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Generalization Of The Secant Method For Nonlinear Equations*

Avram Sidi†

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Abstract

The secant method is a very effective numerical procedure used for solving nonlinear equations of the form $f(x) = 0$. It is derived via a linear interpolation procedure and employs only values of $f(x)$ at the approximations to the root of $f(x) = 0$, hence it computes $f(x)$ only once per iteration. In this note, we generalize it by replacing the relevant linear interpolant by a suitable $(k + 1)$-point polynomial of interpolation, where k is an integer at least 2. Just as the secant method, this generalization too enjoys the property that it computes $f(x)$ only once per iteration. We provide its error in closed form and analyze its order of convergence. We show that this order of convergence is greater than that of the secant method, and it increases as k increases. We also confirm the theory via an illustrative example.

1 Introduction

Let α be the solution to the equation

$$f(x) = 0. \tag{1}$$

An effective iterative method used for solving (1) that makes direct use of $f(x)$ [but no derivatives of $f(x)$] is the *secant method* that is discussed in many books on numerical analysis. See, for example, Atkinson [1], Henrici [2], Ralston and Rabinowitz [3], and Stoer and Bulirsch [5]. See alo the recent note [4] by the author, in which the treatment of the secant method and those of the Newton–Raphson, regula falsi, and Steffensen methods are presented in a unified manner.

This method is derived by a linear interpolation procedure as follows: Starting with two initial approximations x_0 and x_1 to the solution α of (1), we compute a sequence of approximations $\{x_n\}_{n=0}^{\infty}$, such that the approximation x_{n+1} is determined as the point of intersection (in the x-y plane) of the straight line through the points $(x_n, f(x_n))$ and $(x_{n-1}, f(x_{n-1}))$ with the x-axis. Since the equation of this straight line is

$$y = f(x_n) + \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}} (x - x_n), \tag{2}$$

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†Computer Science Department, Technion - Israel Institute of Technology, Haifa 32000, Israel

x_{n+1} is given as

$$x_{n+1} = x_n - \frac{f(x_n)}{\frac{f(x_n)-f(x_{n-1})}{x_n - x_{n-1}}}. \tag{3}$$

In terms of divided differences, (3) can be written in the form

$$x_{n+1} = x_n - \frac{f(x_n)}{f[x_n, x_{n-1}]}. \tag{4}$$

We recall that, provided $f \in C^2(I)$, where I is an open interval containing α, and $f'(\alpha) \neq 0$, and that x_0 and x_1 are sufficiently close to α, the sequence $\{x_n\}_{n=0}^{\infty}$ converges to α of order at least $(1+\sqrt{5})/2 = 1.618\cdots$.

Another way of obtaining the secant method, of interest to us in the present work, is via a variation of the *Newton–Raphson method*. Recall that in the Newton–Raphson method, we start with an initial approximation x_0 and generate a sequence of approximations $\{x_n\}_{n=0}^{\infty}$ to α through

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, \dots . \tag{5}$$

We also recall that, provided $f \in C^2(I)$, where I is an open interval containing α, and $f'(\alpha) \neq 0$, and that x_0 is sufficiently close to α, the sequence $\{x_n\}_{n=0}^{\infty}$ converges to α of order at least 2. As such, the Newton–Raphson method is extremely effective. To avoid computing $f'(x)$ [note that $f'(x)$ may not always be available or may be costly to compute], and to preserve the excellent convergence properties of the Newton–Raphson method, we replace $f'(x_n)$ in (5) by the approximation $f[x_n, x_{n-1}] = [f(x_n) - f(x_{n-1})]/(x_n - x_{n-1})$. This results in (4), that is, in the secant method. The justification for this approach is as follows: When convergence takes place, that is, when $\lim_{n\to\infty} x_n = \alpha$, the difference $x_n - x_{n-1}$ tends to zero, and this implies that, as n increases, the accuracy of $f[x_n, x_{n-1}]$ as an approximation to $f'(x_n)$ increases as well. Note that the Newton-Raphson iteration requires two function evaluations, one of $f(x)$ and another of $f'(x)$, per iteration. The secant method, on the other hand, requires only one function evaluation per iteration, namely, that of $f(x)$.

