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A Survey On Nonlocal Boundary Value Problems*

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Abstract

In this paper, we present a survey of recent results on the existence and multiplicity of solutions of nonlocal boundary value problem involving second order ordinary differential equations.

1 Introduction

Boundary value problems involving ordinary differential equations arise in physical sciences and applied mathematics. In some of these problems, subsidiary conditions are imposed locally. In some other cases, nonlocal conditions are imposed. It is sometimes better to impose nonlocal conditions since the measurements needed by a nonlocal condition may be more precise than the measurement given by a local condition. For example, the classical Robin problem is given by

$$u''(t) + f(t, u(t), u'(t)) = 0, \tag{1}$$

with local conditions

$$u(0) = 0 \text{ and } u'(1) = 0. \tag{2}$$

If the local condition $u'(1) = 0$ in (2) is replaced by the nonlocal condition $u(1) = u(\eta)$ in

$$u(0) = 0 \text{ and } u(1) = u(\eta), \tag{3}$$

(where $\eta \in (0, 1)$), then (1),(3) is a nonlocal problem. By the Rolle theorem, (1),(2) can be thought as the limiting case of (1),(3) as $\eta \to 1^-$. Obviously, the nonlocal problem (1),(3) gives better effect than the local problem (1),(2). In the process of scientific experiment and numerical computation, it is more difficult to determine the value of $u'(1)$ than that of $\frac{u(\eta)-u(1)}{\eta-1}$.

The nonlocal condition $u(1) = u(\eta)$ can be written as a 'difference' $u(1) - u(\eta)$. Therefore, nonlocal problem may be regarded as boundary value problem involving 'continuous equations' and one or more 'discrete multi-point boundary conditions'.

In this paper, we present a survey of recent results on the existence and multiplicity of solutions of nonlocal boundary value problems of second order ordinary differential equations.

More precisely, we will summarize basic results in the literature related to the following four directions:

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- Results at nonresonance.
- Results at resonance.
- Positive solutions of multi-point boundary value problems.
- Global continua of positive and nodal solutions of multi-point BVPs .

2 Results at Nonresonance

In this paper, if a linear differential operator L with certain boundary conditions is invertible, that is, the kernel space $\mathrm{Ker}(L) = \{0\}$, then we say that the corresponding BVPs is at nonresonance. On the other hand, if L is noninvertible, namely, $\mathrm{dimKer}(L) \geq 1$, then we say that the corresponding BVPs is at resonance.

2.1 The Lower Order Singularity Case

The study of multi-point boundary value problems for linear second order ordinary differential equations was initialed by Il'in abd Moiseev in [57, 58]. In 1992, Gupta [36] firstly studied existence of solutions to the nonlinear three-point boundary value problems

$$\begin{gathered} u''(t) = f(t, u(t), u'(t)) + e(t), \qquad 0 < t < 1, \\ u(0) = 0, \quad u(1) = u(\eta), \end{gathered} \tag{4}$$

where $\eta \in (0, 1)$ is a constant, $f : [0, 1] \times R^2 \to R$ satisfies the Carathéodory conditions and some at most linear growth conditions.

Define $Lu = -u''$, $u \in D(L) = \{u \in W^{2,1}(0, 1),\ u(0) = 0,\ u(1) = u(\eta)\}$, then $\mathrm{Ker}(L) = \{0\}$. Hence, (4) is a nonresonance problem. In this section, all problems are at nonresonance, we omit corresponding proofs.

Since then, the existence of solutions of the more general nonlinear multi-point boundary value problems have been investigated by many authors, see [37], [38], [39], [40], [44], [45], [46], [47], [48], [49], [34], [35], [75], [76], [104] for some references along this line.

In this section, we assume that $\alpha \in (0, \infty)$ and $\eta \in (0, 1)$ are given positive constants with

$$\alpha\eta \neq 1. \tag{5}$$

Then (5) implies that the linear three-point boundary value problem

$$x''(t) = y(t), \qquad 0 < t < 1, \tag{6}$$

$$x(0) = 0, \qquad x(1) = \alpha x(\eta) \tag{7}$$

has a unique solution for each $y \in L^1(0, 1)$. So (5) is a nonresonance condition. It is easy to check that (6),(7) is equivalent to the fixed point problem

$$x(t) = \int_0^t (t - s)y(s)ds + \frac{\alpha t}{1 - \alpha\eta} \int_0^\eta (\eta - s)y(s)ds - \frac{1}{1 - \alpha\eta} \int_0^1 (1 - s)y(s)ds. \tag{8}$$

In [44], Gupta, Ntouyas and Tsamatos used the Leray-Schauder continuation theorem [105] to prove an existence result for the three-point boundary value problem

$$x''(t) = f(t, x(t), x'(t)) + e(t), \quad 0 < t < 1, \tag{9}$$

$$x(0) = 0, \quad x(1) = \alpha x(\eta). \tag{10}$$

THEOREM 2.1 [44]. Let $f : [0,1] \times R^2 \to R$ satisfy the Carathéodory conditions. Assume

$$|f(t,u,v)| \leq p(t)|u| + q(t)|v| + r(t), \tag{11}$$

for a.e. $t \in [0,1]$ and $(u,v) \in R^2$. Also let $\alpha \in R$ and $\eta \in (0,1)$ be given. Then the boundary value problem (9),(10) has at least one solution in $C^1[0,1]$ provided

$$\begin{cases} ||p||_1 + ||q||_1 < 1, & \text{if } \alpha \leq 1, \\ ||p||_1 + ||q||_1 < \frac{1-\alpha\eta}{\alpha(1-\eta)}, & \text{if } 1 < \alpha < \frac{1}{\eta}. \end{cases}$$

