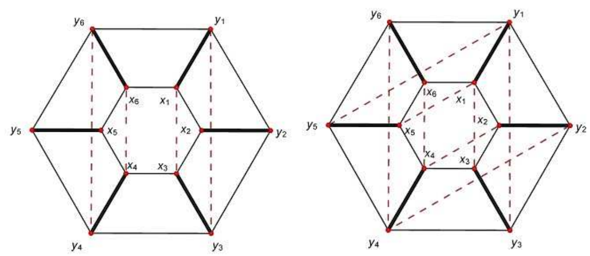


Figure 2

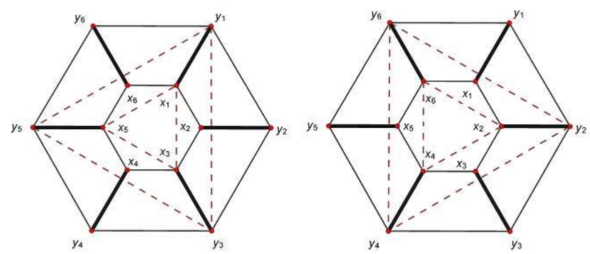


Figure 3

In the above result, we see that there are four groups of synchronized units. These groups may or may not be distinct. For example, consider a neural network where $n = 6$. We consider the special case where $\gamma_t = \gamma$ for all $t$, $f$ is the tent map function, and the Lipschitz constant $\Gamma = 2$. From the Appendix, the condition $\limsup_{t\to\infty} \rho\left(\widehat{B}_t\right) < 1/\Gamma$ in Theorem 4.2 can be replaced by $1/6 < \gamma < 3/14$. Then choosing $\gamma = 0.2$ and

$$\begin{aligned} &\left(x_1^{(0)}, x_2^{(0)}, x_3^{(0)}, x_4^{(0)}, x_5^{(0)}, x_6^{(0)}, y_1^{(0)}, y_2^{(0)}, y_3^{(0)}, y_4^{(0)}, y_5^{(0)}, y_6^{(0)}\right) \\ = \quad &\frac{1}{10}(3,1,3,1,3,1,2,5,2,5,2,5), \end{aligned}$$

we may compute $x_1^{(\infty)} = x_3^{(\infty)} = x_5^{(\infty)} = y_2^{(\infty)} = y_4^{(\infty)} = y_6^{(\infty)} = 0.3323$ and $x_2^{(\infty)} = x_4^{(\infty)} = x_6^{(\infty)} = y_1^{(\infty)} = y_3^{(\infty)} = y_5^{(\infty)} = 0.4747$.

THEOREM 4.3. Suppose $n = 2m$, where $m \geq 3$. For $t \geq 0$, let

$$\widehat{C}_t = (1-4\gamma_t)I_{2m} + \gamma_t \begin{pmatrix} 2J & I_2 & & & & U \\ I_2 & 2J & \ddots & 0 & & 0 \\ & \ddots & \ddots & \ddots & & \vdots \\ & 0 & \ddots & \ddots & I_2 & 0 \\ & & & I_2 & 2J & V \\ U^\dagger & 0 & \cdots & 0 & V^\dagger & -2I_2 \end{pmatrix}_{2m\times 2m}.$$

If $\limsup_{t\to\infty}\rho\left(\widehat{C}_t\right) < 1/\Gamma$, then every solution of (2) is $\{x_1, x_3, \ldots, x_{2m-1}, y_1, y_3, \ldots, y_{2m-1}\}$ and $\{x_2, x_4, \ldots, x_{2m}, y_2, y_4, \ldots, y_{2m}\}$ synchronized.

As in the proof of Theorem 4.2, we may first show that

$$\begin{pmatrix} x_1^{(t+1)} - y_3^{(t+1)} \\ y_1^{(t+1)} - x_3^{(t+1)} \\ x_{2m}^{(t+1)} - y_4^{(t+1)} \\ y_{2m}^{(t+1)} - x_4^{(t+1)} \\ \cdots \\ x_{m+3}^{(t+1)} - y_{m+1}^{(t+1)} \\ y_{m+3}^{(t+1)} - x_{m+1}^{(t+1)} \\ x_2^{(t+1)} - y_2^{(t+1)} \\ x_{m+2}^{(t+1)} - y_{m+2}^{(t+1)} \end{pmatrix} = \widehat{C}_t \begin{pmatrix} f\left(x_1^{(t)}\right) - f\left(y_3^{(t)}\right) \\ f\left(y_1^{(t)}\right) - f\left(x_3^{(t)}\right) \\ f\left(x_{2m}^{(t)}\right) - f\left(y_4^{(t)}\right) \\ f\left(y_{2m}^{(t)}\right) - f\left(x_4^{(t)}\right) \\ \cdots \\ f\left(x_{m+3}^{(t)}\right) - f\left(y_{m+1}^{(t)}\right) \\ f\left(y_{m+3}^{(t)}\right) - f\left(x_{m+1}^{(t)}\right) \\ f\left(x_2^{(t)}\right) - f\left(y_2^{(t)}\right) \\ f\left(x_{m+2}^{(t)}\right) - f\left(y_{m+2}^{(t)}\right) \end{pmatrix}.$$

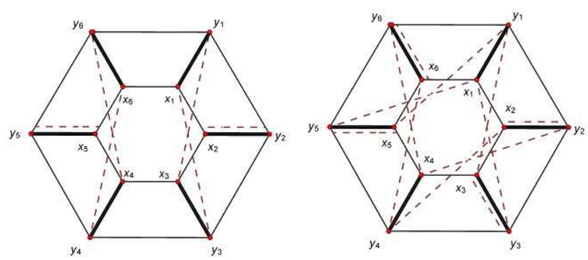


Figure 4

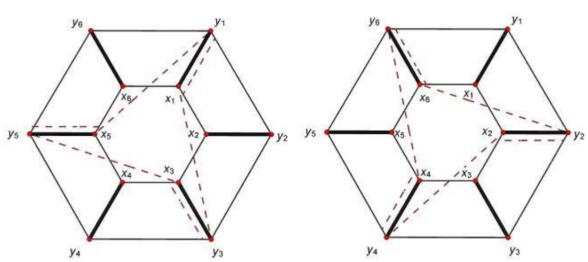


Figure 5

Then the condition $\limsup_{t\to\infty}\rho\left(\widehat{C}_t\right) < 1/\Gamma$ shows that every solution is $\{x_1, y_3\}$, $\{x_3, y_1\}$, $\{x_2, y_2\}$, $\{x_{m+2}, y_{m+2}\}$, $\{x_{2m+4-j}, y_j\}$, and $\{x_j, y_{2m+4-j}\}$ synchronized for

$j = 4, 5, \ldots, m+1$. Then the rotation invariance of (2) shows further that every solution of (2) is $\{x_2, y_4\}$, $\{x_4, y_2\}$, $\{x_1, y_5\}$, $\{x_5, y_1\}$, $\{x_3, y_3\}$, $\{x_{m+3}, y_{m+3}\}$, $\{x_{2m+6-j}, y_j\}$, and $\{x_j, y_{2m+6-j}\}$ synchronized for $j = 6, 7, \ldots, m+2$. By carefully inspecting the connections, we may then conclude the proof of our theorem. These are illustrated in Figures 4 and 5.

THEOREM 4.4. Suppose $n = 2m$, where $m \geq 3$. For $t \geq 0$, let

$$\widehat{D}_t = (1 - 4\gamma_t) I_{2m} + \gamma_t \begin{pmatrix} -I_2 + 2J & I_2 & & & \\ I_2 & 2J & \ddots & 0 & \\ & \ddots & \ddots & \ddots & \\ & 0 & \ddots & 2J & I_2 \\ & & & I_2 & -I_2 + 2J \end{pmatrix}_{2m \times 2m} . \tag{32}$$

If $\limsup_{t\to\infty} \rho\left(\widehat{D}_t\right) < 1/\Gamma$, then every solution of (2) is $\{x_1, \ldots, x_{2m}\}$ and $\{y_1, \ldots, y_{2m}\}$ synchronized.

As in the proof of Theorem 4.2, we may first show that

$$\begin{pmatrix} x_1^{(t+1)} - x_2^{(t+1)} \\ y_1^{(t+1)} - y_2^{(t+1)} \\ x_{2m}^{(t+1)} - x_3^{(t+1)} \\ y_{2m}^{(t+1)} - y_3^{(t+1)} \\ \cdots \\ x_{m+2}^{(t+1)} - x_{m+1}^{(t+1)} \\ y_{m+2}^{(t+1)} - y_{m+1}^{(t+1)} \end{pmatrix} = \widehat{D}_t \begin{pmatrix} f\left(x_1^{(t)}\right) - f\left(x_2^{(t)}\right) \\ f\left(y_1^{(t)}\right) - f\left(y_2^{(t)}\right) \\ f\left(x_{2m}^{(t)}\right) - f\left(x_3^{(t)}\right) \\ f\left(y_{2m}^{(t)}\right) - f\left(y_3^{(t)}\right) \\ \cdots \\ f\left(x_{m+2}^{(t)}\right) - f\left(x_{m+1}^{(t)}\right) \\ f\left(y_{m+2}^{(t)}\right) - f\left(y_{m+1}^{(t)}\right) \end{pmatrix}.$$

Then the condition $\limsup_{t\to\infty} \rho\left(\widehat{D}_t\right) < 1/\Gamma$ shows that every solution is $\{x_1, x_2\}$, $\{y_1, y_2\}$, $\{x_{2m+3-j}, x_j\}$, and $\{y_{2m+3-j}, y_j\}$ synchronized for $j = 3, 4, \ldots, m+1$. Then the rotation invariance of (2) shows further that every solution of (2) is $\{x_2.x_3\}$, $\{y_2.y_3\}$, $\{x_1.x_4\}$, $\{y_1.y_4\}$, $\{x_j, x_{2m+5-j}\}$ and $\{y_j, y_{2m+5-j}\}$ synchronized for $j = 5, 6, \ldots, m+3$. By carefully inspecting the connections, we may then conclude the proof of our theorem. These are illustrated in Figure 6.