In Section 2 of this note, we consider in detail a generalization of the second of the two approaches described above using polynomial interpolation of degree k with $k > 1$. The $(k+1)$-point iterative method that results from this generalization turns out to be very effective. It is of order higher than that of the secant method and requires only one function evaluation per iteration. In Section 3, we analyze this method and determine its order as well. In Section 4, we confirm our theory via a numerical example.

2 Generalization of Secant Method

We start by discussing a known generalization of the secant method (see, for example, Traub [6, Chapters 4, 6, and 10]). In this generalization, we approximate $f(x)$ by the polynomial of interpolation $p_{n,k}(x)$, where $p_{n,k}(x_i) = f(x_i)$, $i = n, n-1, \dots, n-k$, assuming that $x_0, x_1, \dots, x_n$ have all been computed. Following that, we determine

x_{n+1} as a zero of $p_{n,k}(x)$, provided a real solution to $p_{n,k}(x) = 0$ exists. Thus, x_{n+1} is the solution to a polynomial equation of degree k. For $k = 1$, what we have is nothing but the secant method. For $k = 2$, x_{n+1} is one of the solutions to a quadratic equation, and the resulting method is known as the method of Müller. Clearly, for $k \geq 3$, the determination of x_{n+1} is not easy.

This difficulty prompts us to consider the second approach to the secant method we discussed in Section 1, in which we replaced $f'(x_n)$ by the slope of the straight line through the points $(x_n, f(x_n))$ and $(x_{n-1}, f(x_{n-1}))$, that is, by the derivative (at x_n) of the (linear) interpolant to $f(x)$ at x_n and x_{n-1}. We generalize this approach by replacing $f'(x_n)$ by $p'_{n,k}(x_n)$, the derivative at x_n of the polynomial $p_{n,k}(x)$ interpolating $f(x)$ at the points x_{n-i}, $i = 0, 1, \ldots, k$, mentioned in the preceding paragraph, with $k \geq 2$. Because $p_{n,k}(x)$ is a better approximation to $f(x)$ in the neighborhood of x_n, $p'_{n,k}(x_n)$ is a better approximation to $f'(x_n)$ when $k \geq 2$ than $f[x_n, x_{n-1}]$ used in the secant method. In addition, just as the secant method, the new method computes the function $f(x)$ only once per iteration step, the computation being that of $f(x_n)$. Thus, the new method is described by the following $(k+1)$-point iteration:

$$x_{n+1} = x_n - \frac{f(x_n)}{p'_{n,k}(x_n)}, \quad n = k, k+1, \ldots, \tag{6}$$

with $x_0, x_1, \ldots, x_k$ as initial approximations to be provided by the user. Of course, with k fixed, we can start with x_0 and x_1, compute x_2 via the method we have described (with $k = 1$, namely via the secant method), compute x_3 via the method we have described (with $k = 2$), and so on, until we have completed the list $x_0, x_1, \ldots, x_k$.

We now turn to the computational aspects of this method. What we need is a fast method for computing $p'_{n,k}(x_n)$. For this, we write $p_{n,k}(x)$ in Newtonian form as follows:

$$p_{n,k}(x) = f(x_n) + \sum_{i=1}^{k} f[x_n, x_{n-1}, \ldots, x_{n-i}] \prod_{j=0}^{i-1} (x - x_{n-j}). \tag{7}$$

Here $f[x_i, x_{i+1}, \ldots, x_m]$ are divided differences of $f(x)$, and we recall that they can be defined recursively via

$$f[x_i] = f(x_i); \quad f[x_i, x_j] = \frac{f[x_i] - f[x_j]}{x_i - x_j}, \quad x_i \neq x_j, \tag{8}$$

and, for $m > i + 1$, via

$$f[x_i, x_{i+1}, \ldots, x_m] = \frac{f[x_i, x_{i+1}, \ldots, x_{m-1}] - f[x_{i+1}, x_{i+2}, \ldots, x_m]}{x_i - x_m}, \quad x_i \neq x_m. \tag{9}$$

