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A Symmetry Result For A Fourth Order Overdetermined Boundary Value Problem*

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Abstract

This note is concerned with the following fourth order problem.

$$\triangle^2 u_\Omega = 1 \text{ in } \Omega,\ \ u_\Omega = \frac{\partial u_\Omega}{\partial \nu} = 0 \text{ on } \partial\Omega,\ \ \triangle u_\Omega = c \text{ on } \partial\Omega. \quad (1)$$

It is well-known [1] that if (1) admits a solution then Ω is a ball. The aim here is to give an alternative proof of this result which does not use the maximum principle.

To prove symmetry results for overdetermined value problems, three methods have been used: Serrin's method [7], Weinberger method [9] and Duality method [6]. In [1], Bennett used the Weinberger method to show the following result.

THEOREM 1. Let Ω be an open, bounded and connected subset of $\mathbb{R}^N$. Suppose Ω is of class C^2. If (1) admits a solution, then Ω is a ball.

For that purpose, Bennett introduced the auxiliary function

$$\phi(x) = \frac{N-4}{N+2}u + \frac{N-4}{2(N+2)}(\triangle u)^2 + u_{,ij}u_{,ij} - \nabla u.\nabla(\triangle u).$$

Then by using the strong maximum principle, he determined that Ω is a ball with radius $[|c|N(N+2)]^{1/2}$ and the solution of (1) is given by

$$u(x) = \frac{-1}{2N}\left\{\frac{1}{4}(N+2)(Nc)^2 + \frac{Nc}{2}r^2 + \frac{1}{4(N+2)}r^4\right\}.$$

Later, Dalmasso [5] used Serrin's method of moving planes to show that Ω in (1) is a ball and u is radial.

Our aim at present is to prove Theorem 1 without using the maximum principle which is the classical ingredient in many proofs of the earlier results. All we need here is to perform the derivative with respect to domain (also known as the shape derivative), see e.g. [8]. To get similar symmetry results for other problems, this

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notion has been used for instance in [4], see also [2] where it was combined with the Steiner symmetrization.

Before starting our proof, let us remark that (1) is equivalent to

$$\begin{cases} \triangle u_\Omega = c - v_\Omega \text{ in } \overline{\Omega}, \ u_\Omega = \frac{\partial u_\Omega}{\partial \nu} = 0 \text{ on } \partial\Omega, \\ \triangle v_\Omega = -1 \text{ in } \Omega, \ v_\Omega = 0 \text{ on } \partial\Omega \end{cases} \tag{2}$$

v_Ω is called the torsion function relative to the domain Ω.

LEMMA 0. If u_Ω solves (2), then $c > 0$.

PROOF. Since u_Ω solves (2), the Green formula gives

$$0 = \int_{\partial\Omega} \frac{\partial u_\Omega}{\partial \nu} = \int_\Omega \triangle u_\Omega = cV(\Omega) - \int_\Omega v_\Omega$$

which implies

$$\int_\Omega v_\Omega = cV(\Omega).$$

Then, by the maximum principle, $v_\Omega > 0$ in Ω, so

$$c = \frac{\int_\Omega v_\Omega}{V(\Omega)} > 0.$$

Throughout the sequel, let ω be a bounded open connected domain of class C^2 in $\mathbb{R}^N$ $(N \geq 2)$ and let ν be the outward normal to the boundary of ω. Denote by $V(\omega)$ the volume of ω and let v_ω be the torsion function relative to the domain ω.

$$\triangle v_\omega = -1 \text{ in } \omega, \ v_\omega = 0 \text{ on } \partial\omega. \tag{3}$$

Let $\mathcal{B}$ be the class of the open, bounded and connected subsets of $\mathbb{R}^N$. Consider

$$\mathcal{O} = \left\{ \omega \in \mathcal{B},\ \omega \text{ is of class } C^2 : \int_\omega v_\omega \leq cV(\omega) \right\}$$

and

$$J(\omega) = c^2 V(\omega) - \int_\omega v_\omega^2 + \int_\omega u_\omega$$

where v_ω is the solution of (3) and u_ω is the solution of the Dirichlet problem.

$$\triangle u_\omega = c - v_\omega \text{ in } \omega, \ u_\omega = 0 \text{ on } \partial\omega. \tag{4}$$

LEMMA 1. $J(\omega) \geq 0$ for any $\omega \in \mathcal{O}$. Furthermore, if Ω is of class C^2 and u_Ω solves (2) then $J(\Omega) = 0$ and $J(\Omega) = \min\{J(\omega),\ \omega \in \mathcal{O}\}$.

PROOF. Let $\omega \in \mathcal{O}$. According to (3) and (4), the Green formula gives

$$-\int_\omega u_\omega = \int_\omega \triangle v_\omega u_\omega = c\int_\omega v_\omega - \int_\omega v_\omega^2. \tag{5}$$

But $\int_\omega v_\omega \leq cV(\omega)$, so

$$-\int_\omega u_\omega \leq c^2 V(\omega) - \int_\omega v_\omega^2.$$

Thus $J(\omega) \geq 0$. Now in the proof of Lemma 0, we got

$$\int_\Omega v_\Omega = cV(\Omega). \tag{6}$$

This together with the fact that $\Omega \in \mathcal{B}$ and it is of class C^2, implies $\Omega \in \mathcal{O}$.