THEOREM 4.5. Suppose $n = 2m$, where $m \geq 3$. For $t \geq 0$, let

$$\widehat{E}_t = (1 - 4\gamma_t) I_{2m} + \gamma_t \begin{pmatrix} J & I_2 & & & \\ I_2 & 2J & \ddots & 0 & \\ & \ddots & \ddots & \ddots & \\ & 0 & \ddots & 2J & I_2 \\ & & & I_2 & J \end{pmatrix}_{2m \times 2m} .$$

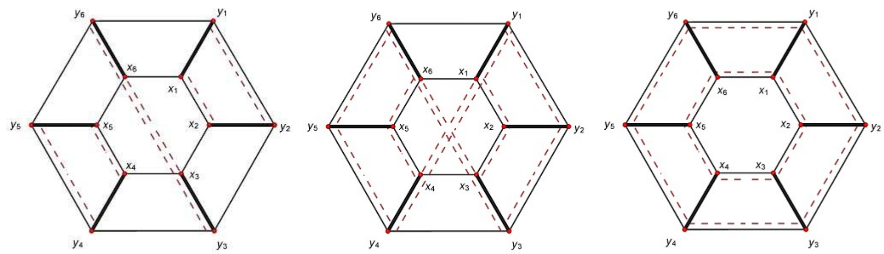


Figure 6

If $\limsup_{t\to\infty} \rho\left(\widehat{E}_t\right) < 1/\Gamma$, then every solution of (2) is $\{x_1, x_3, \ldots, x_{2m-1}, y_2, y_4, \ldots, y_{2m}\}$ and $\{x_2, x_4, \ldots, x_{2m}, y_1, y_3, \ldots, y_{2m-1}\}$ synchronized.

As in the proof of Theorem 4.2, we may first show that

$$\begin{pmatrix} x_1^{(t+1)} - y_2^{(t+1)} \\ y_1^{(t+1)} - x_2^{(t+1)} \\ x_{2m}^{(t+1)} - y_3^{(t+1)} \\ y_{2m}^{(t+1)} - x_3^{(t+1)} \\ \cdots \\ x_{m+2}^{(t+1)} - y_{m+1}^{(t+1)} \\ y_{m+2}^{(t+1)} - x_{m+1}^{(t+1)} \end{pmatrix} = \widehat{E}_t \begin{pmatrix} f\left(x_1^{(t)}\right) - f\left(y_2^{(t)}\right) \\ f\left(y_1^{(t)}\right) - f\left(x_2^{(t)}\right) \\ f\left(x_{2m}^{(t)}\right) - f\left(y_3^{(t)}\right) \\ f\left(y_{2m}^{(t)}\right) - f\left(x_3^{(t)}\right) \\ \cdots \\ f\left(x_{m+2}^{(t)}\right) - f\left(y_{m+1}^{(t)}\right) \\ f\left(y_{m+2}^{(t)}\right) - f\left(x_{m+1}^{(t)}\right) \end{pmatrix}.$$

Then the condition $\limsup_{t\to\infty} \rho\left(\widehat{E}_t\right) < 1/\Gamma$ shows that every solution is $\{x_1, y_2\}$, $\{x_2, y_1\}$, $\{x_{2m+3-j}, y_j\}$, and $\{x_j, y_{2m+3-j}\}$ synchronized for $j = 3, 4, \ldots, m+1$. Then the rotation invariance of (2) shows further that every solution of (2) is $\{x_2, y_3\}$, $\{x_3, y_2\}$, $\{x_1, y_4\}$, $\{x_4, y_1\}$, $\{x_{2m+5-j}, y_j\}$, and $\{x_j, y_{2m+5-j}\}$ synchronized for $j = 5, 6, \ldots, m+2$. By carefully inspecting the connections, we may then conclude the proof of our theorem. These are illustrated in Figures 7 and 8.

In the above result, we see that there are two groups of synchronized units. These groups may or may not be distinct. For example, consider a neural network where $n = 6$. We consider the special case where $\gamma_t = \gamma$ for all $t$, $f$ is the tent map function, and the Lipschitz constant $\Gamma = 2$. From the Appendix, the condition $\limsup_{t\to\infty} \rho\left(\widehat{B}_t\right) < 1/\Gamma$ in Theorem 4.2 can be replaced by $1/6 < \gamma < 3/14$. Then choosing $\gamma = 0.21$ and

$$\begin{aligned} & \left(x_1^{(0)}, x_2^{(0)}, x_3^{(0)}, x_4^{(0)}, x_5^{(0)}, x_6^{(0)}, y_1^{(0)}, y_2^{(0)}, y_3^{(0)}, y_4^{(0)}, y_5^{(0)}, y_6^{(0)}\right) \\ = \quad & \frac{1}{10}(2, 1, 2, 1, 2, 1, 3, 5, 3, 5, 3, 5), \end{aligned}$$

we may compute $x_1^{(\infty)} = x_3^{(\infty)} = x_5^{(\infty)} = y_2^{(\infty)} = y_4^{(\infty)} = y_6^{(\infty)} = 0.8105$ and $x_2^{(\infty)} = x_4^{(\infty)} = x_6^{(\infty)} = y_1^{(\infty)} = y_3^{(\infty)} = y_5^{(\infty)} = 0.5167$.

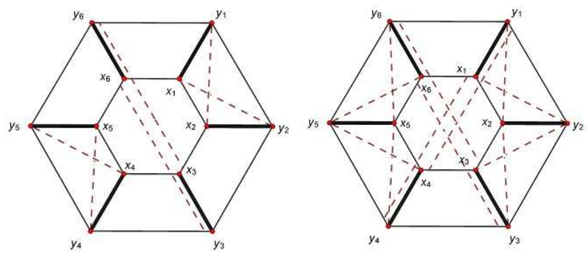


Figure 7

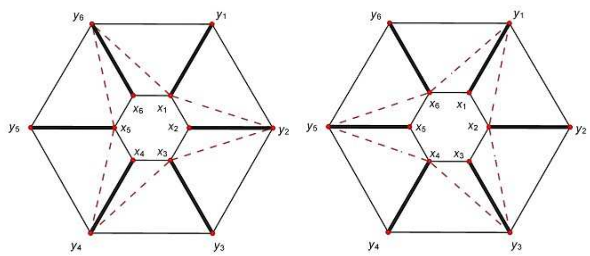


Figure 8

THEOREM 4.6. Suppose $n = 2m$, where $m \geq 3$. For $t \geq 0$, let

$$\widehat{F}_t = (1 - 4\gamma_t) I_{2m} + \gamma_t \begin{pmatrix} 2J & I_2 & & & -I_2 \\ I_2 & 2J & \ddots & 0 & \\ & \ddots & \ddots & \ddots & \\ & 0 & \ddots & 2J & I_2 \\ -I_2 & & & I_2 & 2J \end{pmatrix}_{2m \times 2m} . \tag{33}$$

If $\limsup_{t\to\infty} \rho\left(\widehat{F}_t\right) < 1/\Gamma$, then every solution of (2) is $\{x_j, x_{m+j}\}$ and $\{y_j, y_{m+j}\}$ synchronized for $j = 1, 2, \ldots, m$.

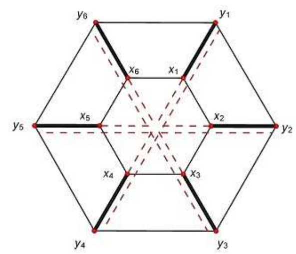


Figure 9

As in the proof of Theorem 4.2, we may first show that

$$\begin{pmatrix} x_1^{(t+1)} - x_{m+1}^{(t+1)} \\ y_1^{(t+1)} - y_{m+1}^{(t+1)} \\ x_2^{(t+1)} - x_{m+2}^{(t+1)} \\ y_2^{(t+1)} - y_{m+2}^{(t+1)} \\ \cdots \\ x_m^{(t+1)} - x_{2m}^{(t+1)} \\ y_m^{(t+1)} - y_{2m}^{(t+1)} \end{pmatrix} = \widehat{F}_t \begin{pmatrix} f\left(x_1^{(t)}\right) - f\left(x_{m+1}^{(t)}\right) \\ f\left(y_1^{(t)}\right) - f\left(y_{m+1}^{(t)}\right) \\ f\left(x_2^{(t)}\right) - f\left(x_{m+2}^{(t)}\right) \\ f\left(y_2^{(t)}\right) - f\left(y_{m+2}^{(t)}\right) \\ \cdots \\ f\left(x_m^{(t)}\right) - f\left(x_{2m}^{(t)}\right) \\ f\left(y_m^{(t)}\right) - f\left(y_{2m}^{(t)}\right) \end{pmatrix},$$

and employing the Banach contracting technique to conclude our proof.

In Figure 9, we consider a neural network where $n = 6$. To illustrate Theorem 4.6, we draw dash lines connecting $x_1$ and $x_4$, $y_1$ and $y_4$, etc. to show synchronization. The rotational invariance of (2), however, leads to no new information.