Now let $\xi_i \in (0,1)$ for $i = 1,2,...,m-2$ satisfy $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$, and $a_i \in R$, $i = 1,2,...,m-2$, have the same sign and $\alpha = \sum_{i=1}^{m-2} a_i \neq 0$, $e \in L^1[0,1]$. Gupta, Ntouyas and Tsamatos studied the nonlinear m-point boundary value problem

$$x''(t) = f(t, x(t), x'(t)) + e(t), \quad 0 < t < 1, \tag{12}$$

$$x(0) = 0, \quad x(1) = \sum_{i=1}^{m-2} a_i x(\xi_i), \tag{13}$$

using the priori estimates that they obtained for the three-point BVP (12),(10). In fact, for every solution $x(t)$ of the BVP (12),(13), let us denote

$$m = \min_{x \in [\xi_1, \xi_{m-2}]} x(t), \quad M = \max_{x \in [\xi_1, \xi_{m-2}]} x(t).$$

If $a_i \in [0, \infty)$, then

$$a_i m \leq a_i x(\xi_i) \leq a_i M, \quad i \in \{1, \cdots, m-2\}.$$

If $a_i \in (-\infty, 0]$, then

$$a_i m \geq a_i x(\xi_i) \geq a_i M, \quad i \in \{1, \cdots, m-2\}.$$

In either case, we have that

$$m \leq \frac{\sum_{i=1}^{m-2} a_i x(\xi_i)}{\sum_{i=1}^{m-2} a_i} \leq M.$$

It follows that there exists $\eta \in [\xi_1, \xi_{m-2}]$, such that $x(\eta) = \frac{x(1)}{\alpha}$, which implies that $x(t)$ is also a solution of the BVP (12),(10).

THEOREM 2.2 [44]. Let $f : [0,1] \times R^2 \to R$ satisfy the Carathéodory conditions. Assume

$$|f(t,u,v)| \leq p(t)|u| + q(t)|v| + r(t), \tag{14}$$

for a.e. $t \in [0,1]$ and $(u,v) \in R^2$. Also let $\alpha = \sum_{i=1}^{m-2} a_i$ and $\eta \in (0,1)$ be given. Then the boundary value problem (12),(13) has at least one solution in $C^1[0,1]$ provided

$$\begin{cases} ||p||_1 + ||q||_1 < 1, & \text{if } \alpha \leq 1, \\ ||p||_1 + ||q||_1 < \frac{1-\alpha\xi_{m-2}}{\alpha(1-\xi_1)}, & \text{if } 1 < \alpha < \frac{1}{\xi_{m-2}}. \end{cases}$$

Feng and Webb established a result in which f is allowed to have nonlinear growth.

THEOREM 2.3 [34]. Assume that $f : [0,1] \times R^2 \to R$ is continuous and has the decomposition

$$f(t,x,p) = g(t,x,p) + h(t,x,p)$$

such that

(1) $pg(t,x,p) \leq 0$ for all $(t,x,p) \in [0,1] \times R^2$;

(2) $|h(t,x,p)| \leq a(t)|x| + b(t)|p| + u(t)|x|^r + v(t)|p|^k + c(t)$ for all $(t,x,p) \in [0,1] \times R^2$, where a, b, u, v, c are in $L^1(0,1)$ and $0 \leq r, k < 1$.

Then, for $\alpha \neq \frac{1}{\eta}$, there exits a solution $x \in C^1[0,1]$ to (9),(10) provided that

$$\begin{cases} ||a||_1 + ||b||_1 < \frac{1}{2}, & \text{if } \alpha \leq 1, \\ ||a||_1 + ||b||_1 < \frac{1}{2}\Big(1 - \frac{(\alpha-1)^2}{\alpha^2(1-\eta)^2}\Big), & \text{if } 1 < \alpha < \frac{1}{\eta}, \\ ||a||_1 + ||b||_1 < \frac{1}{2}\Big(1 - \frac{1}{\alpha^2\eta^2}\Big), & \text{if } \frac{1}{\eta} < \alpha. \end{cases}$$

In [75], Ma used the nonlinear alternative to establish a result on the existence of solutions for the inhomogeneous three-point boundary value problem

$$x''(t) = f(t, x(t), x'(t)) + e(t), \quad 0 < t < 1, \tag{15}$$

$$x(0) = A, \quad x(1) - x(\eta) = B(1-\eta), \tag{16}$$

where $f : [0,1] \times R^2 \to R$ satisfies some sign condition near the constant 'A', but without any growth restriction at ∞.

