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Complements Of Ostrowski Type Inequalities*

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Abstract

Using a new mean value Theorem for the remainder in Taylor's formula, some estimates of difference and sum of two integral means on $[a,b]$, $[c,d]$ with $(a,b)\cap(c,d)=\varnothing$ are obtained. These results are used to obtain complements of two well known generalizations of Ostrowski's inequality for n-time differentiable functions.

1 Introduction

In 1938, Ostrowski proved the following inequality [6]:

THEOREM 1. Let $f:[a,b]\to\mathbf{R}$ be continuous on $[a,b]$ and differentiable on (a,b) and assume that $|f'(x)|\le M$ for all $x\in(a,b)$. Then

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\le\left[\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a)\,M \tag{1}$$

for all $x\in[a,b]$. The constant $\frac{1}{4}$ is the best possible.

For some related results we refer to [4], [5] and [7].

G. V. Milovanovic and J. E. Pecaric (see for example [5, p. 468]), and P. Cerone, S. S. Dragomir and J. Roumeliotis (see [2]) proved respectively the following generalizations of Ostrowski's inequality for n-time differentiable functions:

THEOREM 2. Let $f:[a,b]\to\mathbf{R}$ be a n-time differentiable function, $n\ge 1$, such that $\left\|f^{(n)}\right\|_\infty<\infty$. Then for all $x\in[a,b]$,

$$\begin{aligned}&\left|\frac{f(x)}{n}+\sum_{k=1}^{n-1}\frac{(n-k)}{k!n}\frac{(x-a)^k f^{(k-1)}(a)-(x-b)^k f^{(k-1)}(b)}{(b-a)}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\\ \le\;&\frac{\left\|f^{(n)}\right\|_\infty}{(n+1)!n}.\frac{(x-a)^{n+1}+(b-x)^{n+1}}{b-a},\end{aligned} \tag{2}$$

and

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(x)\,dx-\sum_{k=0}^{n-1}\frac{(b-x)^{k+1}+(-1)^k(x-a)^{k+1}}{(k+1)!\,(b-a)}f^{(k)}(x)\right|\\ \le\;&\frac{\left\|f^{(n)}\right\|_\infty}{(n+1)!}\frac{(x-a)^{n+1}+(b-x)^{n+1}}{b-a}.\end{aligned} \tag{3}$$

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In [1] N. S. Barnet, P. Cerone, S. S. Dragomir and A. M. Fink estimated the difference of two integral means:

THEOREM 3. Let $f : [a, b] \to \mathbf{R}$ be an absolutely continuous mapping with the property that $f' \in L_\infty [a, b]$. Then for $[c, d] \subset [a, b]$, we have the inequality

$$\left| \frac{1}{b-a} \int_a^b f(t)\, dt - \frac{1}{d-c} \int_c^d f(t)\, dt \right| \leq \frac{1}{2} (b + c - a - d) \|f'\|_\infty . \tag{4}$$

The constant $\frac{1}{2}$ is the best possible.

For $c = d = x$ we can assume that $\frac{1}{d-c} \int_c^d f(x)\, dx = f(x)$, as a limit case, so that (4) reduces to the Ostrowski inequality (1). So inequality (4) can be regarded as a generalization of (1).

In the recent paper [3], a complement of inequality (4) is obtained:

THEOREM 4. Let $c < d \leq a < b$ and F be a continuous function on $[a, b]$ with bounded derivative on (a, b). Then,

$$\begin{aligned} \frac{1}{2} (a + b - c - d) \inf_{x \in (a,b)} F'(x) &\leq \frac{1}{b-a} \int_a^b F(t)\, dt - \frac{1}{d-c} \int_c^d F(t)\, dt \\ &\leq \frac{1}{2} (a + b - c - d) \sup_{x \in (a,b)} F'(x) . \end{aligned} \tag{5}$$

The constant $\frac{1}{2}$ in (5) is the best possible.