Now replacing in (5) ω by Ω, we obtain

$$c\int_\Omega v_\Omega - \int_\Omega v_\Omega^2 + \int_\Omega u_\Omega = 0$$

and by (6),

$$J(\Omega) = c^2 V(\Omega) - \int_\Omega v_\Omega^2 + \int_\Omega u_\Omega = 0.$$

It then follows that Ω minimizes J on $\mathcal{O}$.

As it is mentioned above, the use of the shape derivative will allow us to prove that Ω is a ball. Before doing this, let us recall the definition of the domain derivative, see for instance [8]. Consider a deformation field $V \in C^2\left(\mathbb{R}^N; \mathbb{R}^N\right)$ and set $\omega_t = \{x + tV(x),\ x \in \Omega\}$, $t > 0$. The application $Id + tV$ is a perturbation of the identity which is a Lipschitz diffeomorphism for t small enough. By definition, the derivative of J at ω in the direction V is

$$dJ(\omega, V) = \lim_{t \to 0} \frac{J(\omega_t) - J(\omega)}{t}.$$

Since the functional J depends on the domain ω through the solution of the Dirichlet problems (3) and (4) we need to define also the domain derivative of u_ω (resp. v_ω). If u' (resp. v') denotes the domain derivative of u_ω (resp. v_ω) then

$$u' = \lim_{t \to 0} \frac{u_{\omega_t} - u_\omega}{t}.$$

and

$$v' = \lim_{t \to 0} \frac{v_{\omega_t} - v_\omega}{t}.$$

Furthermore, we can prove ([8], [**?**]) the following lemma.

LEMMA 2. u' satisfies

$$-\Delta u' = 0 \ \text{ in } \ \omega \ \text{ and } \ u' = -\frac{\partial u_\omega}{\partial \nu} V \cdot \nu \ \text{ on } \ \partial\omega. \tag{7}$$

and v' satisfies

$$-\Delta v' = 0 \ \text{ in } \ \omega \ \text{ and } \ v' = -\frac{\partial v_\omega}{\partial \nu} V \cdot \nu \ \text{ on } \ \partial\omega. \tag{8}$$

Now since $J(\omega) = c^2 V(\omega) - j_1(\omega) + j_2(\omega)$ where $j_1(\omega) = \int_\omega v_\omega^2$ and $j_2(\omega) = \int_\omega u_\omega$, we need to perform the derivative of a functional in the form $F(\omega) = \int_\omega f(w_\omega)$ where w_ω is the solution of some Dirichlet problem on ω and w' satisfies

$$-\Delta w' = 0 \text{ in } \omega \text{ and } w' = -\frac{\partial w_\omega}{\partial \nu} V \cdot \nu \text{ on } \partial\omega. \tag{9}$$

$$dF(\omega, V) = \int_\omega f'(w_\omega)\, w' dx + \int_{\partial\omega} f(w_\omega) V \cdot \nu \, d\sigma. \tag{10}$$

LEMMA 3. Let $\omega \in \mathcal{O}$ then for any direction V:

$$dJ(\omega, V) = \int_{\partial\omega} [c^2 - 2c(\frac{\partial v_\omega}{\partial \nu})^2 - \frac{\partial u_\omega}{\partial \nu}\frac{\partial v_\omega}{\partial \nu}] V \cdot \nu d\sigma.$$

PROOF. In (10), one can take $f \equiv 1$ and obtain the derivative of the volume, i.e

$$dV(\omega, V) = \int_{\partial\omega} V \cdot \nu d\sigma. \tag{11}$$

By replacing in (10), w_ω by v_ω and putting $f(t) = t^2$ we obtain

$$dj_1(\omega, V) = 2\int_\omega v_\omega v' dx + \int_{\partial\omega} v_\omega^2 V \cdot \nu \, d\sigma. \tag{12}$$

Since v_ω vanishes on $\partial\omega$,

$$dj_1(\omega, V) = 2\int_\omega v_\omega v' dx. \tag{13}$$

By (4)

$$dj_1(\omega, V) = 2c\int_\omega v' dx - 2\int_\omega \triangle u_\omega v' dx. \tag{14}$$

On one hand using (3) and (8), the Green formula gives

$$\int_\omega v' dx = -\int_\omega \triangle v_\omega v' dx = \int_{\partial\omega} (\frac{\partial v_\omega}{\partial \nu})^2 V \cdot \nu d\sigma. \tag{15}$$

On the other hand according to (4) and (8), the Green formula gives

$$\int_\omega \triangle u_\omega v' dx = -\int_{\partial\omega} \frac{\partial v_\omega}{\partial \nu}\frac{\partial u_\omega}{\partial \nu} V \cdot \nu d\sigma. \tag{16}$$

Combining (15) with (16), we get

$$dj_1(\omega, V) = \int_{\partial\omega} [2c(\frac{\partial v_\omega}{\partial \nu})^2 - 2\frac{\partial v_\omega}{\partial \nu}\frac{\partial u_\omega}{\partial \nu}] V \cdot \nu d\sigma. \tag{17}$$

Now if we replace in (10), w_ω by u_ω and put $f(t) = t$ we obtain

$$dj_2(\omega, V) = \int_\omega u' dx = -\int_\omega \triangle v_\omega u' dx. \tag{18}$$

Using (3) and (7), the Green formula gives

$$dj_2(\omega, V) = \int_{\partial\omega} \frac{\partial v_\omega}{\partial \nu}\frac{\partial u_\omega}{\partial \nu} V \cdot \nu d\sigma. \tag{19}$$

The result follows then from (11) combined with (17) and (19).