THEOREM 2.4 [75]. Let $f : [0,1] \times R^2 \to R$ be continuous. Suppose there are constants $L_1, L_2 : L_2 < B < L_1$ such that

(1) $f(t,x,L_1) \geq 0$ for $(t,x) \in [0,1] \times [A - |L_2|, A - |L_1|]$;

(2) $f(t,x,L_2) \leq 0$ for $(t,x) \in [0,1] \times [A - |L_2|, A - |L_1|]$;

(3) $\frac{L_2 - B}{1-\eta} \leq f(t,x,p) \leq \frac{L_1 - B}{1-\eta}$ for $(t,x,p) \in [0,1] \times [A - |L_2|, A - |L_1|] \times [L_2, L_1]$.

Then the problem (15),(16) has at least one solution x such that $L_2 \leq x' \leq L_1$.

In [76], Ma obtained two results on the existence of the Robin type m-point boundary value problem

$$x''(t) = f_1(t, x(t), x'(t)) + e_1(t), \quad 0 < t < 1, \tag{17}$$

$$x'(0) = 0, \quad x(1) = \sum_{i=1}^{m-2} a_i x(\xi_i), \tag{18}$$

with the nonresonance condition $\alpha = \sum_{i=1}^{m-2} a_i \neq 1$.

THEOREM 2.5 [76]. Let $\alpha \leq 0$ and $f : [0,1] \times R^2 \to R$ be continuous. Suppose there are constants $L_1, L_2 : L_2 < 0 < L_1$ such that

(1) $f(t,x,L_1) + e(t) \leq 0$ for $(t,x) \in [0,1] \times [-L, L]$;

(2) $f(t,x,L_2) + e(t) \geq 0$ for $(t,x) \in [0,1] \times [-L, L]$ where $L := \max\{L_1, -L_2\}$.

Then the problem (17),(18) has at least one solution satisfying $L_2 \leq x' \leq L_1$.

THEOREM 2.6 [76] let $0 < \alpha \neq 1$ and $f : [0,1] \times R^2 \to R$ be continuous. Suppose there are constants $L_1, L_2 : L_2 < 0 < L_1$ such that

(1) $f(t,x,L_1) + e(t) \leq 0$ for $(t,x) \in [0,1] \times [-\bar{L}, \bar{L}]$;

(2) $f(t,x,L_2) + e(t) \geq 0$ for $(t,x) \in [0,1] \times [-\bar{L}, \bar{L}]$ where

$$\bar{L} > \big(\frac{1-\xi_1}{|\alpha - 1|} + 1\big) \max\{-L_2, L_1\}.$$

Then the problem (17),(18) has at least one solution satisfying $L_2 \leq x' \leq L_1$.

2.2 The Higher Order Singularity Case

In 2005, Ma and Thompson [101] obtained an existence result for the second order m-point boundary value problem (12),(13) in which f and e have a higher order singularity at $t = 0$ and $t = 1$. They made the following assumptions:

(H0) $a_i \in R$ and $\xi_i \in (0,1)$ for $i = 1, 2, ..., m-2$ where $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$ and

$$\sum_{i=1}^{m-2} a_i \xi_i \neq 1.$$

(H1) There exist $q(t) \in L^1[0,1]$ and $p(t),\ r(t) \in L^1_{\text{loc}}(0,1)$ so that $t(1-t)p(t),\ t(1-t)r(t) \in L^1[0,1]$, and

$$|f(t,u,v)| \leq p(t)|u| + q(t)|v| + r(t), \qquad \text{a.e. } t \in [0,1],\ (u,v) \in \mathbb{R}^2,$$

where

$$L^1_{\text{loc}}(0,1) = \big\{u \mid u|_{[c,d]} \in L^1[c,d] \text{ for every compact interval } [c,d] \subset (0,1)\big\}.$$

(H2) The function $e : [0,1] \to R$ satisfies $\int_0^1 t(1-t)|e(t)|dt < \infty$.

THEOREM 2.7 [101]. Let $f : [0,1] \times R^2 \to R$ satisfy the Carathéodory conditions. Assume that (H0), (H1) and (H2) hold. Then problem (12),(13) has at least one solution in

$$X := \{u \in C^1(0,1) \mid u \in C[0,1],\ \lim_{t\to 1}(1-t)u'(t) \text{ and } \lim_{t \to 0} tu'(t) \text{ exist}\}$$

provided

$$||p||_E \big(1 + \frac{\sum_{i=1}^{m-2} |a_i|}{|1 - \sum_{i=1}^{m-2} a_i \xi_i|}\big) + ||q||_{L^1} < 1,$$

where E is the Banach space

$$E = \{y \in L^1_{\text{loc}}(0,1) \mid t(1-t)y(t) \in L^1[0,1]\}$$

equipped with the norm

$$||y||_E = \int_0^1 t(1-t)|y(t)|dt.$$

REMARK 2.1. Let us consider the three-point boundary value problem

$$\begin{array}{l} x'' = g(t, x, x'), \\ x(0) = 0, \quad x(1) = x\left(\frac{1}{3}\right) - x\left(\frac{2}{3}\right), \end{array} \tag{19}$$

where

$$g(t, u, v) = \frac{\alpha}{t(1-t)} \sin(u+v)u + \beta v + \frac{1}{t(1-t)}[1 + \cos(u^{200} + v^{30})].$$