As an example, we consider a neural network where $n = 6$. We consider the special case where $\gamma_t = \gamma$ for all $t$, $f$ is the tent map function, and the Lipschitz constant $\Gamma = 2$. From the Appendix, the condition $\limsup_{t\to\infty} \rho\left(\widehat{F}_t\right) < 1/\Gamma$ in Theorem 4.6 is replaced by $1/8 < \gamma < 3/16$. By choosing $\gamma = 0.16$ and

$$\begin{aligned} & \left(x_1^{(0)}, x_2^{(0)}, x_3^{(0)}, x_4^{(0)}, x_5^{(0)}, x_6^{(0)}, y_1^{(0)}, y_2^{(0)}, y_3^{(0)}, y_4^{(0)}, y_5^{(0)}, y_6^{(0)}\right) \\ = \; & \frac{1}{10}\left(2, 1, 6, 2, 1, 6, 7, 4, 5, 7, 4, 5\right), \end{aligned}$$

we may compute $x_1^{(\infty)} = x_4^{(\infty)} = y_1^{(\infty)} = y_4^{(\infty)} = 0.3409$, $x_2^{(\infty)} = x_5^{(\infty)} = y_2^{(\infty)} = y_5^{(\infty)} = 0.3651$, $x_3^{(\infty)} = x_6^{(\infty)} = y_3^{(\infty)} = y_6^{(\infty)} = 0.3351$.

THEOREM 4.7. Suppose $n = 2m$, where $m \geq 3$. For $t \geq 0$, let

$$\widehat{G}_t = (1-4\gamma_t)I_{2m} + \gamma_t \begin{pmatrix} 2J & I_2 & & & -J \\ I_2 & 2J & \ddots & 0 & \\ & \ddots & \ddots & \ddots & \\ & 0 & \ddots & 2J & I_2 \\ -J & & & I_2 & 2J \end{pmatrix}_{2m\times 2m}.$$

If $\limsup_{t\to\infty} \rho\left(\widehat{G}_t\right) < 1/\Gamma$, then every solution of (2) is $\{x_j, y_{m+j}\}$ and $\{x_{m+j}, y_j\}$ synchronized for $j = 1, 2, \ldots, m$.

This follows easily from the fact that

$$\begin{pmatrix} x_1^{(t+1)} - y_{m+1}^{(t+1)} \\ y_1^{(t+1)} - x_{m+1}^{(t+1)} \\ x_2^{(t+1)} - y_{m+2}^{(t+1)} \\ y_2^{(t+1)} - x_{m+2}^{(t+1)} \\ \cdots \\ x_m^{(t+1)} - y_{2m}^{(t+1)} \\ y_m^{(t+1)} - x_{2m}^{(t+1)} \end{pmatrix} = \widehat{G}_t \begin{pmatrix} f\left(x_1^{(t)}\right) - f\left(y_{m+1}^{(t)}\right) \\ f\left(y_1^{(t)}\right) - f\left(x_{m+1}^{(t)}\right) \\ f\left(x_2^{(t)}\right) - f\left(y_{m+2}^{(t)}\right) \\ f\left(y_2^{(t)}\right) - f\left(x_{m+2}^{(t)}\right) \\ \cdots \\ f\left(x_m^{(t)}\right) - f\left(y_{2m}^{(t)}\right) \\ f\left(y_m^{(t)}\right) - f\left(x_{2m}^{(t)}\right) \end{pmatrix}.$$

In Figure 10, we consider a neural network where $n = 6$. To illustrate Theorem 4.7, we draw dash lines connecting $x_1$ and $y_4$, $x_2$ and $y_5$, etc. to show synchronization. The rotation invariance of (2), however, leads to no new information.

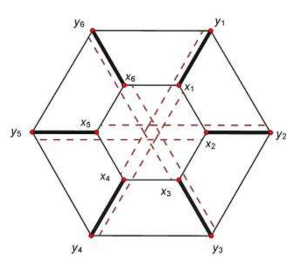


Figure 10

THEOREM 4.8. Suppose $n = 2m+1$, where $m \geq 2$. For $t \geq 0$, let

$$\widehat{H}_t = (1-4\gamma_t)I_{2m} + \gamma_t \begin{pmatrix} 2J & I_2 & & & \\ I_2 & 2J & \ddots & 0 & \\ & \ddots & \ddots & \ddots & \\ & 0 & \ddots & 2J & I_2 \\ & & & I_2 & -I_2+2J \end{pmatrix}_{2m\times 2m}. \tag{34}$$

If $\limsup_{t\to\infty} \rho\left(\widehat{H}_t\right) < 1/\Gamma$, then every solution of (2) is $\{x_1, x_2, \ldots, x_{2m+1}\}$ and $\{y_1, y_2, \ldots, y_{2m+1}\}$ synchronized.

As in the proof of Theorem 4.2, we may first show that

$$\begin{pmatrix} x_1^{(t+1)} - x_3^{(t+1)} \\ y_1^{(t+1)} - y_3^{(t+1)} \\ x_{2m+1}^{(t+1)} - x_4^{(t+1)} \\ y_{2m+1}^{(t+1)} - y_4^{(t+1)} \\ \cdots \\ x_{m+3}^{(t+1)} - x_{m+2}^{(t+1)} \\ y_{m+3}^{(t+1)} - y_{m+2}^{(t+1)} \end{pmatrix} = \widehat{H}_t \begin{pmatrix} f\left(x_1^{(t)}\right) - f\left(x_3^{(t)}\right) \\ f\left(y_1^{(t)}\right) - f\left(y_3^{(t)}\right) \\ f\left(x_{2m+1}^{(t)}\right) - f\left(x_4^{(t)}\right) \\ f\left(y_{2m+1}^{(t)}\right) - f\left(y_4^{(t)}\right) \\ \cdots \\ f\left(x_{m+3}^{(t)}\right) - f\left(x_{m+2}^{(t)}\right) \\ f\left(y_{m+3}^{(t)}\right) - f\left(y_{m+2}^{(t)}\right) \end{pmatrix}.$$

Then the condition $\limsup_{t\to\infty} \rho\left(\widehat{H}_t\right) < 1/\Gamma$ implies that every solution is $\{x_1, x_3\}$, $\{y_1, y_3\}$, and $\{x_{2m+5-j}, x_j\}$, $\{x_{2m+5-j}, x_j\}$, where $j = 4, 5, \ldots, m+2$, synchronized. The rotational invariance of (2) shows further that every solution of (2) is $\{x_2, x_4\}$, $\{y_2, y_4\}$, $\{x_1, x_5\}$, $\{y_1, y_5\}$, $\{x_{2m+7-j}, x_j\}$, and $\{y_{2m+7-j}, y_j\}$ synchronized for $j = 6, 7, \ldots, m+3$. Then careful inspection of the connections completes our proof. These are illustrated in Figure 11.

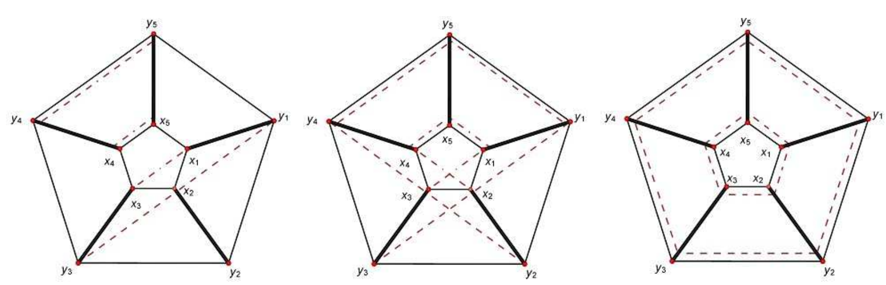


Figure 11

THEOREM 4.9. Suppose $n = 2m+1$, where $m \geq 2$. For $t \geq 0$, let

$$\widehat{K}_t = (1-4\gamma_t) I_{2m+1} + \gamma_t \begin{pmatrix} 2J & I_2 & & & & -\widehat{u} \\ I_2 & 2J & \ddots & & 0 & 0 \\ & \ddots & \ddots & \ddots & & \vdots \\ & 0 & \ddots & 2J & I_2 & \vdots \\ & & & I_2 & J & 0 \\ -\widehat{u}^\dagger & 0 & \cdots & \cdots & 0 & -2 \end{pmatrix}_{(2m+1)\times(2m+1)}.$$

If $\limsup_{t\to\infty} \rho\left(\widehat{K}_t\right) < 1/\Gamma$, then every solution of (2) is (fully) synchronized.

As in the proof of Theorem 4.2, we may first show that

$$\begin{pmatrix} x_1^{(t+1)} - y_3^{(t+1)} \\ y_1^{(t+1)} - x_3^{(t+1)} \\ x_{2m+1}^{(t+1)} - y_4^{(t+1)} \\ y_{2m+1}^{(t+1)} - x_4^{(t+1)} \\ \cdots \\ x_{m+3}^{(t+1)} - y_{m+2}^{(t+1)} \\ y_{m+3}^{(t+1)} - x_{m+2}^{(t+1)} \\ x_2^{(t+1)} - y_2^{(t+1)} \end{pmatrix} = \widehat{K}_t \begin{pmatrix} f\left(x_1^{(t)}\right) - f\left(y_3^{(t)}\right) \\ f\left(y_1^{(t)}\right) - f\left(x_3^{(t)}\right) \\ f\left(x_{2m+1}^{(t)}\right) - f\left(y_4^{(t)}\right) \\ f\left(y_{2m+1}^{(t)}\right) - f\left(x_4^{(t)}\right) \\ \cdots \\ f\left(x_{m+3}^{(t)}\right) - f\left(y_{m+2}^{(t)}\right) \\ f\left(y_{m+3}^{(t)}\right) - f\left(x_{m+2}^{(t)}\right) \\ f\left(x_2^{(t)}\right) - f\left(y_2^{(t)}\right) \end{pmatrix}.$$

Then the condition $\limsup_{t\to\infty} \rho\left(\widehat{K}_t\right) < 1/\Gamma$ shows that every solution of (2) is $\{x_1, y_3\}$, $\{x_3, y_1\}$, $\{x_2, y_2\}$, $\{x_{2m+5-j}, y_j\}$, and $\{x_j, y_{2m+5-j}\}$ synchronized for $j = 4, 5, \ldots, m+2$. The rotation invariance of (2) then shows that every solution of (2) is $\{x_2, y_4\}$, $\{y_2, x_4\}$, $\{x_1, y_5\}$, $\{y_1, x_5\}$, $\{x_3, y_3\}$, $\{x_{2m+7-j}, y_j\}$, and $\{x_j, y_{2m+7-j}\}$ synchronized for $j = 6, 7, \ldots, m+3$. Careful inspection of the connections then completes our proof. These are illustrated in Figure 12.