We also recall that $f[x_i, x_{i+1}, \ldots, x_m]$ is a symmetric function of its arguments, that is, it has the same value under any permutation of $\{x_i, x_{i+1}, \ldots, x_m\}$. Thus, in (7),

$$f[x_n, x_{n-1}, \ldots, x_{n-i}] = f[x_{n-i}, x_{n-i+1}, \ldots, x_n].$$

In addition, when $f \in C^m(I)$, where I is an open interval containing the points $z_0, z_1, \ldots, z_m$, whether these are distinct or not, there holds

$$f[z_0, z_1, \ldots, z_m] = \frac{f^{(m)}(\xi)}{m!} \quad \text{for some } \xi \in (\min\{z_i\}, \max\{z_i\}).$$

x_0	f_0			
		f_{01}		
x_1	f_1		f_{012}	
		f_{12}		f_{0123}
x_2	f_2		f_{123}	
		f_{23}		f_{1234}
x_3	f_3		f_{234}	
		f_{34}		f_{2345}
x_4	f_4		f_{345}	
		f_{45}		f_{3456}
x_5	f_5		f_{456}	
		f_{56}		f_{4567}
x_6	f_6		f_{567}	
		f_{67}		
x_7	f_7			

Table 1: Table of divided differences over $\{x_0, x_1, \ldots, x_7\}$ for use to compute x_8 via $p_{7,3}(x)$ in (6). Note that $f_{i,i+1,\ldots,m}$ stands for $f[x_i, x_{i+1}, \ldots, x_m]$ throughout.

Going back to (7), we note that $p_{n,k}(x)$ there is computed by ordering the x_i as $x_n, x_{n-1}, \ldots, x_{n-k}$. This ordering enables us to compute $p'_{n,k}(x_n)$ easily. Indeed, differentiating $p_{n,k}(x)$ in (7), and letting $x = x_n$, we obtain

$$p'_{n,k}(x_n) = f[x_n, x_{n-1}] + \sum_{i=2}^{k} f[x_n, x_{n-1}, \ldots, x_{n-i}] \prod_{j=1}^{i-1} (x_n - x_{n-j}). \tag{10}$$

In addition, note that the relevant divided difference table need not be computed anew each iteration; what is needed is adding a new diagonal (from the south-west to north-east) at the bottom of the existing table. To make this point clear, let us look at the following example: Suppose $k = 3$ and and we have computed x_i, $i = 0, 1, \ldots, 7$. To compute x_8, we use the divided difference table in Table 1. Letting $f_{i,i+1,\ldots,m}$ stand for $f[x_i, x_{i+1}, \ldots, x_m]$, we have

$$x_8 = x_7 - \frac{f(x_7)}{p'_{7,3}(x_7)} = x_7 - \frac{f_7}{f_{67} + f_{567}(x_7 - x_6) + f_{4567}(x_7 - x_6)(x_7 - x_5)}.$$

To compute x_9, we will need the divided differences $f_8, f_{78}, f_{678}, f_{5678}$. Computing first $f_8 = f(x_8)$ with the newly computed x_8, the rest of these divided differences can be computed from the bottom diagonal of Table 1 via the recursion relations

$$f_{78} = \frac{f_7 - f_8}{x_7 - x_8}, \quad f_{678} = \frac{f_{67} - f_{78}}{x_6 - x_8}, \quad f_{5678} = \frac{f_{567} - f_{678}}{x_5 - x_8}, \quad \text{in this order,}$$

and appended to the bottom of Table 1. Actually, we can do even better: Since we need only the bottom diagonal of Table 1 to compute x_8, we need to save only this diagonal, namely, only the entries $f_7, f_{67}, f_{567}, f_{4567}$. Once we have computed x_8 and $f_8 = f(x_8)$, we can overwrite $f_7, f_{67}, f_{567}, f_{4567}$ with $f_8, f_{78}, f_{678}, f_{5678}$. Thus, in general, to be able to compute x_{n+1} via (6), after x_n has been determined, we need to store only the entries $f_n, f_{n-1,n}, \dots, f_{n-k,n-k+1,\dots,n-1,n}$ along with $x_n, x_{n-1}, \dots, x_{n-k}$.