It is easy to see that

$$|g(t, u, v)| \le p(t)|u| + q(t)|v| + r(t)$$

with $p(t) = \frac{\alpha}{t(1-t)}$, $q(t) = \beta$ and $r(t) = \frac{2}{t(1-t)}$ Clearly, $||p||_E = |\alpha|$, $||q||_{L^1} = |\beta|$, $||r||_E = 2$, and

$$\frac{\sum_{i=1}^{m-2} |a_i|}{|1 - \sum_{i=1}^{m-2} a_i|} = \frac{1+1}{|1 - (1 \times \frac{1}{3} - 1 \times \frac{2}{3})|} = \frac{3}{2}.$$

By Theorem 2.7, (19) has at least one solution in

$$X = \{u \in C^1(0,1) \mid u \in C[0,1], \lim_{t\to 1}(1-t)u'(t) \text{ and } \lim_{t\to 0} tu'(t) \text{ exist}\}$$

provided

$$\frac{5}{2}|\alpha| + |\beta| < 1.$$

3 Results at Resonance

In the following we shall give existence results for BVP

$$x''(t) = f(t, x(t), x'(t)) + e(t), \quad 0 < t < 1, \tag{20}$$

$$x(0) = 0, \quad x(1) = \alpha x(\eta) \tag{21}$$

when $\alpha\eta = 1$.

Define $Lx = -x'', x \in D(L) := \{x \in W^{2,1}(0,1), x(0) = 0, x(1) = \alpha x(\eta)\}$. Then $\text{Ker}(L) = \{ct \mid c \in R\}$. Hence, (20),(21) is at resonance.

In this case, Leray-Schauder continuation theorem cannot be used.

In [35], Feng and Webb applied the Mawhin continuation theorem to prove the existence results for (20),(21) at resonance.

THEOREM 3.1 [35]. Let $f : [0,1] \times R^2 \to R$ be continuous. Assume that

(1) There exist functions p, q, r in $L^1[0,1]$ such that

$$|f(t, u, v)| \le p(t)|u| + q(t)|v| + r(t), \quad \text{for } t \in [0,1] \text{ and } (u, v) \in R^2.$$

(2) There exists $N > 0$ such that for $v \in R$ with $|v| > N$, one has

$$|f(t,u,v)| \geq -l|u| + n|v| - M, \quad \text{for } t \in [0,1], u \in R$$

where $n > l \geq 0,\ M \geq 0$.

(3) There exists $R > 0$ such that for $|v| > R$ one has either

$$vf(t,vt,v) \leq 0, \qquad t \in [0,1]$$

or

$$vf(t,vt,v) \geq 0, \qquad t \in [0,1].$$

Then, for every continuous function e, the BVP (20),(21) with $\alpha\eta = 1$ has at least one solution in $C^1[0,1]$ provided that

$$2(||p||_1 + 2||q||_1) + \frac{l}{n} < 1.$$

In [91], Ma considered the m-point BVP

$$u''(t) = f(t,u), \quad 0 < t < 1, \tag{22}$$

$$u'(0) = 0, \quad u(1) = \sum_{i=1}^{m-2} a_i u(\eta_i), \tag{23}$$

with the resonance condition $\sum_{i=1}^{m-2} a_i = 1$. He developed the methods of lower and upper solutions by the connectivity properties of the solution set of parameterized families of compact vector fields.

DEFINITION 3.1. We say that the function $x \in C^2[0,1]$ is a upper solution of (22),(23) if

$$x''(t) \leq f(t,x), \quad 0 < t < 1, \tag{24}$$

$$x'(0) \leq 0, \quad x(1) - \sum_{i=1}^{m-2} a_i x(\eta_i) \geq 0, \tag{25}$$

and $y \in C^2[0,1]$ is a lower solution of (22),(23) if

$$y''(t) \geq f(t,y), \quad 0 < t < 1 \tag{26}$$

$$y'(0) \geq 0, \quad y(1) - \sum_{i=1}^{m-2} a_i y(\eta_i) \leq 0. \tag{27}$$

If the inequalities in (24) and (26) are strict, then x and y are called strict upper and lower solutions.

Applying the same method to prove Theorem 2.2 in [86] with some obvious changes, we have

THEOREM 3.2 If $f : [0,1] \times R \to R$ is continuous. Assume that x and y are respectively strict upper and strict lower solutions of (22),(23) satisfying $x(t) \geq y(t)$ on $[0,1]$. Then (22),(23) has a solution u satisfying

$$y(t) \leq u(t) \leq x(t), \quad t \in [0,1].$$

THEOREM 3.3 [91]. If $f : [0,1] \times R \to R$ is continuous. Assume that one of the following sets of conditions is fulfilled.

(A1) There exist $p, r \in L^1(0,1)$ with $||p||_1 < \frac{1}{2}$ such that

$$|f(t,u)| \leq p(t)|u| + r(t).$$

Assume that x and y are strict upper solution and strict lower solution of (22),(23) satisfying $x(t) \leq y(t)$ on $[0,1]$.

(A2) There exist a strict lower solution α and a strict upper solution β such that

$$\alpha(t) < x(t) < y(t) < \beta(t), \quad t \in [0,1].$$

Then (22),(23) has a solution u satisfying

$$x(t_u) \leq u(t_u) \leq y(t_u), \quad \text{for some } t_u \in [0,1].$$

4 Positive Solutions of Multi-Point BVPs

In this section, we discuss the existence and multiplicity of positive solutions of nonlinear multi-point boundary value problems. There is much attention focused on question of positive solutions of BVPs for ordinary differential equations. Much of the interest is due to the applicability of certain Krasnosel'skii fixed point theorem. Here we present some of the results on positive solutions of some nonlocal problems.