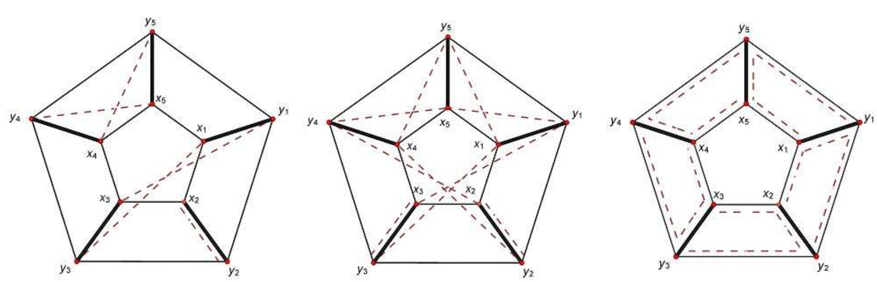


Figure 12

# 5 Sharpness

We will now consider the special case where $\gamma_t = \gamma$ for all $t$, $f$ is the identity function, and the Lipschitz constant $\Gamma = 1$.

- Since
$$\max_{1\le k\le n}\left\{6-2\cos\frac{2k\pi}{n}\right\} = \begin{cases} 8 & \text{if } n \text{ is even} \\ 6-2\cos\frac{(n-1)\pi}{n} & \text{if } n \text{ is odd}\end{cases},$$
the condition $\limsup_{t\to\infty}\rho\left(\widehat{A}_t\right) < 1/\Gamma$ in Theorem 4.1 is replaced by
$$0<\gamma<\frac{2}{\max_{1\le k\le n}\left(6-2\cos\frac{2k\pi}{n}\right)} = \begin{cases} 1/4 & \text{if } n \text{ is even} \\ 1/\left(3-\cos\frac{(n-1)\pi}{n}\right) & \text{if } n \text{ is odd}\end{cases}.$$
First note that this condition is sharp when $n$ is even. Indeed, take $f$ to be the identity function. When $\gamma = 1/4$, (28) becomes
$$x_i^{(t+1)} - y_i^{(t+1)} = \sum_{k=1}^{n}\left(\frac{-1}{2}I_n + \frac{1}{4}A_n\right)_{ik}\left(x_k^{(t)} - y_k^{(t)}\right) \text{ for } i = 1,2,\ldots,n.$$
The number $-1$ is an eigenvalue of $\frac{-1}{2}I_n + \frac{1}{4}A_n$ and the corresponding eigenvector is $\widetilde{u} = (-1,1,-1,\ldots,1)^{\dagger}$. If we let
$$x_{2k}^{(0)} = y_{2k-1}^{(0)} = 1,\ x_{2k-1}^{(0)} = y_{2k}^{(0)} = 0 \text{ for } k = 1,2,\ldots,n/2,$$
then since
$$\begin{aligned}\begin{pmatrix} x_1^{(t)} - y_1^{(t)} \\ \cdots \\ x_n^{(t)} - y_n^{(t)}\end{pmatrix} &= \left(\frac{-1}{2}I_n + \frac{1}{4}A_n\right)^t \begin{pmatrix} x_1^{(0)} - y_1^{(0)} \\ \cdots \\ x_n^{(0)} - y_n^{(0)}\end{pmatrix} \\ &= \left(\frac{-1}{2}I_n + \frac{1}{4}A_n\right)^t \widetilde{u} = (-1)^t\,\widetilde{u},\end{aligned}$$
we see that $\left\{\left|x_i^{(t)} - y_i^{(t)}\right|\right\}_{t=0}^{\infty}$ does not converge to zero for $i = 1,\ldots,n$. Next, when $n$ is odd, this condition is also sharp since if $\gamma = 1/\left(3-\cos\frac{(n-1)\pi}{n}\right)$, the number $-1$ is an eigenvalue of the matrix and the corresponding eigenvector is
$$\widetilde{v} = \left(c_1\cos\frac{(n-1)\pi}{n} + c_2\sin\frac{(n-1)\pi}{n},\ldots,c_1\cos(n-1)\pi + c_2\sin(n-1)\pi\right)^{\dagger},$$
where $c_1$ and $c_2$ are not both equal to zero. If we take $f$ to be the identity function and let
$$x_k^{(0)} = c_1\cos\frac{(n-1)k\pi}{n},\ y_k^{(0)} = -c_2\sin\frac{(n-1)k\pi}{n},\ \text{for } k = 1,\ldots,n,$$

then since

$$\left(\widehat{A}_t\right)^t \widetilde{v} = (-1)^t \widetilde{v},$$

we see that $\left\{\left|x_i^{(t)} - y_i^{(t)}\right|\right\}_{t=0}^{\infty}$ does not converge to zero for $i = 1, \ldots, n$.

- Since

$$3 - 2\cos\frac{(m-1)\pi}{m} < 6 - 2\cos\frac{(m-1)\pi}{m},$$

we have

$$\max_{1\le k\le m-1}\left\{6 - 2\cos\frac{k\pi}{m},\ 2 - 2\cos\frac{k\pi}{m}\right\} = 6 - 2\cos\frac{(m-1)\pi}{m}.$$

Thus, the condition $\limsup_{t\to\infty}\rho\left(\widehat{B}_t\right) < 1/\Gamma$ in Theorem 4.2 is replaced by $0 < \gamma < 1/\left(3 - \cos\frac{(m-1)\pi}{m}\right)$. This condition is sharp since when $\gamma$ is equal to $1/\left(3 - \cos\frac{(m-1)\pi}{m}\right)$, the number $-1$ is an eigenvalue of the matrix $\widehat{B}_t$ and the corresponding eigenvector is

$$\widetilde{u} = \left(\sin\frac{(m-1)\pi}{m}\widehat{u}^\dagger, \sin\frac{2(m-1)\pi}{m}\widehat{u}^\dagger, \ldots, \sin\frac{(m-1)^2\pi}{m}\widehat{u}^\dagger\right)^\dagger.$$

If we take $f$ to be the identity function and let

$$y_1^{(0)} = \sin\frac{(m-1)\pi}{m},\ x_1^{(0)} = y_3^{(0)} = x_{2m+4-k}^{(0)} = y_k^{(0)} = 0,\ k = 4, 5, \ldots, m-1,$$

$$x_k^{(0)} = \sin\frac{(k-2)(m-1)\pi}{m},\ k = 3, 4, \ldots, m+1,$$

and

$$y_{2m+2-k}^{(0)} = \sin\frac{(m-1)k\pi}{m},\ k = 2, 3, \ldots, m-1,$$

then since

$$\begin{pmatrix} x_1^{(t)} - x_3^{(t)} \\ y_1^{(t)} - y_3^{(t)} \\ x_{2m}^{(t)} - x_4^{(t)} \\ y_{2m}^{(t)} - y_4^{(t)} \\ \cdots \\ x_{m+3}^{(t)} - x_{m+1}^{(t)} \\ y_{m+3}^{(t)} - y_{m+1}^{(t)} \end{pmatrix} = \left(\widehat{B}_t\right)^t \begin{pmatrix} x_1^{(0)} - x_3^{(0)} \\ y_1^{(0)} - y_3^{(0)} \\ x_{2m}^{(0)} - x_4^{(0)} \\ y_{2m}^{(0)} - y_4^{(0)} \\ \cdots \\ x_{m+3}^{(0)} - x_{m+1}^{(0)} \\ y_{m+3}^{(0)} - y_{m+1}^{(0)} \end{pmatrix} = \left(\widehat{B}_t\right)^t \widetilde{u} = (-1)^t \widetilde{u},$$

we see that neither $\left\{\left|x_1^{(t)} - x_3^{(t)}\right|\right\}_{t=0}^{\infty}$, $\left\{\left|y_1^{(t)} - y_3^{(t)}\right|\right\}_{t=0}^{\infty}$, $\left\{\left|x_j^{(t)} - x_{2m+4-j}^{(t)}\right|\right\}_{t=0}^{\infty}$ nor $\left\{\left|y_j^{(t)} - y_{2m+4-j}^{(t)}\right|\right\}_{t=0}^{\infty}$ converges to zero for $j = 4, 5, \ldots, m+1$.