Now we are ready to state and prove the following proposition.

PROPOSITION 1. If Ω is of class C^2 and u_Ω solves (1), then there exists a Lagrange multiplier λ such that

$$\left(\frac{\partial v_\Omega}{\partial \nu}\right)^2 = \frac{c(c+\lambda)}{2c+\lambda} \text{ on } \partial\Omega.$$

PROOF. Put $j(\omega) = \int_\omega v_\omega$. Since Ω is the minimum of J on $\mathcal{O}$ then there exists a Lagrange multiplier λ such that for any direction V

$$dJ(\Omega, V) = \lambda(dj(\Omega, V) - cdV(\Omega, V)).$$

But $\frac{\partial u_\Omega}{\partial \nu} = 0$ on $\partial\Omega$, so according to Lemma 3,

$$dJ(\Omega, V) = \int_{\partial\Omega} [c^2 - 2c(\frac{\partial v_\Omega}{\partial \nu})^2] V \cdot \nu d\sigma.$$

Then by (15), we obtain: for any direction V

$$\int_{\partial\Omega} [c^2 + c\lambda - (2c+\lambda)(\frac{\partial v_\Omega}{\partial \nu})^2] V \cdot \nu d\sigma = 0.$$

Then using the density of the functions $V \cdot \nu$ in $L^2(\partial\Omega)$, we get the result.

Now the previous proposition says that v_Ω is a solution to the Serrin problem that is to say that Ω is a ball with radius $N[\frac{c(c+\lambda)}{2c+\lambda}]^{1/2}$, v_Ω and u_Ω are radially symmetric.

REMARK 1. Suppose $c < 0$. Let $\omega \in \mathcal{B}$. By the maximum principle, $v_\omega > 0$ in ω, so $\int_\omega v_\omega > cV(\omega)$. Therefore $J(\omega) > 0$ for any $\omega \in \mathcal{B}$. Unfortunately, the value 0 cannot be reached by Ω the solution of (2), i.e. Ω cannot minimize J on $\mathcal{B}$.

REMARK 2. Since we want to get the optimality condition, we need to perform the shape derivative at the minimum Ω which requires only the C^2 regularity of Ω. So, we can consider $\mathcal{O} = \{\omega \in \mathcal{B} \ : \ \int_\omega v_\omega \leq cV(\omega)\}$ and suppose ω of class C^2 in Lemma 3.

REMARK 3. One can replace in $\mathcal{O}$, the inequality by its converse and obtain a maximization problem since the optimal shape Ω leads to an equality. Then using the same arguments as above, we reach the same conclusion.

REMARK 4. Let v_ω and u_ω be respectively the solution of (3) and (4). Consider the functional $G(\omega) = c\int_\omega v_\omega dx - \frac{1}{2}\int_\omega v_\omega^2 dx$. Denote by O_{ad} some class of the admissible domains, for example the class of the domains with the ε-cone property [3]. One can show the existence of a minimum Ω of G on O_{ad}. Then, if Ω is of class C^2, the shape derivative of G gives (for any admissible direction V)

$$\int_{\partial\Omega} \frac{\partial u_\Omega}{\partial \nu}\frac{\partial v_\Omega}{\partial \nu} V \cdot \nu = 0.$$

It then follows that $\frac{\partial u_\Omega}{\partial \nu} = 0$ on $\partial\Omega$ since $\frac{\partial v_\Omega}{\partial \nu} < 0$ on $\partial\Omega$. This means that Ω is a solution of (2).

REMARK 5. Consider the fourth order problem:

$$\triangle^2 w_\Omega = 1 \text{ in } \Omega, \quad w_\Omega = \triangle w_\Omega = 0 \text{ on } \partial\Omega, \quad \frac{\partial w_\Omega}{\partial \nu} = c\frac{\partial v_\Omega}{\partial \nu} \text{ on } \partial\Omega,$$

where v_Ω is the torsion function relative to Ω. As above, this problem is equivalent to:

$$-\triangle w_\Omega = v_\Omega \text{ in } \overline{\Omega}, \quad w_\Omega = 0 \text{ and } \frac{\partial w_\Omega}{\partial \nu} = c\frac{\partial v_\Omega}{\partial \nu} \text{ on } \partial\Omega.$$

Now if we put $u_\Omega = w_\Omega - cv_\Omega$, it is simple to see that u_Ω solves (1) and then Ω is a ball.

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