Consider the differential equation

$$x'' + a(t)f(x) = 0, \quad t \in (0,1), \tag{28}$$

$$x(0) = 0, \qquad x(1) = \alpha x(\eta), \tag{29}$$

where $\eta \in (0,1)$ is a given constant, and a, f satisfy

(C1) $a : [0,1] \to [0,\infty)$ is continuous and $a(t) \not\equiv 0$ on $[0,1]$;

(C2) $f : [0,\infty) \to [0,\infty)$ is continuous.

In [77], Ma gave the following existence result for positive solutions to (28),(29) by using the Krasnosel'skii fixed point theorem, the fixed point index theory and the fact that (28),(29) is equivalent to the integral equation

$$\begin{aligned} x(t) \;&=\; -\int_0^t (t-s)a(s)f(x(s))ds - \frac{\alpha t}{1-\alpha\eta}\int_0^\eta (\eta - s)a(s)f(x(s))ds \\ &\quad + \frac{t}{1-\alpha\eta}\int_0^1 (1-s)a(s)f(x(s))ds. \end{aligned} \tag{30}$$

THEOREM 4.1 [77]. Let (C1) and (C2) hold, and let

$$0 < \eta < \frac{1}{\alpha}. \tag{31}$$

Assume that

$$f_0 = \lim_{u \to 0^+} \frac{f(u)}{u}, \quad f_\infty = \lim_{u \to \infty} \frac{f(u)}{u} \tag{32}$$

exist. Then (28),(29) has at least one positive solution in the case

(i) $f_0 = 0,\ f_\infty = \infty$ (superlinear case); or

(ii) $f_0 = \infty,\ f_\infty = 0$ (sublinear case).

Let $a, b \in (0,1)$ be such that

$$\int_a^b a(s)ds > 0.$$

Let

$$k(t,\ s) = \frac{1}{1-\alpha\eta} t(1-s) - \begin{cases} \frac{\alpha t}{1-\alpha\eta}(\eta - s) & s \le \eta \\ 0 & s > \eta \end{cases} - \begin{cases} t-s & s \le t \\ 0 & s > t \end{cases} \tag{33}$$

In 2001, Webb [124] used the cone

$$K = \{x \in C[0,1] : x \ge 0,\ \min\{x(t) : a \le t \le b\} \ge c||x||_\infty\}$$

to study the existence and multiplicity of positive solutions of (28),(29). By taking

$$c = \begin{cases} \min\{a, \alpha\eta, 4a(1-\eta), \alpha(1-\eta)\}, & \alpha < 1 \\ \min\{a\eta, 4a(1-\alpha\eta), \eta(1-\alpha\eta)\}, & \alpha \ge 1 \end{cases}$$

and finding a function $\Phi(s)$:

$$\begin{array}{ll} k(t,s) \le \Phi(s), & \text{for every } t, s \in [0,1], \\ k(t,s) \ge c\Phi(s), & \text{for every } s \in [0,1], t \in [a,b], \end{array}$$

he established the following

THEOREM 4.2 [124]. Let $0 < \eta < \frac{1}{\alpha}$ and let

$$m = \Big(\max_{0 \le t \le 1} \int_0^1 k(t,s)a(s)ds\Big)^{-1}, \quad M = \Big(\min_{a \le t \le b} \int_a^b k(t,s)a(s)ds\Big)^{-1}.$$

Then (28),(29) has at least one solution if either

(h1) $0 \le \limsup_{x\to 0} \frac{f(x)}{x} < m, \quad M < \liminf_{x \to \infty} \frac{f(x)}{x} \le \infty$, or

(h2) $0 \le \limsup_{x\to \infty} \frac{f(x)}{x} < m, \quad M < \liminf_{x \to 0} \frac{f(x)}{x} \le \infty$,

and has at least two positive solutions if there is $\rho > 0$ such that either

$$(E_1) \quad \begin{cases} 0 \le \limsup_{x\to\infty} \frac{f(x)}{x} < m, \\ \min\left\{\frac{f(x)}{\rho} : x \in [c\rho, \rho]\right\} \ge cM,\ x \ne Tx \ \text{ for } x \in \partial\Omega_\rho, \\ 0 \le \limsup_{x \to 0} \frac{f(x)}{x} < m, \end{cases}$$

or

$$(E_2)\quad \begin{cases} M < \liminf_{x\to 0} \frac{f(x)}{x} \le \infty, \\ \max\left\{\frac{f(x)}{\rho} : x \in [0,\rho]\right\} \le m,\ x \ne Tx \ \text{ for } x \in \partial\Omega_\rho, \\ M < \liminf_{x\to\infty} \frac{f(x)}{x} \le \infty, \end{cases}$$

where

$$\Omega_\rho = \{x \in K:\ c||x||_\infty \le \min_{a\le t\le b} x(t) < c\rho\}.$$

A more general three-point BVP was studied by Ma and Wang. In [102], they studied the existence of positive solutions of the following BVP

$$x'' + a(t)x'(t) + b(t)x(t) + h(t)f(x) = 0, \quad t \in (0,1), \tag{34}$$

$$x(0) = 0, \qquad x(1) = \alpha x(\eta) \tag{35}$$

under the assumptions:

(H1) $h : [0,1] \to [0,\infty)$ is continuous and $h(t) \not\equiv 0$ on any subinterval of $[0,1]$;

(H2) $f : [0,\infty) \to [0,\infty)$ is continuous;

(H3) $a : [0,1] \to R$, $b : [0,1] \to (-\infty, 0)$ are continuous;

(H4) $0 < \alpha\phi_1(\eta) < 1$, where ϕ_1 be the unique solution of the boundary value problem

$$\phi'' + a(t)\phi'(t) + b(t)\phi(t) = 0, \quad t \in (0,1),$$

$$\phi(0) = 0, \quad \phi(1) = 1.$$

THEOREM 4.3 [102]. Let (H1),(H2), (H3) and (H4) hold. Then (34),(35) has at least one positive solution in the case

(i) $f_0 = 0,\ f_\infty = \infty$ (superlinear case); or

(ii) $f_0 = \infty,\ f_\infty = 0$ (sublinear case).

In [84], Ma considered the existence of positive solutions for superlinear semipositone m-point boundary value problems

$$(p(t)u')' - q(t)u + \lambda f(t,u) = 0, \qquad r < t < R, \tag{36}$$

$$\begin{array}{l} au(r) - bp(r)u'(r) = \sum_{i=1}^{m-2} \alpha_i u(\xi_i), \\ cu(R) + dp(R)u'(R) = \sum_{i=1}^{m-2} \beta_i u(\xi_i), \end{array} \tag{37}$$

where $p, q \in C([r,R],(0,\infty))$, $a,b,c,d \in [0,\infty)$, $\xi_i \in (r,R)$, $\alpha_i,\ \beta_i \in [0,\infty)$ (for $i \in \{1,\cdots m-2\}$) are given constants.

Let

(A1) $p \in C^1([r,R],(0,\infty))$, $q \in C([r,R],(0,\infty))$; and

(A2) $a,b,c,d \in [0,\infty)$ with $ac+ad+bc > 0$; $\alpha_i,\ \beta_i \in [0,\infty)$ for $i \in \{1,...,m-2\}$.

And let ψ and ϕ be the solutions of the linear problems

$$\begin{cases} (p(t)\psi'(t))' - q(t)\psi(t) = 0, \\ \psi(r) = b, \qquad p(r)\psi'(r) = a \end{cases}$$

and

$$\begin{cases} (p(t)\phi'(t))' - q(t)\phi(t) = 0, \\ \phi(R) = d, \qquad p(R)\phi'(R) = -c \end{cases}$$

respectively. Set

$$\rho := p(r)\begin{vmatrix} \phi(r) & \psi(r) \\ \phi'(r) & \psi'(r) \end{vmatrix}, \quad \Delta := \begin{vmatrix} -\sum_{i=1}^{m-2}\alpha_i\psi(\xi_i) & \rho-\sum_{i=1}^{m-2}\alpha_i\phi(\xi_i) \\ \rho-\sum_{i=1}^{m-2}\beta_i\psi(\xi_i) & -\sum_{i=1}^{m-2}\beta_i\phi(\xi_i) \end{vmatrix}.$$

THEOREM 4.4 [84]. Let (A1), (A2) hold. Assume that

(A3) $\Delta < 0$, $\rho - \sum_{i=1}^{m-2}\alpha_i\phi(\xi_i) > 0$, $\rho - \sum_{i=1}^{m-2}\beta_i\psi(\xi_i) > 0$;

(A4) $f : [r,R]\times[0,\infty) \to R$ is continuous and there exists an $M > 0$ such that $f(t,u) \geq -M$ for every $t \in [r,R]$, $u \geq 0$.

(A5) $\lim_{u\to\infty}\frac{f(t,u)}{u} = \infty$ uniformly on a compact subinterval $[\alpha,\beta]$ of (r,R).

Then (36),(37) has a positive solution for $\lambda > 0$ sufficiently small.

REMARK 4.1. It is worth remarking that (A3) can be reduced to (31) if the special problem (28),(29) is considered.

REMARK 4.2. The Green's function in (33) contains two negative terms and one positive term, it is *not a good form* in the study of positive solutions. Fortunately, we can construct Green's functions for multi-point BVPs (34),(35) and (36),(37) via the Green's functions of the corresponding two-point BVPs, see [102, 84]. For example, (33) can be rewritten as

$$k(t,\ s) = G(t,\ s) + \frac{\alpha}{1-\alpha\eta}G(\eta,\ s), \tag{38}$$

where

$$G(t,\ s) = \begin{cases} (1-t)s, & 0 \leq s \leq t \leq 1, \\ (1-s)t, & 0 \leq t \leq s \leq 1. \end{cases}$$

Obviously, (38) contains only two nonnegative terms. It is convenient for us to check the strongly positivity of the related integral operators.

5 Global Continua of Positive Solutions and Nodal Solutions of Multi-Point BVPs

The results on the existence of positive solutions of the nonlinear multi-point BVP

$$\begin{aligned} &u'' + h(t)f(u) = 0, \\ &u(0) = 0,\ u(1) = \alpha u(\eta) \end{aligned} \tag{39}$$

has also been introduced in Theorem 4.1. However Theorem 4.1 gives no information on the interesting problem as to what happens to the norms of positive solutions of (39) as α varies in $[0, \frac{1}{\eta})$. Ma and Thompson [100] gave an answer to this question.