3 Convergence Analysis

We now turn to the analysis of the sequence $\{x_n\}_{n=0}^{\infty}$ that is generated via (6). Since we already know everything concerning the case $k = 1$, namely, the secant method, we treat the case $k \geq 2$. The following theorem gives the main convergence result for the generalized secant method.

THEOREM 3.1. Let α be the solution to the equation $f(x) = 0$. Assume $f \in C^{k+1}(I)$, where I in an open interval containing α, and assume also that $f'(\alpha) \neq 0$, in addition to $f(\alpha) = 0$. Let $x_0, x_1, \dots, x_k$ be distinct initial approximations to α, and generate x_n, $n = k+1, k+2, \dots$, via

$$x_{n+1} = x_n - \frac{f(x_n)}{p'_{n,k}(x_n)}, \quad n = k, k+1, \dots, \tag{11}$$

where $p_{n,k}(x)$ is the polynomial of interpolation to $f(x)$ at the points $x_n, x_{n-1}, \dots, x_{n-k}$. Then, provided $x_0, x_1, \dots, x_k$ are in I and sufficiently close to α, the sequence $\{x_n\}_{n=0}^{\infty}$ converges to α, and

$$\lim_{n\to\infty} \frac{\epsilon_{n+1}}{\prod_{i=0}^{k} \epsilon_{n-i}} = \frac{(-1)^{k+1}}{(k+1)!} \frac{f^{(k+1)}(\alpha)}{f'(\alpha)} \equiv L; \quad \epsilon_n = x_n - \alpha \quad \forall n. \tag{12}$$

The order of convergence is s_k, $1 < s_k < 2$, where s_k is the only positive root of the equation $s^{k+1} = \sum_{i=0}^{k} s^i$ and satisfies

$$2 - 2^{-k-1}e < s_k < 2 - 2^{-k-1} \quad \text{for } k \geq 2; \quad s_k < s_{k+1}; \quad \lim_{k\to\infty} s_k = 2, \tag{13}$$

where e is the base of natural logarithms, and

$$\lim_{n\to\infty} \frac{|\epsilon_{n+1}|}{|\epsilon_n|^{s_k}} = |L|^{(s_k-1)/k}. \tag{14}$$

REMARK. Note that, in Theorem 3.1, $s_2 \doteq 1.839$, $s_3 \doteq 1.928$, $s_4 \doteq 1.966$, etc., rounded to three significant figures. For the secant method, we have $s_1 \doteq 1.618$, rounded to three significant figures.

PROOF. Below, we shall use the short-hand notation

$$\mathrm{int}(a_1, \dots, a_m) = (\min\{a_1, \dots, a_m\}, \max\{a_1, \dots, a_m\}).$$

We start by deriving a closed-form expression for the error in x_{n+1}. Subtracting α from both sides of (11), and noting that

$$f(x_n) = f(x_n) - f(\alpha) = f[x_n, \alpha](x_n - \alpha),$$

we have

$$x_{n+1} - \alpha = \left(1 - \frac{f[x_n, \alpha]}{p'_{n,k}(x_n)}\right)(x_n - \alpha) = \frac{p'_{n,k}(x_n) - f[x_n, \alpha]}{p'_{n,k}(x_n)}\,(x_n - \alpha). \tag{15}$$