A limit case of inequality (5) leads to the following complement of Ostrowski's inequality (see [3]):

THEOREM 5. Let f be a differentiable mapping with bounded f' on the interior $\overset{\circ}{I}$ of an interval $I \subset \mathbf{R}$ and let $a, b \in I$ with $b > a$. Then for all $x \in I - (a, b)$,

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)\, dt \right| \leq \frac{|b + a - 2x|}{2} \|f'\|_{\infty,\ (\min\{a,x\}, \max\{b,x\})} . \tag{6}$$

The constant $\frac{1}{2}$ in (6) is the best possible.

In this paper, we combine a new mean value Theorem for the remainder in Taylor's formula with an integral identity involving the difference of two Taylor's remainder in order to generalize inequality (5) for n-time differentiable mappings. These results are used to obtain sharp complements of the inequalities (2) and (3), which are also generalizations of inequality (6).

2 Estimates of Difference and Sum of Two Integral Means

Let $\{.\}$ be a finite subset of $\mathbf{R}$. Then we denote by $(\{.\})$ the open interval $(\min \{.\}, \max \{.\})$ and by $[\{.\}]$ the closed interval $[\min \{.\}, \max \{.\}]$.

As usual, $R_n(f;a,x)$ denotes the remainder in Taylor's formula, that is,

$$R_n(f;a,x) = f(x) - \sum_{k=0}^{n} \frac{(x-a)^k}{k!} f^{(k)}(a). \tag{7}$$

LEMMA 1. Let a, b be any two distinct real numbers. Let $f \in C^n[\{a,b\}]$ be a mapping, which is $(n+1)$-time differentiable on $(\{a,b\})$. Then for any $t \in [\{a,b\}]$ there exist a number $\xi \in (\{a,b\})$ such that

$$R_n(f;a,b) - R_n(f;a,t) = \frac{(b-a)^{n+1} - (t-a)^{n+1}}{(n+1)!} f^{(n+1)}(\xi). \tag{8}$$

PROOF. Consider the mappings $R, h : [\{a,b\}] \to \mathbf{R}$ defined by $R(x) = R_n(f;a,x)$ and $h(x) = (x-a)^{n+1}$. Then we have

$$R'(x) = f'(x) - \sum_{k=1}^{n} \frac{(x-a)^{k-1}}{(k-1)!} f^{(k)}(a),$$

which by $k-1=i$ can be rewritten as

$$R'(x) = f'(x) - \sum_{i=0}^{n-1} \frac{(x-a)^i}{i!} f'^{(i)}(a) = R_{n-1}(f';a,x). \tag{9}$$

Further, from the condition $t \in [\{a,b\}]$ we have $a \notin (\{t,b\})$. So, $h'(x) = (n+1)(x-a)^n \neq 0$ for all $x \in (\{t,b\})$. Therefore we can apply the Cauchy's mean value theorem for the functions R, h on $[\{t,b\}]$ to obtain

$$\frac{R(b) - R(t)}{h(b) - h(t)} = \frac{R'(\rho)}{h'(\rho)}$$

for some $\rho \in (\{t,b\})$, which, by using (9), can be rewritten as

$$\frac{R_n(f;a,b) - R_n(f;a,t)}{(b-a)^{n+1} - (t-a)^{n+1}} = \frac{R_{n-1}(f';a,\rho)}{(n+1)(\rho-a)^n}. \tag{10}$$

Now, using the Taylor-Lagrange formula we have that for some $\xi \in (\{a,\rho\}) \subset (\{a,b\})$ holds

$$R_{n-1}(f';a,\rho) = \frac{(\rho-a)^n}{n!} f^{(n+1)}(\xi). \tag{11}$$

Finally, setting (11) in (10), we get (8).