Denote by Σ the closure of the set

$$\{(\lambda, u) \in [0, \frac{1}{\eta}) \times C[0,1]\ |u \text{ is a positive solution of (39)}\}$$

in $R \times C[0,1]$, and assume that

(A1) $h \in C([0,1],[0,\infty))$ does not vanish on any subinterval of $[0,1]$;

(A2) $f \in C([0,\infty),[0,\infty))$ and $f(s) > 0$ for $s > 0$;
(A3) $\alpha > 0$ and $\eta \in (0,1)$ are given constants satisfying

$$0 < \alpha < \frac{1}{\eta}.$$

THEOREM 5.1 [100]. Let (A1), (A2) and (A3) hold. Let $f_0 = 0$, $f_\infty = \infty$ (*superlinear*). Then Σ contains a continuum which joins $\{0\} \times C[0,1]$ with $(\frac{1}{\eta}, 0)$.

THEOREM 5.2 [100]. Let (A1), (A2) and (A3) hold. Let $f_0 = \infty$, $f_\infty = 0$ (*sublinear*). Then Σ contains a continuum which joins $\{0\} \times C[0,1]$ with $(\frac{1}{\eta}, \infty)$.

In 2004, Ma and Thompson [97] considered the existence and multiplicity of nodal solutions (*u is called a nodal solution if each zero of u in the open interval* $(0,1)$ *is simple*) to the problem

$$u''(t) + rh(t)f(u) = 0, \quad t \in (0,1), \tag{40}$$

$$u(0) = u(1) = 0 \tag{41}$$

under the assumptions:
(H1) $f \in C(R,R)$ with $sf(s) > 0$ for $s \neq 0$;
(H2) there exist $f_0,\ f_\infty \in (0,\infty)$ such that

$$f_0 = \lim_{|s|\to 0} \frac{f(s)}{s}, \qquad f_\infty = \lim_{|s|\to\infty} \frac{f(s)}{s}.$$

Let λ_k be the k-th eigenvalue of

$$\begin{aligned} \varphi'' + \lambda h(t)\varphi &= 0, \quad 0 < t < 1, \\ \varphi(0) &= \varphi(1) = 0, \end{aligned}$$

and let φ_k be an eigenfunction corresponding to λ_k. It is well-known that

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_k < \lambda_{k+1} < \cdots, \quad \lim_{k\to\infty} \lambda_k = \infty$$

and that φ_k has exactly $k-1$ zeros in $(0,1)$. By applying the bifurcation theorem of Rabinowitz [115], they established the following result.

THEOREM 5.3 [97]. Let (H1), (H2) and (A1) hold. Assume that for some $k \in \mathbb{N}$, either

$$\frac{\lambda_k}{f_\infty} < r < \frac{\lambda_k}{f_0}$$

or

$$\frac{\lambda_k}{f_0} < r < \frac{\lambda_k}{f_\infty}.$$

Then (40),(41) has two solutions u_k^+ and u_k^- such that u_k^+ has exactly $k-1$ zero in $(0,1)$ and is positive near 0, and u_k^- has exactly $k-1$ zero in $(0,1)$ and is negative near 0.

REMARK 5.1. Since a positive solution can be thought as a nodal solution whose number of nodal points is 0, Theorem 5.3 generalizes and unifies many known results on the existence of positive solutions for nonlinear two-point BVPs.

To study the nodal solutions of nonlinear m-point BVPs

$$u'' + f(u) = 0, \quad t \in (0,1), \tag{42}$$

$$u(0) = 0, \;\; u(1) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i), \tag{43}$$

we firstly consider the *spectral properties* of the linear eigenvalue problem

$$u'' + \lambda u = 0, \quad t \in (0,1), \tag{44}$$

$$u(0) = 0, \;\; u(1) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i) \tag{45}$$

under the assumptions:

(G0) $\eta_i = \frac{p_i}{q_i} \in \mathbb{Q} \cap (0,1)$ $(i = 1, \cdots, m-2)$ with $p_i,\ q_i \in \mathbb{N}$ and $(p_i, q_i) = 1$;

(G1) $\alpha_i \in (0, \infty)$, $(i = 1, 2, \cdots, m-2)$ with $0 < \sum_{i=1}^{m-2} \alpha_i \le 1$;

(G2) $f \in C^1(R, R)$ with $sf(s) > 0$ for $s \neq 0$ and $f_0, f_\infty \in (0, \infty)$ exist.