- Since

$$2-2\cos\frac{(m-1)\,\pi}{m}<4<6-2\cos\frac{(m-1)\,\pi}{m}<8,$$

we have

$$\max_{1\le k\le m-1}\left\{4,\ 8,\ 6-2\cos\frac{k\pi}{m},\ 2-2\cos\frac{k\pi}{m}\right\}=8.$$

Thus, the condition $\limsup_{t\to\infty}\rho\left(\widehat{C}_t\right)<1/\Gamma$ in Theorem 4.3 is replaced by $0<\gamma<1/4$. This condition is sharp since when $\gamma=1/4$, the number $-1$ is an eigenvalue of the matrix $\widehat{C}_t$ and the corresponding eigenvector is

$$\widetilde{u}=\begin{cases}\left(-\widehat{u}^\dagger,\widehat{u}^\dagger,\ldots,-\widehat{u}^\dagger,\widehat{u}^\dagger,\widehat{u}^\dagger\right)^\dagger & \text{if } m \text{ is odd}\\ \left(\widehat{u}^\dagger,-\widehat{u}^\dagger,\ldots,\widehat{u}^\dagger,-\widehat{u}^\dagger,\widehat{u}^\dagger,\widehat{v}^\dagger\right)^\dagger & \text{if } m \text{ is even}\end{cases}.$$

If we take $f$ to be the identity function and let

$$\begin{cases}x_{2k-1}^{(0)}=y_{2k}^{(0)}=1,\ x_{2k}^{(0)}=y_{2k-1}^{(0)}=0 \text{ for } k=1,\ldots,m & \text{if } m \text{ is odd}\\ x_{2k}^{(0)}=y_{2k-1}^{(0)}=1,\ x_{2k-1}^{(0)}=y_{2k}^{(0)}=0 \text{ for } k=1,\ldots,m & \text{if } m \text{ is even}\end{cases},$$

then since

$$\begin{pmatrix} x_1^{(t)}-y_3^{(t)}\\ y_1^{(t)}-x_3^{(t)}\\ x_{2m}^{(t)}-y_4^{(t)}\\ y_{2m}^{(t)}-x_4^{(t)}\\ \cdots\\ x_{m+3}^{(t)}-y_{m+1}^{(t)}\\ y_{m+3}^{(t)}-x_{m+1}^{(t)}\\ x_2^{(t)}-y_2^{(t)}\\ x_{m+2}^{(t)}-y_{m+2}^{(t)}\end{pmatrix}=\left(\widehat{C}_t\right)^t\begin{pmatrix} x_1^{(0)}-y_3^{(0)}\\ y_1^{(0)}-x_3^{(0)}\\ x_{2m}^{(0)}-y_4^{(0)}\\ y_{2m}^{(0)}-x_4^{(0)}\\ \cdots\\ x_{m+3}^{(0)}-y_{m+1}^{(0)}\\ y_{m+3}^{(0)}-x_{m+1}^{(0)}\\ x_2^{(0)}-y_2^{(0)}\\ x_{m+2}^{(0)}-y_{m+2}^{(0)}\end{pmatrix}=\left(\widehat{C}_t\right)^t\widetilde{u}=(-1)^t\,\widetilde{u},$$

we see that neither $\left\{\left|x_1^{(t)}-y_3^{(t)}\right|\right\}_{t=0}^{\infty}$, $\left\{\left|x_3^{(t)}-y_1^{(t)}\right|\right\}_{t=0}^{\infty}$, $\left\{\left|x_2^{(t)}-y_2^{(t)}\right|\right\}_{t=0}^{\infty}$, nor $\left\{\left|x_{m+2}^{(t)}-y_{m+2}^{(t)}\right|\right\}_{t=0}^{\infty}$, $\left\{\left|x_{2m+4-j}^{(t)}-y_j^{(t)}\right|\right\}_{t=0}^{\infty}$ $\left\{\left|x_j^{(t)}-y_{2m+4-j}^{(t)}\right|\right\}_{t=0}^{\infty}$, where $j=4,5,\ldots,m+1$, can converge to zero.

- Since

$$\max_{1\le k\le m}\left\{6-2\cos\frac{k\pi}{m},\ 2-2\cos\frac{k\pi}{m}\right\}=\max\{8,4\}=8,$$

Thus, the condition $\limsup_{t\to\infty}\rho\left(\widehat{D}_t\right)<1/\Gamma$ in Theorem 4.4 is replaced by $0<\gamma<1/4$. This condition is sharp since when $\gamma=1/4$, the number $-1$ is an eigenvalue of the matrix $\widehat{D}_t$ and the corresponding eigenvector is

$$\widetilde{u}=\left(\widehat{u}^\dagger,-\widehat{u}^\dagger,\widehat{u}^\dagger,-\widehat{u}^\dagger,\ldots,(-1)^{m+1}\,\widehat{u}^\dagger\right)^\dagger.$$

If we take $f$ to be the identity function and let

$$x_{2k}^{(0)} = y_{2k-1}^{(0)} = 1,\ x_{2k-1}^{(0)} = y_{2k}^{(0)} = 0 \text{ for } k = 1, \ldots, m,$$

then since

$$\begin{pmatrix} x_1^{(t)} - x_2^{(t)} \\ y_1^{(t)} - y_2^{(t)} \\ x_{2m}^{(t)} - x_3^{(t)} \\ y_{2m}^{(t)} - y_3^{(t)} \\ \cdots \\ x_{m+2}^{(t)} - x_{m+1}^{(t)} \\ y_{m+2}^{(t)} - y_{m+1}^{(t)} \end{pmatrix} = \left(\widehat{D}_t\right)^t \begin{pmatrix} x_1^{(0)} - x_2^{(0)} \\ y_1^{(0)} - y_2^{(0)} \\ x_{2m}^{(0)} - x_3^{(0)} \\ y_{2m}^{(0)} - y_3^{(0)} \\ \cdots \\ x_{m+2}^{(0)} - x_{m+1}^{(0)} \\ y_{m+2}^{(0)} - y_{m+1}^{(0)} \end{pmatrix} = \left(\widehat{D}_t\right)^t \widetilde{u} = (-1)^t \widetilde{u},$$

we see that neither $\left\{\left|x_1^{(t)} - x_2^{(t)}\right|\right\}_{t=0}^{\infty}$, $\left\{\left|y_1^{(t)} - y_2^{(t)}\right|\right\}_{t=0}^{\infty}$, $\left\{\left|x_{2m+3-j}^{(t)} - x_j^{(t)}\right|\right\}_{t=0}^{\infty}$ nor $\left\{\left|y_{2m+3-j}^{(t)} - y_j^{(t)}\right|\right\}_{t=0}^{\infty}$ can converge to zero for $j = 3, 4, \ldots, m+1$.

- Since

$$4 < 6 - 2\cos\frac{(m-1)\pi}{m} < 8,$$

we have

$$\max_{1\le k\le m}\left\{6 - 2\cos\frac{(k-1)\pi}{m},\ 2 - 2\cos\frac{k\pi}{m}\right\} = 6 - 2\cos\frac{(m-1)\pi}{m}.$$

Thus, the condition $\limsup_{t\to\infty} \rho\left(\widehat{E}_t\right) < 1/\Gamma$ in Theorem 4.5 is replaced by $0 < \gamma < 1/\left(3 - \cos\frac{(m-1)\pi}{m}\right)$. This condition is sharp since when $\gamma$ is equal to $1/\left(3 - \cos\frac{(m-1)\pi}{m}\right)$, the number $-1$ is an eigenvalue of the matrix $\widehat{E}_t$ and the corresponding eigenvector is

$$\widetilde{u} = \left(\cos\frac{(m-1)\pi}{m}\widehat{u}^\dagger, \cos\frac{3(m-1)\pi}{m}\widehat{u}^\dagger, \ldots, \cos\frac{(2m-1)(m-1)\pi}{m}\widehat{u}^\dagger\right)^\dagger.$$

If we take $f$ to be the identity function and let

$$-x_1^{(0)} = y_1^{(0)} = \cos\frac{(m-1)\pi}{m},\ \ -x_{2m}^{(0)} = y_{2m}^{(0)} = \cos\frac{3(m-1)\pi}{m},$$

$$-x_{2m+2-k}^{(0)} = y_{2m+2-k}^{(0)} = \cos\frac{(m-1)(2k-1)\pi}{m},\ \text{for } k = 3, 4, \ldots, m,$$

and

$$x_k^{(0)} = y_k^{(0)} = 0,\ \text{for } k = 2, 3, \ldots, m+1,$$

then since

$$\begin{pmatrix} x_1^{(t)} - y_2^{(t)} \\ y_1^{(t)} - x_2^{(t)} \\ x_{2m}^{(t)} - y_3^{(t)} \\ y_{2m}^{(t)} - x_3^{(t)} \\ \cdots \\ x_{m+2}^{(t)} - y_{m+1}^{(t)} \\ y_{m+2}^{(t)} - x_{m+1}^{(t)} \end{pmatrix} = \left(\widehat{E}_t\right)^t \begin{pmatrix} x_1^{(0)} - y_2^{(0)} \\ y_1^{(0)} - x_2^{(0)} \\ x_{2m}^{(0)} - y_3^{(0)} \\ y_{2m}^{(0)} - x_3^{(0)} \\ \cdots \\ x_{m+2}^{(0)} - y_{m+1}^{(0)} \\ y_{m+2}^{(0)} - x_{m+1}^{(0)} \end{pmatrix} = \left(\widehat{E}_t\right)^t \widetilde{u} = (-1)^t \widetilde{u},$$

we see that neither $\left\{\left|x_1^{(t)} - y_2^{(t)}\right|\right\}_{t=0}^{\infty}$, $\left\{\left|x_2^{(t)} - y_1^{(t)}\right|\right\}_{t=0}^{\infty}$, $\left\{\left|x_{2m+3-j}^{(t)} - y_j^{(t)}\right|\right\}_{t=0}^{\infty}$ nor $\left\{\left|x_j^{(t)} - y_{2m+3-j}^{(t)}\right|\right\}_{t=0}^{\infty}$ converges to zero for $j = 3, 4, \ldots, m+1$.