LEMMA 2. Let f be a $(n+1)$ −time differentiable mapping on (a,b) with $f^{(n+1)}$ integrable on (a,b). Then for any $u, w, y, z \in [a,b]$ with $w \neq u$ and $z \neq y$ the following identity holds:

$$\begin{aligned} &\frac{1}{(w-u)(z-y)} \int_u^w \left(R_n(f;x,z) - R_n(f;x,y)\right) dx \\ = \; &\frac{f(z) - f(y)}{z-y} - n\frac{f(w) - f(u)}{w-u} - \sum_{k=1}^{n-1} \frac{n-k}{(k+1)!} L_k(f',z,y;w,u), \end{aligned} \tag{12}$$

where $L_k(.,.,.;.,.)$, $k \geq 1$, is defined by

$$\begin{aligned} & L_k(g;\alpha,\beta;\gamma,\delta) \\ = & \frac{\left((\alpha-\gamma)^{k+1}-(\beta-\gamma)^{k+1}\right)g^{(k-1)}(\gamma)-\left((\alpha-\delta)^{k+1}-(\beta-\delta)^{k+1}\right)g^{(k-1)}(\delta)}{(\alpha-\beta)(\gamma-\delta)}. \end{aligned}$$

PROOF. From the definition of $R_n(f;x,z)$ by (7), we have

$$\begin{aligned} & \int_u^w (R_n(f;x,z)-R_n(f;x,y))\,dx \\ = & \int_u^w \left(f(z)-f(y)-\sum_{k=1}^{n}\frac{(z-x)^k-(y-x)^k}{k!}f^{(k)}(x)\right)dx \\ = & (w-u)(f(z)-f(y))-\sum_{k=1}^{n} I_k, \qquad (13) \end{aligned}$$

where

$$I_k=\int_u^w \frac{(z-x)^k-(y-x)^k}{k!}f^{(k)}(x)\,dx.$$

For $k \geq 1$, by using integration by parts we obtain

$$\begin{aligned} & I_{k+1}-I_k \\ = & \frac{\left((z-w)^{k+1}-(y-w)^{k+1}\right)f^{(k)}(w)-\left((z-u)^{k+1}-(y-u)^{k+1}\right)f^{(k)}(u)}{(k+1)!} \\ = & \frac{(w-u)(z-y)}{(k+1)!}L_k(f',z,y;w,u). \end{aligned}$$

Therefore we have

$$\begin{aligned} \sum_{k=1}^{n} I_k = & nI_1+\sum_{k=1}^{n-1}(n-k)(I_{k+1}-I_k) \\ = & n(z-y)(f(w)-f(u))+\sum_{k=1}^{n-1}\frac{n-k}{(k+1)!}(w-u)(z-y)L_k(f',z,y;w,u) \qquad (14) \end{aligned}$$

Putting (14) in (13) and dividing the resulting identity by $(w-u)(z-y)$ we get identity (12).

THEOREM 6. Let u,w,y,z be real numbers such that $u < w \leq y < z$. Let $f \in C^{n-1}[u,z]$ be a n-time differentiable mapping on the interval (u,z) with $f^{(n)}$ bounded and integrable on (u,z). Then we have the inequalities,

$$\begin{aligned} & T_{n+2}(z,y;w,u)\gamma_n \\ \leq & (-1)^n\left(\frac{1}{w-u}\int_u^w f(s)\,ds-\frac{n}{z-y}\int_y^z f(s)\,ds-\sum_{k=1}^{n-1}\frac{n-k}{(k+1)!}L_k(f,w,u;z,y)\right) \\ \leq & T_{n+2}(z,y;w,u)\Gamma_n, \qquad (15) \end{aligned}$$

and

$$\begin{aligned}
&T_{n+2}(z,y;w,u)\,\gamma_n \\
\leq\ & \frac{1}{z-y}\int_y^z f(s)\,ds - \frac{n}{w-u}\int_u^w f(s)\,ds - \sum_{k=1}^{n-1}\frac{n-k}{(k+1)!}L_k(f,z,y;w,u) \\
\leq\ & T_{n+2}(z,y;w,u)\,\Gamma_n, \qquad (16)
\end{aligned}$$

where $\gamma_n := \inf_{t\in(u,z)} f^{(n)}(t)$, $\Gamma_n := \sup_{t\in(u,z)} f^{(n)}(t)$ and

$$T_{n+2}(z,y;w,u) := \frac{(z-u)^{n+2}-(z-w)^{n+2}-(y-u)^{n+2}+(y-w)^{n+2}}{(n+2)!\,(z-y)(w-u)}.$$

The inequalities (15) and (16) are sharp.