THEOREM 5.4 [96]. Let (G0) and (G1) hold, and let

$$q^* := \min\{\hat{q} \in \mathbb{N} \mid \Gamma(s + 2\hat{q}\pi) = \Gamma(s),\ \forall s \in \mathbb{R}\},$$

where

$$\Gamma(s) = \sin(s) - \sum_{i=1}^{m-2} \alpha_i \sin(\eta_i s),$$

and

$$l = \sharp\{t \mid \Gamma(t) = 0,\ t \in (0, 2q^*\pi]\}$$

respectively. Assume that the sequence of positive solutions of $\Gamma(s) = 0$ is

$$s_1 < s_2 < \cdots < s_n < \cdots .$$

Then

(1) The sequence of positive eigenvalues of (44),(45) are exactly given by

$$\lambda_n = s_n^2, \qquad n = 1, 2, ...;$$

(2) For each $n \in \mathbb{K}$, the eigenfunction corresponding to λ_n is

$$\varphi_n(t) = \sin(\sqrt{\lambda_n}\, t);$$

(3) For each $n = kl + j$ with $k \in \mathbb{N} \cup \{0\}$ and $j \in \{1, \cdots, l\}$,

$$\sqrt{\lambda_{lk+j}} = 2kq^*\pi + \sqrt{\lambda_j}.$$

THEOREM 5.5 [122] Let (G0) hold and assume that

(G3) $\alpha_i \in (0, \infty)$, $(i = 1, 2, \cdots, m-2)$ with $0 < \sum_{i=1}^{m-2} \alpha_i < 1$.

Assume that the sequence of positive solutions of $\Gamma(s) = 0$ is

$$s_1 < s_2 < \cdots < s_n < \cdots .$$

Then the sequence of positive characteristic values of the operator K (the integral operator corresponding the problems (44),(45)) is

$$s_1^2 < s_2^2 < \cdots < s_n^2 < \cdots .$$

Moreover, the characteristic values s_n^2 have algebraic multiplicity one, and the corresponding eigenfunction is

$$\varphi_n(t) = \sin(s_n t).$$

Combining the above *spectral properties* and applying the Rabinowitz global bifurcation theorem, Ma and O'Regan proved the following

THEOREM 5.6 [96]. Let

$$Z_n := \{t \in (0,1) \mid \sin(\sqrt{\lambda}_n t) = 0\}$$

and

$$\mu_n := \sharp Z_n.$$

Let (G2) and (G3) hold and assume that

(G4) $\eta_i = \frac{p_i}{q_i} \in \mathbb{Q} \cap (0, \frac{1}{2}]$, $i = 1, ..., , m-2$, with $p_i,\ q_i \in \mathbb{N}$ and $(p_i, q_i) = 1$.

Assume that either

$$f_0 < \lambda_{kl+1} < f_\infty$$

or

$$f_\infty < \lambda_{kl+1} < f_0$$

for some $k \in \mathbb{N} \cup \{0\}$.

Then problem (42),(43) has two solutions u^+_{kl+1} and u^-_{kl+1}; u^+_{kl+1} has exactly μ_{kl+1} zeros in $(0,1)$ and is positive near $t = 0$, and u^-_{kl+1} has exactly μ_{kl+1} zeros in $(0,1)$ and is negative near $t = 0$.

THEOREM 5.7 [96]. Let (G2) and(G3) and (G4). Assume that either (i) or (ii) holds for some $k \in \mathbb{N} \cup \{0\}$ and $j \in \{0\} \cup \mathbb{N}$:

(i) $f_0 < \lambda_{kl+1} < \cdots < \lambda_{(k+j)l+1} < f_\infty$;

(ii) $f_\infty < \lambda_{kl+1} < \cdots < \lambda_{(k+j)l+1} < f_0$.

Then problem (42),(43) has $2(j+1)$ solutions $u^+_{(k+i)l+1}$, $u^-_{(k+i)l+1}$, $i = 0, ..., j$; $u^+_{(k+i)l+1}$ has exactly $\mu_{(k+i)l+1}$ zeros in $(0,1)$ and is positive near $t = 0$, $u^-_{(k+i)l+1}$ has exactly $\mu_{(k+i)l+1}$ zeros in $(0,1)$ and is negative near $t = 0$.

Very recently, Rynne studied the linear eigenvalue problem (44),(45). He proved the following

THEOREM 5.8 [117]. Let $m \geq 3$, $\eta_i \in (0,1)$ and $\alpha_i > 0$ for $i = 1, ..., m-2$, with

$$\sum_{i=1}^{m-2} \alpha_i < 1.$$

Then the eigenvalues of (44),(45) form a strictly increasing sequence

$$0 < \lambda_1 < \lambda_1 < \cdots < \lambda_k < \cdots$$

with corresponding eigenfunctions $\phi_k(x) = \sin(\lambda_k^{1/2} x)$. In addition

(1) $\lim_{k\to\infty} \lambda_k = \infty$;

(2) $\phi_k \in T_k^+$, for each $k \geq 1$, and ϕ_1 is strictly positive on $(0,1)$, where T_k^ν ($\nu = \{\pm\}$) is the set of function $n : [0,1] \to \mathbb{R}$ satisfying

(i) $u(0) = 0$, $\nu u'(0) > 0$ and $u'(1) \neq 0$;

(ii) u' has only simple zeros in $(0,1)$, and has exactly k such zeros;

(iii) u has a zero strictly between each two consecutive zeros of u'.

These spectral properties were used to prove a Rabinowitz-type global bifurcation theorem for a bifurcation problem related the nonlinear m-point BVP (42),(43). Moreover, he obtained the following

THEOREM 5.9 [117]. Let $f \in C^1(R, R)$ with $f(0) = 0$. Assume that f_∞ is finite. If, for some $k \in \mathbb{N}$,

$$(\lambda_k - f_0)(\lambda_k - f_\infty) < 0.$$

Then (42),(43) has solutions $u_k^\pm \in T_k^\pm$.

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