- Since

$$\max_{1\le k\le m}\left\{6 - 2\cos\frac{(2k-1)\pi}{m}\right\} = \begin{cases} 8 & \text{if } m \text{ is odd} \\ 6 - 2\cos\frac{(m-1)\pi}{m} & \text{if } m \text{ is even} \end{cases},$$

and

$$\max_{1\le k\le m}\left\{2 - 2\cos\frac{(2k-1)\pi}{m}\right\} = \begin{cases} 4 & \text{if } m \text{ is odd} \\ 2 - 2\cos\frac{(m-1)\pi}{m} & \text{if } m \text{ is even} \end{cases},$$

we have

$$\max_{1\le k\le m}\left\{6 - 2\cos\frac{(2k-1)\pi}{m},\ 2 - 2\cos\frac{(2k-1)\pi}{m}\right\} = 8.$$

Thus, the condition $\limsup_{t\to\infty}\rho\left(\widehat{F}_t\right) < 1/\Gamma$ in Theorem 4.6 is replaced by $0 < \gamma < 1/4$. This condition is sharp since when $\gamma = 1/4$, the number $-1$ is an eigenvalue of the matrix $\widehat{F}_t$ and the corresponding eigenvector is

$$\widetilde{u} = \left(-\widehat{u}^\dagger, \widehat{u}^\dagger, \ldots, -\widehat{u}^\dagger, \widehat{u}^\dagger, -\widehat{u}^\dagger\right)^\dagger.$$

If we take $f$ to be the identity function and let

$$x_{2k-1}^{(0)} = y_{2k}^{(0)} = 1,\ x_{2k}^{(0)} = y_{2k-1}^{(0)} = 0,\ \text{for } k = 1, 2, \ldots, m,$$

then since

$$\begin{pmatrix} x_1^{(t)} - x_{m+1}^{(t)} \\ y_1^{(t)} - y_{m+1}^{(t)} \\ x_2^{(t)} - x_{m+2}^{(t)} \\ y_2^{(t)} - y_{m+2}^{(t)} \\ \cdots \\ x_m^{(t)} - x_{2m}^{(t)} \\ y_m^{(t)} - y_{2m}^{(t)} \end{pmatrix} = \left(\widehat{F}_t\right)^t \begin{pmatrix} x_1^{(0)} - x_{m+1}^{(0)} \\ y_1^{(0)} - y_{m+1}^{(0)} \\ x_2^{(0)} - x_{m+2}^{(0)} \\ y_2^{(0)} - y_{m+2}^{(0)} \\ \cdots \\ x_m^{(0)} - x_{2m}^{(0)} \\ y_m^{(0)} - y_{2m}^{(0)} \end{pmatrix} = \left(\widehat{F}_t\right)^t \widetilde{u} = (-1)^t \widetilde{u},$$

we see that neither $\left\{\left|x_j^{(t)}-x_{m+j}^{(t)}\right|\right\}_{t=0}^{\infty}$ nor $\left\{\left|y_j^{(t)}-y_{m+j}^{(t)}\right|\right\}_{t=0}^{\infty}$ converges to zero for $j=1,2,\ldots,m$.

- Since

$$\max_{1\le k\le m}\left\{6-2\cos\frac{2k\pi}{m}\right\}=\begin{cases}6-2\cos\frac{(m-1)\pi}{m} & \text{if } m \text{ is odd}\\ 8 & \text{if } m \text{ is even}\end{cases}$$

and

$$\max_{1\le k\le m}\left\{2-2\cos\frac{(2k-1)\pi}{m}\right\}=\begin{cases}4 & \text{if } m \text{ is odd}\\ 2-2\cos\frac{(m-1)\pi}{m} & \text{if } m \text{ is even}\end{cases},$$

we have

$$\max_{1\le k\le m}\left\{6-2\cos\frac{2k\pi}{m},\ 2-2\cos\frac{(2k-1)\pi}{m}\right\}=8.$$

Thus, the condition $\limsup_{t\to\infty}\rho\left(\widehat{G}_t\right)<1/\Gamma$ in Theorem 4.7 is replaced by $0<\gamma<1/4$. This condition is sharp since when $\gamma=1/4$, the number $-1$ is an eigenvalue of the matrix $\widehat{G}_t$ and the corresponding eigenvector is

$$\widetilde{u}=\left(-\widehat{u}^\dagger,\widehat{u}^\dagger,\ldots,-\widehat{u}^\dagger,\widehat{u}^\dagger\right)^\dagger.$$

If we take $f$ to be the identity function and let

$$x_{2k-1}^{(0)}=y_{2k}^{(0)}=1,\ x_{2k}^{(0)}=y_{2k-1}^{(0)}=0,\ \text{for } k=1,2,\ldots,m,$$

then since

$$\begin{pmatrix}x_1^{(t)}-y_{m+1}^{(t)}\\ y_1^{(t)}-x_{m+1}^{(t)}\\ x_2^{(t)}-y_{m+2}^{(t)}\\ y_2^{(t)}-x_{m+2}^{(t)}\\ \cdots\\ x_m^{(t)}-y_{2m}^{(t)}\\ y_m^{(t)}-x_{2m}^{(t)}\end{pmatrix}=\left(\widehat{G}_t\right)^t\begin{pmatrix}x_1^{(0)}-y_{m+1}^{(0)}\\ y_1^{(0)}-x_{m+1}^{(0)}\\ x_2^{(0)}-y_{m+2}^{(0)}\\ y_2^{(0)}-x_{m+2}^{(0)}\\ \cdots\\ x_m^{(0)}-y_{2m}^{(0)}\\ y_m^{(0)}-x_{2m}^{(0)}\end{pmatrix}=\left(\widehat{G}_t\right)^t\widetilde{u}=(-1)^t\,\widetilde{u},$$

we see that neither $\left\{\left|x_j^{(t)}-y_{m+j}^{(t)}\right|\right\}_{t=0}^{\infty}$ nor $\left\{\left|x_{m+j}^{(t)}-y_j^{(t)}\right|\right\}_{t=0}^{\infty}$ converges to zero for $j=1,2,\ldots,m$.

- Since

$$2-2\cos\frac{2m\pi}{2m+1}<6-2\cos\frac{2m\pi}{2m+1},$$

we have

$$\max_{1\le k\le m}\left\{6-2\cos\frac{2k\pi}{2m+1},\ 2-2\cos\frac{2k\pi}{2m+1}\right\}=6-2\cos\frac{2m\pi}{2m+1}.$$

Thus, the condition $\limsup_{t\to\infty} \rho\left(\widehat{H}_t\right) < 1/\Gamma$ in Theorem 4.8 is replaced by $0 < \gamma < 1/\left(3 - \cos\frac{2m\pi}{2m+1}\right)$. This condition is sharp since when $\gamma = 1/\left(3 - \cos\frac{2m\pi}{2m+1}\right)$, the number $-1$ is an eigenvalue of the matrix $\widehat{H}_t$ and the corresponding eigenvector is

$$\widetilde{u} = \left(\sin\frac{2m\pi}{2m+1}\widehat{u}^\dagger, \sin\frac{4m\pi}{2m+1}\widehat{u}^\dagger, \ldots, \sin\frac{2m^2\pi}{2m+1}\widehat{u}^\dagger\right)^\dagger.$$

If we take $f$ to be the identity function and let

$$y_1 = \sin\frac{2m\pi}{2m+1},\ \ y_{2m+1} = \sin\frac{4m\pi}{2m+1},$$

$$x_{k+2}^{(0)} = \sin\frac{2mk\pi}{2m+1},\ \text{ for } k = 1, 2, \ldots, m,$$

and

$$y_{2m+3-k}^{(0)} = \sin\frac{2mk\pi}{2m+1},\ \text{ for } k = 3, 4, \ldots, m,$$

then since

$$\begin{pmatrix} x_1^{(t)} - x_3^{(t)} \\ y_1^{(t)} - y_3^{(t)} \\ x_{2m+1}^{(t)} - x_4^{(t)} \\ y_{2m+1}^{(t)} - y_4^{(t)} \\ \cdots \\ x_{m+3}^{(t)} - x_{m+2}^{(t)} \\ y_{m+3}^{(t)} - y_{m+2}^{(t)} \end{pmatrix} = \left(\widehat{H}_t\right)^t \begin{pmatrix} x_1^{(0)} - x_3^{(0)} \\ y_1^{(0)} - y_3^{(0)} \\ x_{2m+1}^{(0)} - x_4^{(0)} \\ y_{2m+1}^{(0)} - y_4^{(0)} \\ \cdots \\ x_{m+3}^{(0)} - x_{m+2}^{(0)} \\ y_{m+3}^{(0)} - y_{m+2}^{(0)} \end{pmatrix} = \left(\widehat{H}_t\right)^t \widetilde{u} = (-1)^t\,\widetilde{u},$$

we see that neither $\left\{\left|x_1^{(t)} - x_3^{(t)}\right|\right\}_{t=0}^\infty$, $\left\{\left|y_1^{(t)} - y_3^{(t)}\right|\right\}_{t=0}^\infty$, $\left\{\left|x_{2m+5-j}^{(t)} - x_j^{(t)}\right|\right\}_{t=0}^\infty$ nor $\left\{\left|y_{2m+5-j}^{(t)} - y_j^{(t)}\right|\right\}_{t=0}^\infty$ converges to zero for $j = 4, 5, \ldots, m+2$.