We now note that

$$p'_{n,k}(x_n) - f[x_n, \alpha] = \big\{p'_{n,k}(x_n) - f'(x_n)\big\} + \big\{f'(x_n) - f[x_n, \alpha]\big\},$$

which, by

$$f'(x_n) - f[x_n, \alpha] = f[x_n, x_n] - f[x_n, \alpha] = f[x_n, x_n, \alpha](x_n - \alpha) = \frac{f^{(2)}(\eta_n)}{2!}(x_n - \alpha)$$

for some $\eta_n \in \operatorname{int}(x_n, \alpha)$, and by

$$\begin{aligned} f'(x_n) - p'_{n,k}(x_n) &= f[x_n, x_n, x_{n-1}, \ldots, x_{n-k}] \prod_{i=1}^{k}(x_n - x_{n-i}) \\ &= \frac{f^{(k+1)}(\xi_n)}{(k+1)!} \prod_{i=1}^{k}(x_n - x_{n-i}) \end{aligned} \tag{16}$$

for some $\xi_n \in \operatorname{int}(x_n, x_{n-1}, \ldots, x_{n-k})$, becomes

$$p'_{n,k}(x_n) - f[x_n, \alpha] = -\frac{f^{(k+1)}(\xi_n)}{(k+1)!} \prod_{i=1}^{k}(\epsilon_n - \epsilon_{n-i}) + \frac{f^{(2)}(\eta_n)}{2!}\epsilon_n. \tag{17}$$

Substituting (16) and (17) in (15), and letting

$$\widehat{D}_n = -\frac{f^{(k+1)}(\xi_n)}{(k+1)!} \quad \text{and} \quad \widehat{E}_n = \frac{f^{(2)}(\eta_n)}{2!}, \tag{18}$$

we finally obtain

$$\epsilon_{n+1} = C_n \epsilon_n; \quad C_n \equiv \frac{p'_{n,k}(x_n) - f[x_n, \alpha]}{p'_{n,k}(x_n)} = \frac{\widehat{D}_n \prod_{i=1}^{k}(\epsilon_n - \epsilon_{n-i}) + \widehat{E}_n \epsilon_n}{f'(x_n) + \widehat{D}_n \prod_{i=1}^{k}(\epsilon_n - \epsilon_{n-i})}. \tag{19}$$

We now prove that convergence takes place. Without loss of generality, we can assume that $I = (\alpha - T, \alpha + T)$ for some $T > 0$, and that $m_1 = \min_{x \in I} |f'(x)| > 0$. This is possible since $\alpha \in I$ and $f'(\alpha) \neq 0$. Let $M_s = \max_{x \in I} |f^{(s)}(x)|/s!$, $s = 1, 2, \ldots$, and choose the interval $J = (\alpha - t/2, \alpha + t/2)$ sufficiently small to ensure that $m_1 > 2M_{k+1}t^k + M_2 t/2$. It can now be shown that, provided x_{n-i}, $i = 0, 1, \ldots, k$, are all in J, there holds

$$|C_n| \leq \frac{M_{k+1} \prod_{i=1}^{k} |\epsilon_n - \epsilon_{n-i}| + M_2|\epsilon_n|}{m_1 - M_{k+1} \prod_{i=1}^{k} |\epsilon_n - \epsilon_{n-i}|} \leq \frac{M_{k+1} \prod_{i=1}^{k}(|\epsilon_n| + |\epsilon_{n-i})| + M_2|\epsilon_n|}{m_1 - M_{k+1} \prod_{i=1}^{k}(|\epsilon_n| + |\epsilon_{n-i}|)} \leq \overline{C},$$

where

$$\overline{C} \equiv \frac{M_{k+1}t^k + M_2 t/2}{m_1 - M_{k+1}t^k} < 1.$$

Consequently, by (19), $|\epsilon_{n+1}| < |\epsilon_n|$, which implies that $x_{n+1} \in J$, just like x_{n-i}, $i = 0, 1, \ldots, k$. Therefore, if $x_0, x_1, \ldots, x_k$ are chosen in J, then $|C_n| \leq \overline{C} < 1$ and hence $x_{n+1} \in J$, for all $n \geq k$. Consequently, by (19), $\lim_{n\to\infty} x_n = \alpha$.