PROOF. Let $F:[u,z]\to\mathbf{R}$ be a function defined by

$$F(x) = \int_u^x f(t)\,dt. \qquad (17)$$

Clearly we have that $F\in C^n[u,z]$, F is $(n+1)$-time differentiable on the interval (u,z) with $F^{(n+1)}$ bounded and integrable on (u,z).

Let s be any number in (y,z). Then we have that $w\in(u,s)$. Therefore we can apply Lemma 1 for F by choosing $a=s$, $b=u$, $t=w$, to obtain that for some $\rho\in(u,s)\subseteq(u,z)$,

$$R_n(F;s,u)-R_n(F;s,w) = \frac{(u-s)^{n+1}-(w-s)^{n+1}}{(n+1)!}F^{(n+1)}(\rho),$$

which by using (17) can be rewritten as

$$(-1)^n\left(R_n(F;s,w)-R_n(F;s,u)\right) = \frac{(s-u)^{n+1}-(s-w)^{n+1}}{(n+1)!}f^{(n)}(\rho). \qquad (18)$$

Now from $s>y\geq w>u$ we have that $(s-u)^{n+1}-(s-w)^{n+1}>0$ for all $s\in[y,z]$. Thus, for $s\in[y,z]$ we have

$$\begin{aligned}
\frac{(s-u)^{n+1}-(s-w)^{n+1}}{(n+1)!}\gamma_n &\leq \frac{(s-u)^{n+1}-(s-w)^{n+1}}{(n+1)!}f^{(n)}(\rho) \\
&\leq \frac{(s-u)^{n+1}-(s-w)^{n+1}}{(n+1)!}\Gamma_n,
\end{aligned}$$

which combined with (18) gives,

$$\begin{aligned}
\frac{(s-u)^{n+1}-(s-w)^{n+1}}{(n+1)!}\gamma_n &\leq (-1)^n\left(R_n(F;s,w)-R_n(F;s,u)\right) \\
&\leq \frac{(s-u)^{n+1}-(s-w)^{n+1}}{(n+1)!}\Gamma_n
\end{aligned}$$

for all $s \in [y, z]$.

Integrating this latter estimation with respect to s from y to z and using the identity (12) of Lemma 2, and taking into account that from (17) we have $F' = f$, we get the first conclusion (15). Let now s be any number in (u, w). Since $y \in (s, z)$, similar to above, by using Lemma 1, there is a $\xi \in (s, z) \subset (u, z)$ such that

$$R_n (F; s, z) - R_n (F; s, y) = \frac{(z-s)^{n+1} - (y-s)^{n+1}}{(n+1)!} f^{(n)} (\xi),$$

and since $(z-s)^{n+1} - (y-s)^{n+1} > 0$ for all $s \in [u, w]$, we conclude, that for all $s \in [u, w]$ the following estimation holds

$$\begin{aligned} \frac{(z-s)^{n+1} - (y-s)^{n+1}}{(n+1)!} \gamma_n &\leq R_n (F; s, z) - R_n (F; s, y) \\ &\leq \frac{(z-s)^{n+1} - (y-s)^{n+1}}{(n+1)!} \Gamma_n. \end{aligned} \tag{19}$$

Integrating inequality (19) with respect to s from u to w, we get the second conclusion (16). Choosing $f(x) = x^n$ in (15) and (16), we see that the equalities hold. So inequalities (15) and (16) are sharp.

REMARK 1. For $n = 1$ and $u = c$, $w = d$, $y = a$, $z = b$, both inequalities (15) and (16) are reduced to (5).