- Since

$$2 - 2\cos\frac{2m\pi}{2m+1} < 4 < 6 - 2\cos\frac{2m\pi}{2m+1},$$

we have

$$\max_{1\le k\le m}\left\{6 - 2\cos\frac{2k\pi}{2m+1},\ 4,\ 2 - 2\cos\frac{2k\pi}{2m+1}\right\} = 6 - 2\cos\frac{2m\pi}{2m+1}.$$

Thus, the condition $\limsup_{t\to\infty} \rho\left(\widehat{K}_t\right) < 1/\Gamma$ in Theorem 4.9 is replaced by $0 < \gamma < 1/\left(3 - \cos\frac{2m\pi}{2m+1}\right)$. This condition is sharp since when $\gamma = 1/\left(3 - \cos\frac{2m\pi}{2m+1}\right)$, the number $-1$ is an eigenvalue of the matrix $\widehat{K}_t$ and the corresponding eigenvector is

$$\widetilde{u} = \left(-\cos\frac{2m\pi}{2m+1}\widehat{u}^\dagger, -\cos\frac{4m\pi}{2m+1}\widehat{u}^\dagger, \ldots, -\cos\frac{2m^2\pi}{2m+1}\widehat{u}^\dagger, 1\right)^\dagger.$$

If we take $f$ to be the identity function and let

$$x_1^{(0)} = -y_1^{(0)} = \cos\frac{2m\pi}{2m+1},\ x_{2m+1}^{(0)} = -y_{2m+1}^{(0)} = \cos\frac{4m\pi}{2m+1},\ x_2^{(0)} = 1,\ y_2^{(0)} = 0,$$

$$x_{2m+3-k}^{(0)} = -y_{2m+3-k}^{(0)} = \cos\frac{2mk\pi}{2m+1},\ \text{for } k = 3, 4, \ldots, m,$$

and

$$x_k^{(0)} = y_k^{(0)} = 0,\ \text{for } k = 3, 4, \ldots, m+2,$$

then since

$$\begin{pmatrix} x_1^{(t)} - y_3^{(t)} \\ y_1^{(t)} - x_3^{(t)} \\ x_{2m+1}^{(t)} - y_4^{(t)} \\ y_{2m+1}^{(t)} - x_4^{(t)} \\ \cdots \\ x_{m+3}^{(t)} - y_{m+2}^{(t)} \\ y_{m+3}^{(t)} - x_{m+2}^{(t)} \\ x_2^{(t)} - y_2^{(t)} \end{pmatrix} = \left(\widehat{K}_t\right)^t \begin{pmatrix} x_1^{(0)} - y_3^{(0)} \\ y_1^{(0)} - x_3^{(0)} \\ x_{2m+1}^{(0)} - y_4^{(0)} \\ y_{2m+1}^{(0)} - x_4^{(0)} \\ \cdots \\ x_{m+3}^{(0)} - y_{m+2}^{(0)} \\ y_{m+3}^{(0)} - x_{m+2}^{(0)} \\ x_2^{(0)} - y_2^{(0)} \end{pmatrix} = \left(\widehat{K}_t\right)^t \widetilde{u} = (-1)^t \widetilde{u},$$

we see that neither $\left\{\left|x_1^{(t)} - y_3^{(t)}\right|\right\}_{t=0}^{\infty}$, $\left\{\left|x_3^{(t)} - y_1^{(t)}\right|\right\}_{t=0}^{\infty}$, $\left\{\left|x_2^{(t)} - y_2^{(t)}\right|\right\}_{t=0}^{\infty}$ nor $\left\{\left|x_{2m+5-j}^{(t)} - y_j^{(t)}\right|\right\}_{t=0}^{\infty}$, $\left\{\left|x_j^{(t)} - y_{2m+5-j}^{(t)}\right|\right\}_{t=0}^{\infty}$, where $j = 4, 5, \ldots, m+2$, can converge to zero.

We conclude our investigation with the following remark: Suppose $n = 2m$ where $m \geq 3$. When $0 < \gamma < 1/4$, every solution of (2) is (fully) synchronized; and when $1/4 \leq \gamma < 2/7$, every solution of (2) is $\{x_1, x_3, \ldots, x_{2m-1}, y_2, y_4, \ldots, y_{2m}\}$ and $\{x_2, x_4, \ldots, x_{2m}, y_1, y_3, \ldots, y_{2m-1}\}$ synchronized. But in general, (2) is not $\{x_1, x_3, \ldots, x_{2m-1}, y_1, y_3, \ldots, y_{2m-1}\}$, $\{x_2, x_4, \ldots, x_{2m}, y_2, y_4, \ldots, y_{2m}\}$, $\{x_1, x_2, \ldots, x_{2m}\}$, nor $\{y_1, y_2, \ldots, y_{2m}\}$ synchronized. In general, $\left\{\left|x_i^{(t)} - y_i^{(t)}\right|\right\}$ does not converge to 0 for all $i = 1, \ldots, n$; $\left\{\left|x_j^{(t)} - x_{m+j}^{(t)}\right|\right\}$ does not converge to 0 for all $j = 1, \ldots, m$ (if $m$ is odd); $\left\{\left|y_j^{(t)} - y_{m+j}^{(t)}\right|\right\}$ does not converge to zero for all $j = 1, 2, \ldots, m$ (if $m$ is odd); $\left\{\left|x_j^{(t)} - y_{m+j}^{(t)}\right|\right\}$ does not converge to zero for $j = 1, 2, \ldots, m$ (if $m$ is even); $\left\{\left|x_{m+j}^{(t)} - y_j^{(t)}\right|\right\}$ does not converge to zero for $j = 1, 2, \ldots, m$ (if $m$ is even). We give some data for a neural network in which $n = 6$:

| $\gamma = 0.25$ | $x_1^{(t)}$ | $x_2^{(t)}$ | $x_3^{(t)}$ | $x_4^{(t)}$ | $x_5^{(t)}$ | $x_6^{(t)}$ |
|---|---|---|---|---|---|---|
| $t = 0$ | 1 | 3 | 1 | 3 | 1 | 3 |
| $t = 1$ | 2 | 3 | 2 | 3 | 2 | 3 |
| $t = 2$ | 2 | 3 | 2 | 3 | 2 | 3 |
| $t = 10$ | 2 | 3 | 2 | 3 | 2 | 3 |
| $t = 100$ | 2 | 3 | 2 | 3 | 2 | 3 |

| $\gamma = 0.25$ | $y_1^{(t)}$ | $y_2^{(t)}$ | $y_3^{(t)}$ | $y_4^{(t)}$ | $y_5^{(t)}$ | $y_6^{(t)}$ |
|---|---|---|---|---|---|---|
| $t = 0$ | 1 | 5 | 1 | 5 | 1 | 5 |
| $t = 1$ | 3 | 2 | 3 | 2 | 3 | 2 |
| $t = 2$ | 3 | 2 | 3 | 2 | 3 | 2 |
| $t = 10$ | 3 | 2 | 3 | 2 | 3 | 2 |
| $t = 100$ | 3 | 2 | 3 | 2 | 3 | 2 |

This is an example for $x_1^{(\infty)} = x_3^{(\infty)} = x_5^{(\infty)} = x_2^{(\infty)} = x_4^{(\infty)} = x_6^{(\infty)} = 2$ and $x_2^{(\infty)} = x_4^{(\infty)} = x_6^{(\infty)} = x_1^{(\infty)} = x_3^{(\infty)} = x_5^{(\infty)} = 3$. But $x_i^{(\infty)} \neq y_i^{(\infty)}$ for $i = 1, 2, \ldots, 6$, $x_j^{(\infty)} \neq x_{3+j}^{(\infty)}$, and $y_j^{(\infty)} \neq y_{3+j}^{(\infty)}$ for $j = 1, 2, 3$.

If $n = 2m + 1$, where $m \geq 2$, every solution of (2) is synchronized if $0 < \gamma < 1/\left(3 - \cos \frac{2m\pi}{2m+1}\right)$. If we let $m \to \infty$, then whether $n$ is even or odd, when $0 < \gamma < 1/4$, every solution of (2) is (fully) synchronized. Furthermore, the conditions are sharp.

# 6 Appendix

We collect here the eigenvalues and eigenvectors of matrices used in the previous discussions. They can be verified in a straightforward manner and details can be found in [6]. We denote the $n$ by $n$ identity matrix by $I_n$, $J$, $U$ and $V$ are respectively

$$J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; U = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} \text{ and } V = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}.$$

The matrix $J$ has eigenvalues $-1$ and 1 with the corresponding (independent) eigenvectors $\widehat{u} = (-1, 1)^\dagger$ and $\widehat{v} = (1, 1)^\dagger$ respectively. The matrix $U$ has eigenvalue 0 and 1 with corresponding eigenvectors $(0, 1)^\dagger$ and $\widehat{u}$ respectively. The matrix $V$ has eigenvalues 0 and $-1$ with eigenvectors $(1, 0)^\dagger$ and $\widehat{u}$ respectively. The matrix $U^\dagger$ has eigenvalues 0 and 1 with corresponding eigenvectors $\widehat{v}$ and $(1, 0)^\dagger$ respectively. The matrix $V^\dagger$ has eigenvalues 0 and $-1$ with corresponding eigenvectors $\widehat{v}$ and $(0, 1)^\dagger$ respectively.

- First of all, the eigenvalues and the corresponding eigenvectors of the matrix

$$A_n = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}_{n \times n}, \; n \geq 3,$$

  are well known [3, 4, 5] and given respectively by

$$\lambda_k(A_n) = 2 \cos \frac{2k\pi}{n}, \; k = 1, 2, \ldots, n$$

  and $u^{(k)} = \left(u_1^{(k)}, \ldots, u_n^{(k)}\right)^\dagger$, $1 \leq k \leq n$, where

$$u_j^{(k)} = c_1 \cos \frac{2jk\pi}{n} + c_2 \sin \frac{2jk\pi}{n}, \; j = 1, 2, \ldots, n,$$

where $c_1$ and $c_2$ are not both equal to zero.

- The matrix
$$\begin{pmatrix} 2J & I_2 & & 0 \\ I_2 & 2J & \ddots & \\ & \ddots & \ddots & I_2 \\ 0 & & I_2 & 2J \end{pmatrix}_{(2m-2)\times(2m-2)}, \; m \geq 3,$$
has the eigenvalues
$$-2+2\cos\frac{k\pi}{m} \text{ and } 2+2\cos\frac{k\pi}{m}, \;\; k=1,\ldots,m-1$$
with the corresponding eigenvectors
$$u^{(k)} = \left(u_1^{(k)}\widehat{u}^\dagger,\ldots,u_{m-1}^{(k)}\widehat{u}^\dagger\right)^\dagger \text{ and } v^{(k)} = \left(v_1^{(k)}\widehat{v}^\dagger,\ldots,v_{m-1}^{(k)}\widehat{v}^\dagger\right)^\dagger$$
respectively, where
$$u_j^{(k)} = v_j^{(k)} = \sin\frac{kj\pi}{m}, \;\; j=1,\ldots,m-1,$$
for $k=1,\ldots,m-1$.