As for (12), we proceed as follows: By the fact that $\lim_{n\to\infty} x_n = \alpha$, we first note that $\lim_{n\to\infty} p'_{n,k}(x_n) = f'(\alpha) = \lim_{n\to\infty} f[x_n, \alpha]$, and thus $\lim_{n\to\infty} C_n = 0$. This means that $\lim_{n\to\infty}(\epsilon_{n+1}/\epsilon_n) = 0$ and, equivalently, that $\{x_n\}_{n=0}^{\infty}$ converges of order greater than 1. As a result, $\lim_{n\to\infty}(\epsilon_n/\epsilon_{n-i}) = 0$, for all $i \geq 1$, and $\epsilon_n/\epsilon_{n-i} = o(\epsilon_n/\epsilon_{n-j})$ as $n \to \infty$, for $j < i$. Consequently, expanding in (19) the product $\prod_{i=1}^{k}(\epsilon_n - \epsilon_{n-i})$, we have

$$\begin{aligned}\prod_{i=1}^{k}(\epsilon_n - \epsilon_{n-i}) &= \prod_{i=1}^{k}\big(-\epsilon_{n-i}[1 - \epsilon_n/\epsilon_{n-i}]\big) \\ &= (-1)^k\left(\prod_{i=1}^{k}\epsilon_{n-i}\right)\big[1 + O\big(\epsilon_n/\epsilon_{n-1}\big)\big] \quad \text{as } n \to \infty. \end{aligned} \tag{20}$$

Substituting (20) in (19), and defining

$$D_n = \frac{\widehat{D}_n}{p'_{n,k}(x_n)}, \quad E_n = \frac{\widehat{E}_n}{p'_{n,k}(x_n)}, \tag{21}$$

we obtain

$$\epsilon_{n+1} = (-1)^k D_n\left(\prod_{i=0}^{k}\epsilon_{n-i}\right)\big[1 + O\big(\epsilon_n/\epsilon_{n-1}\big)\big] + E_n\epsilon_n^2 \quad \text{as } n \to \infty. \tag{22}$$

Dividing both sides of (22) by $\prod_{i=0}^{k}\epsilon_{n-i}$, and defining

$$\sigma_n = \frac{\epsilon_{n+1}}{\prod_{i=0}^{k}\epsilon_{n-i}}, \tag{23}$$

we have

$$\sigma_n = (-1)^k D_n\big[1 + O\big(\epsilon_n/\epsilon_{n-1}\big)\big] + E_n\sigma_{n-1}\epsilon_{n-k-1} \quad \text{as } n \to \infty. \tag{24}$$

Now,

$$\lim_{n\to\infty} D_n = -\frac{1}{(k+1)!}\frac{f^{(k+1)}(\alpha)}{f'(\alpha)}, \quad \lim_{n\to\infty} E_n = \frac{f^{(2)}(\alpha)}{2f'(\alpha)}. \tag{25}$$

Since $\lim_{n\to\infty} D_n$ and $\lim_{n\to\infty} E_n$ are finite, $\lim_{n\to\infty} \epsilon_n/\epsilon_{n-1} = 0$ and $\lim_{n\to\infty} \epsilon_{n-k-1} = 0$, it follows that there exist a positive integer N and positive constants $\beta < 1$ and D, with $|E_n\epsilon_{n-k-1}| \leq \beta$ when $n \geq N$, for which (24) gives

$$|\sigma_n| \leq D + \beta|\sigma_{n-1}| \quad \text{for all } n \geq N. \tag{26}$$

Using (26), it is easy to show that

$$|\sigma_{N+s}| \leq D\frac{1-\beta^s}{1-\beta} + \beta^s|\sigma_N|, \quad s = 1, 2, \ldots,$$

which, by the fact that $\beta < 1$, implies that $\{\sigma_n\}$ is a bounded sequence. Making use of this fact, we have $\lim_{n\to\infty} E_n\sigma_{n-1}\epsilon_{n-k-1} = 0$. Substituting this in (24), and invoking (25), we next obtain $\lim_{n\to\infty}\sigma_n = (-1)^k \lim_{n\to\infty} D_n = L$, which is precisely (12).