COROLLARY 1. Let u, w, x, y, f, γ_n, Γ_n as in Theorem 6. Then we have the estimations:

$$\begin{aligned} 2T_{n+2}(z, y; w, u)\gamma_n &\leq ((-1)^n - n) \left(\frac{1}{w-u} \int_u^w f(s)\,ds + \frac{(-1)^n}{z-y} \int_y^z f(s)\,ds \right) \\ &\quad - \sum_{k=1}^{n-1} \frac{n-k}{(k+1)!} \left((-1)^n L_k (f, w, u; z, y) - L_k (f, z, y; w, u) \right) \\ &\leq 2T_{n+2}(z, y; w, u)\Gamma_n, \end{aligned}$$

$$\begin{aligned} &\left| (n + (-1)^n) \left(\frac{1}{z-y} \int_y^z f(s)\,ds - \frac{(-1)^n}{w-u} \int_u^w f(s)\,ds \right) \right. \\ &\left. - \sum_{k=1}^{n-1} \frac{n-k}{(k+1)!} \left((-1)^n L_k (f, z, y; w, u) - L_k (f, w, u; z, y) \right) \right| \\ \leq\ & T_{n+2}(z, y; w, u)(\Gamma_n - \gamma_n), \end{aligned}$$

and the first estimation is sharp.

Indeed, adding inequalities (15) and (16), we get the first conclusion, while multiplying inequality (15) by -1 and adding the resulting estimation with (16), we readily get the second conclusion.

3 Complements of Ostrowski's Inequality

We have the following result.

THEOREM 7. Let f be a continuous function on an interval $I \subset \mathbf{R}$ such that $f^{(n)} \in L_\infty \overset{\circ}{I}$ and let $a, b \in \overset{\circ}{I}$ $(a < b)$. Then for all $x \in I - (a, b)$ the following inequality holds:

$$\begin{aligned} & \left| \frac{f(x)}{n} - \sum_{k=1}^{n-1} \frac{n-k}{k!n} \cdot \frac{(x-b)^k f^{(k-1)}(b) - (x-a)^k f^{(k-1)}(a)}{b-a} - \frac{1}{b-a} \int_a^b f(s)\, ds \right| \\ \leq \quad & \frac{\left\| f^{(n)} \right\|_{\infty,\ [\min\{a,x\}, \max\{x,b\}]}}{(n+1)!n} \frac{\left| (b-x)^{n+1} - (a-x)^{n+1} \right|}{b-a}, \end{aligned} \tag{20}$$

and inequality (20) is sharp.

PROOF. If $x \geq b$, then by choosing $t \in I$ such that $a < b \leq x < t$ and applying inequality (16) of Theorem 6 for $u = a,\ w = b,\ y = x,\ z = t$, we get

$$\begin{aligned} & \left| \frac{1}{t-x} \int_x^t f(s)\, ds - \frac{n}{b-a} \int_a^b f(s)\, ds - \sum_{k=1}^{n-1} \frac{n-k}{(k+1)!} L_k(f; t, x; b, a) \right| \\ \leq \quad & T_{n+2}(t, x; b, a) \left\| f^{(n)} \right\|_{\infty,\ [a,t]}. \end{aligned}$$

Now letting $t \to x+$, and dividing the resulting estimation with n we readily get (20), and we will omit the details.

If $x \leq a$, then by choosing $t \in I$ such that $x < t \leq a < b$ and applying inequality (15) of Theorem 6 for $u = x,\ w = t,\ y = a,\ z = b$, we get

$$\begin{aligned} & \left| \frac{1}{t-x} \int_x^t f(s)\, ds - \frac{n}{b-a} \int_a^b f(s)\, ds - \sum_{k=1}^{n-1} \frac{n-k}{(k+1)!} L_k(f, t, x; b, a) \right| \\ \leq \quad & T_{n+2}(b, a; t, x) \left\| f^{(n)} \right\|_{\infty,\ [x,b]}. \end{aligned}$$

Now letting $t \to x+$, and dividing the resulting estimation with n we also get (20).

Finally, an easy calculation yields that for $f(x) = x^n$ the equality in (20) holds. So inequality (20) is sharp.