- The matrix
$$\begin{pmatrix} 2J & I_2 & & & & U \\ I_2 & 2J & \ddots & 0 & & 0 \\ & \ddots & \ddots & \ddots & & \vdots \\ & 0 & \ddots & \ddots & I_2 & 0 \\ & & & I_2 & 2J & V \\ U^\dagger & 0 & \cdots & 0 & V^\dagger & -2I_2 \end{pmatrix}_{2m\times 2m}, \; m \geq 3,$$
has the eigenvalues
$$-2+2\cos\frac{k\pi}{m} \text{ and } 2\cos\frac{k\pi}{m}, \;\; k=1,\ldots,m$$
with the corresponding eigenvectors
$$u^{(k)} = \begin{cases} \left(u_1^{(k)}\widehat{u}^\dagger,\ldots,u_{m-1}^{(k)}\widehat{u}^\dagger,\widehat{u}^\dagger\right)^\dagger & \text{if } m \text{ is odd} \\ \left(u_1^{(k)}\widehat{u}^\dagger,\ldots,u_{m-1}^{(k)}\widehat{u}^\dagger,\widehat{v}^\dagger\right)^\dagger & \text{if } m \text{ is even} \end{cases}, \; 1 \leq k \leq m,$$
and
$$v^{(k)} = \left(v_1^{(k)}\widehat{v}^\dagger,\ldots,v_{m-1}^{(k)}\widehat{v}^\dagger,0\widehat{v}^\dagger\right)^\dagger, \; 1 \leq k \leq m-1,$$

$$v^{(m)} = \left(-\widehat{u}^{\dagger}, \ldots, -\widehat{u}^{\dagger}, \widehat{v}^{\dagger}\right)^{\dagger}$$

respectively, where

$$u_j^{(k)} = \begin{cases} \cos\frac{kj\pi}{m} & \text{if } m \text{ is odd} \\ -\cos\frac{kj\pi}{m} & \text{if } m \text{ is even} \end{cases}, \; j = 1, \ldots, m,$$

for $k = 1, 2, \ldots, m$, and

$$v_j^{(k)} = \sin\frac{kj\pi}{m}, \; j = 1, \ldots, m-1,$$

for $k = 1, 2, \ldots, m-1$.

- Then the matrix

$$\begin{pmatrix} -I_2 + 2J & I_2 & & & \\ I_2 & 2J & \ddots & 0 & \\ & \ddots & \ddots & \ddots & \\ & 0 & \ddots & 2J & I_2 \\ & & & I_2 & -I_2 + 2J \end{pmatrix}_{2m\times 2m}, \; m \geq 3,$$

has the eigenvalues

$$-2 + 2\cos\frac{k\pi}{m} \text{ and } 2 + 2\cos\frac{k\pi}{m}, \; k = 1, \ldots, m$$

with the corresponding eigenvectors

$$u^{(k)} = \left(u_1^{(k)}\widehat{u}^{\dagger}, \ldots, u_m^{(k)}\widehat{u}^{\dagger}\right)^{\dagger} \text{ and } v^{(k)} = \left(v_1^{(k)}\widehat{v}^{\dagger}, \ldots, v_m^{(k)}\widehat{v}^{\dagger}\right)^{\dagger}$$

respectively, where

$$u_j^{(k)} = v_j^{(k)} = \sin\frac{k\,(2j-1)\,\pi}{2m}, \; j = 1, \ldots, m,$$

for $k = 1, 2, \ldots, m$.

- The matrix

$$\begin{pmatrix} J & I_2 & & & \\ I_2 & 2J & \ddots & 0 & \\ & \ddots & \ddots & \ddots & \\ & 0 & \ddots & 2J & I_2 \\ & & & I_2 & J \end{pmatrix}_{2m\times 2m}, \; m \geq 3,$$

has the eigenvalues

$$-2 + 2\cos\frac{(k-1)\,\pi}{m} \text{ and } 2 + 2\cos\frac{k\pi}{m}, \; k = 1, \ldots, m,$$

with the corresponding eigenvectors

$$u^{(k)} = \left(u_1^{(k)}\widehat{u}^\dagger, \ldots, u_m^{(k)}\widehat{u}^\dagger\right)^\dagger \text{ and } v^{(k)} = \left(v_1^{(k)}\widehat{v}^\dagger, \ldots, v_m^{(k)}\widehat{v}^\dagger\right)^\dagger,$$

where

$$u_j^{(k)} = \cos\frac{(k-1)(2j-1)\pi}{m} \text{ and } v_j^{(k)} = \sin\frac{k(2j-1)\pi}{m},\ j = 1, \ldots, m$$

for $k = 1, 2, \ldots, m$.

- The matrix

$$\begin{pmatrix} 2J & I_2 & & & -I_2 \\ I_2 & 2J & \ddots & 0 & \\ & \ddots & \ddots & \ddots & \\ & 0 & \ddots & 2J & I_2 \\ -I_2 & & & I_2 & 2J \end{pmatrix}_{2m\times 2m},\ m \geq 3,$$

has the eigenvalues

$$-2 + 2\cos\frac{(2k-1)\pi}{m} \text{ and } 2 + 2\cos\frac{(2k-1)\pi}{m},\ k = 1, \ldots, m$$

with the corresponding eigenvectors

$$u^{(k)} = \left(u_1^{(k)}\widehat{u}^\dagger, \ldots, u_m^{(k)}\widehat{u}^\dagger\right)^\dagger \text{ and } v^{(k)} = \left(v_1^{(k)}\widehat{v}^\dagger, \ldots, v_m^{(k)}\widehat{v}^\dagger\right)^\dagger,\ 1 \leq k \leq m$$

where

$$u_j^{(k)} = v_j^{(k)} = c_1\cos\frac{(2k-1)j\pi}{m} + c_2\sin\frac{(2k-1)j\pi}{m},\ j = 1, \ldots, m,$$

where $c_1$ and $c_2$ are not both equal to zero.

- The matrix

$$\begin{pmatrix} 2J & I_2 & & & -J \\ I_2 & 2J & \ddots & 0 & \\ & \ddots & \ddots & \ddots & \\ & 0 & \ddots & 2J & I_2 \\ -J & & & I_2 & 2J \end{pmatrix}_{2m\times 2m},\ m \geq 3,$$

has the eigenvalues

$$-2 + 2\cos\frac{2k\pi}{m} \text{ and } 2 + 2\cos\frac{(2k-1)\pi}{m},\ k = 1, \ldots, m$$

with the corresponding eigenvectors

$$u^{(k)} = \left(u_1^{(k)}\widehat{u}^\dagger, \ldots, u_m^{(k)}\widehat{u}^\dagger\right)^\dagger \text{ and } v^{(k)} = \left(v_1^{(k)}\widehat{v}^\dagger, \ldots, v_m^{(k)}\widehat{v}^\dagger\right)^\dagger,\ 1 \leq k \leq m$$

where

$$u_j^{(k)} = c_1 \cos \frac{2jk\pi}{m} + c_2 \sin \frac{2jk\pi}{m},\ j = 1, \ldots, m,$$

and

$$v_j^{(k)} = c_1 \cos \frac{(2k-1)\, j\pi}{m} + c_2 \sin \frac{(2k-1)\, j\pi}{m},\ j = 1, ..., m,$$

where $c_1$ and $c_2$ are not both equal to zero.

- The matrix

$$\begin{pmatrix} 2J & I_2 & & & \\ I_2 & 2J & \ddots & 0 & \\ & \ddots & \ddots & \ddots & \\ & 0 & \ddots & 2J & I_2 \\ & & & I_2 & -I_2 + 2J \end{pmatrix}_{2m\times 2m},\ m \geq 2,$$

has eigenvalues

$$-2 + 2\cos \frac{2k\pi}{2m+1} \text{ and } 2 + 2\cos \frac{2k\pi}{2m+1},\ k = 1, \ldots, m$$

with the corresponding eigenvectors

$$u^{(k)} = \left(u_1^{(k)}\widehat{u}^\dagger, \ldots, u_m^{(k)}\widehat{u}^\dagger\right)^\dagger \text{ and } v^{(k)} = \left(v_1^{(k)}\widehat{v}^\dagger, \ldots, v_m^{(k)}\widehat{v}^\dagger\right)^\dagger,\ 1 \leq k \leq m$$

where

$$u_j^{(k)} = v_j^{(k)} = \sin \frac{2jk\pi}{2m+1},\ j = 1, \ldots, m,$$

for $k = 1, 2, ..., m$.

- The matrix

$$\begin{pmatrix} 2J & I_2 & & & & -\widehat{u} \\ I_2 & 2J & \ddots & & 0 & 0 \\ & \ddots & \ddots & \ddots & & \vdots \\ & 0 & \ddots & 2J & I_2 & \vdots \\ & & & I_2 & J & 0 \\ -\widehat{u}^\dagger & 0 & \cdots & \cdots & 0 & -2 \end{pmatrix}_{(2m+1)\times(2m+1)},\ m \geq 2,$$

has eigenvalues

$$-2 + 2\cos \frac{2k\pi}{2m+1},\ k = 0, 1, \ldots, m,$$

and

$$2 + 2\cos \frac{2k\pi}{2m+1},\ k = 1, \ldots, m,$$

with the corresponding eigenvectors

$$u^{(k)} = \left(u_1^{(k)}\widehat{u}^\dagger, \ldots, u_m^{(k)}\widehat{u}^\dagger, 1\right)^\dagger, \ 0 \le k \le m,$$

and

$$v^{(k)} = \left(v_1^{(k)}\widehat{v}^\dagger, \ldots, v_m^{(k)}\widehat{v}^\dagger, 0\right)^\dagger, \ 1 \le k \le m$$

where

$$u_j^{(k)} = -\cos\frac{2kj\pi}{2m+1} \text{ and } v_j^{(k)} = \sin\frac{2kj\pi}{2m+1}, \ j = 1, \ldots, m.$$

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