That the order of the method is s_k, as defined in the statement of the theorem, follows from [6, Chapter 3]. A weaker version can be proved by letting $\sigma_n = L$ for all n and showing that $|\epsilon_{n+1}| = Q|\epsilon_n|^{s_k}$ is possible for s_k a solution to the equation $s^{k+1} = \sum_{i=0}^{k} s^i$ and $Q = |L|^{(s_k-1)/k}$. The proof of this is easy and is left to the reader. This completes our proof.

4 A Numerical Example

We apply the method described in Sections 2 and 3 to the solution of the equation $f(x) = 0$, where $f(x) = x^3 - 8$, whose solution is $\alpha = 2$. We take $k = 2$ in our method. We also choose $x_0 = 5$ and $x_1 = 4$, and compute x_2 via one step of the secant method, namely,

$$x_2 = x_1 - \frac{f(x_1)}{f[x_0, x_1]}. \tag{27}$$

Following that, we compute $x_3, x_4, \ldots,$ via

$$x_{n+1} = x_n - \frac{f(x_n)}{f[x_n, x_{n-1}] + f[x_n, x_{n-1}, x_{n-2}](x_n - x_{n-1})}, \quad n = 2, 3, \ldots . \tag{28}$$

Our computations were done in quadruple-precision arithmetic (approximately 35-decimal-digit accuracy), and they are given in Table 2. Note that in order to verify the theoretical results concerning iterative methods of order greater that unity, we need to use computer arithmetic of high precision (preferably, of variable precision, if available) because the number of correct significant decimal digits increases dramatically from one iteration to the next as we are approaching the solution.

From Theorem 3.1,

$$\lim_{n\to\infty} \frac{\epsilon_{n+1}}{\epsilon_n\epsilon_{n-1}\epsilon_{n-2}} = \frac{(-1)^3}{3!}\,\frac{f^{(3)}(2)}{f'(2)} = -\frac{1}{12} = -0.08333\cdots$$

and

$$\lim_{n\to\infty} \frac{\log|\epsilon_{n+1}/\epsilon_n|}{\log|\epsilon_n/\epsilon_{n-1}|} = s_2 = 1.83928\cdots,$$

and these seem to be confirmed in Table 2. Also, x_9 should have a little under 50 correct significant figures, even though we do not see this in Table 2 due to the fact that the arithmetic we have used to generate Table 2 can provide an accuracy of at most 35 digits approximately.

n	x_n	ϵ_n	L_n	Q_n
0	5.0000000000000000000000000000000000	3.000×10^{0}		
1	4.0000000000000000000000000000000000	2.000×10^{0}		1.515
2	3.08196721311475409836065573770491792	1.082×10^{0}	0.0441	2.164
3	2.28621882971781130732266803773062580	2.862×10^{-1}	0.1670	2.497
4	2.01034420943787831264152973172014271	1.034×10^{-2}	−0.6370	1.182
5	1.99979593345266992578358353656798415	-2.041×10^{-4}	−0.1196	2.024
6	2.00000007223139333059960671366229837	7.223×10^{-8}	−0.1005	1.934
7	2.00000000000001531923884491258853168	1.532×10^{-14}	−0.0838	1.784
8	2.00000000000000000000000001893448134	1.893×10^{-26}	*	*
9	2.0000000000000000000000000000000000	*	*	*

Table 2: Results obtained by applying the generalized secant method with $k = 2$, as shown in (27) and (28), to the equation $x^3 - 8 = 0$. Here, $L_n = \epsilon_{n+1}/(\epsilon_n \epsilon_{n-1} \epsilon_{n-2})$ and $Q_n = \big(\log|\epsilon_{n+1}/\epsilon_n|\big)/\big(\log|\epsilon_n/\epsilon_{n-1}|\big)$.

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