THEOREM 8. Let f be as in Theorem 7. Then for all $x \in I - (a, b)$ we have:

$$\begin{aligned} & \left| \frac{1}{b-a} \int_a^b f(s)\, ds - \sum_{k=0}^{n-1} \frac{(b-x)^{k+1} + (-1)^k (x-a)^{k+1}}{(k+1)!\,(b-a)} f^{(k)}(x) \right| \\ \leq & \frac{\left| (b-x)^{n+1} - (a-x)^{n+1} \right|}{(n+1)!\,(b-a)} \left\| f^{(n)} \right\|_{\infty,\ [\min\{a,x\}, \max\{x,b\}]}, \end{aligned} \tag{21}$$

and inequality (21) is sharp.

PROOF. If $x \geq b$, then by choosing $t \in I$ such that $a < b \leq x < t$ and applying inequality (15) for $u = a$, $w = b$, $y = x$, $z = t$, we clearly get the following inequality:

$$\begin{aligned}
& \int_a^b f(s)\,ds - \frac{n}{t-x}\int_x^t f(s)\,ds - \sum_{k=1}^{n-1} \frac{n-k}{(k+1)!} L_k(f;b,a;t,x) \\
\leq \;& T_{n+2}(t,x;b,a) \left\| f^{(n)} \right\|_{\infty,\ [a,t]}.
\end{aligned}$$

Now letting $t \to x+$, and taking into account, that $L_k(f;b,a;t,x)$ can be rewritten as

$$\begin{aligned}
L_k(f;b,a;t,x) \;=\;& \frac{(b-t)^{k+1}-(a-t)^{k+1}}{b-a}\,\frac{f^{(k-1)}(t)-f^{(k-1)}(x)}{t-x} \\
& + \frac{(b-t)^{k+1}-(a-t)^{k+1}-\left((b-x)^{k+1}-(a-x)^{k+1}\right)}{t-x}\,\frac{f^{(k-1)}(x)}{b-a},
\end{aligned}$$

we get,

$$\begin{aligned}
& \left| \frac{1}{b-a}\int_a^b f(s)\,ds - nf(x) - \sum_{k=1}^{n-1} \frac{(n-k)\left((b-x)^{k+1}-(a-x)^{k+1}\right)}{(k+1)!\,(b-a)} f^{(k)}(x) \right. \\
& \left. + \sum_{k=1}^{n-1} \frac{(n-k)\left((b-x)^{k}-(a-x)^{k}\right)}{k!\,(b-a)} f^{(k-1)}(x) \right| \leq \frac{(x-a)^{n+1}-(x-b)^{n+1}}{(n+1)!\,(b-a)} \left\| f^{(n)} \right\|_{\infty,\ [a,x]}.
\end{aligned} \tag{22}$$

Further, replacing $k-1$ by k in the second sum of (22), we have

$$\begin{aligned}
& \sum_{k=1}^{n-1} \frac{(n-k)\left((b-x)^{k}-(a-x)^{k}\right)}{k!\,(b-a)} f^{(k-1)}(x) \\
& = \sum_{k=0}^{n-2} \frac{(n-k-1)\left((b-x)^{k+1}-(a-x)^{k+1}\right) f^{(k)}(x)}{(k+1)!\,(b-a)} \\
& = (n-1)\,f(x) + \sum_{k=1}^{n-2} \frac{(n-k-1)\left((b-x)^{k+1}-(a-x)^{k+1}\right) f^{(k)}(x)}{(k+1)!\,(b-a)}.
\end{aligned}$$

Finally, using this later relation in (22), we get the desired estimation (21).

If $x \leq a$, then we choose $t \in I$ such that $x < t \leq a < b$ and apply inequality (16) for $u = a$, $w = b$, $y = x$, $z = t$, similarly to above we may prove that inequality (21) also holds and we will omit the details.

Now, an easy calculation yields that for $f(x) = x^n$ the equality in (21) holds. So inequality (21) is sharp.

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