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An Existence Result For A Free Boundary Problem For The p-Laplace Operator*

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Abstract

This paper deals with a free boundary problem for the p-Laplace operator. We will use the compactness-continuity result for the solution of a non linear Dirichlet problem, due to D. Bucur and P. Trebeschi, and prove the existence of solution (which is of class C^2) for the associated shape optimization problem. The shape derivative and Hopf's comparison principle allow us to give a sufficient condition of existence for the free boundary problem.

1 Introduction

Let D be an open ball of $\mathbb{R}^N$ $(N \geq 2)$ which will contain all the sets we use in this paper. Given an L^∞-function $f \geq 0$ which has a compact support K with a nonempty interior. Let k be a parameter, $k > 0$. We look for an open and bounded set Ω $(\supset K)$, such that there exists a function u_Ω, satisfying the following overdetermined problem (FL)

$$-\Delta_p u_\Omega = -\mathrm{div}(|\nabla u_\Omega|^{p-2}\,\nabla u_\Omega) = f \text{ in } \Omega,\ u_\Omega = 0 \text{ and } |\nabla u_\Omega| = k \text{ on } \partial\Omega.$$

Most of existing results for the problem (FL) assume that $p = 2$, e.g [7], [2]. For other values of p, this is an open question.

In [8], the authors showed, by using the moving plane method [6], that if the problem (FL) admits a solution (Ω, u_Ω) such that Ω is of class C^2 and $u_\Omega \in C^2(\overline{\Omega}\setminus K)\cap C^1\left(\overline{\Omega}\right)$, then all the inward normals at the boundary $\partial\Omega$ of Ω meet C (the convex hull of K). Since we relate the existence of a solution for Problem (FL) to the existence of a minimum of some shape optimization problem, it is natural to solve this one in a class of domains with this geometric normal property (see below).

Using the shape derivative, the problem (FL) can be seen as the Euler equation of the following problem of minimization, e.g. [11]:

$$(OP)\ \text{ Find } \Omega \in \mathcal{O}_C \text{ such that } J(\Omega) = \min_{\omega\in\mathcal{O}_C} J(\omega),$$

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where $\mathcal{O}_C = \{\omega \subset D : \omega$ satisfies C-GNP$\}$ and

$$J(\omega) = \int_\omega \left(\frac{1}{p} |\nabla u_\omega|^p - f u_\omega + \frac{k^p}{p} \right) dx$$

with u_ω the solution of the Dirichlet problem $P(\omega, f)$:

$$-\Delta_p u_\omega = f \;\text{ in }\; \omega, \;\; u_\omega = 0 \;\text{ on }\; \partial\omega.$$

This paper deals with the problem (FL). We will use the compactness-continuity result for the solution of a non linear Dirichlet problem, due to Bucur and Trebeschi [4], and prove the existence of solution (for the shape optimization problem (OP)) which is of class C^2. Then the shape derivative and Hopf's comparison principle allow us to give a sufficient condition of existence of solution for our free boundary problem as in the case of Laplace operator [2].

2 Preliminaries

We need a few definitions.

DEFINITION 1. Let K_1 and K_2 be two compact subsets of D. We call a Hausdorff distance of K_1 and K_2 (or briefly $d_H(K_1, K_2)$), the following positive number:

$$d_H(K_1, K_2) = \max\left[\rho(K_1, K_2), \rho(K_2, K_1)\right],$$

where $\rho(K_i, K_j) = \max_{x \in K_i} d(x, K_j)$ $\; i, \; j = 1, \; 2$ and $d(x, K_j) = \min_{y \in K_i} |x - y|$.

DEFINITION 2. Let ω_n be a sequence of open subsets of D and ω be an open subset of D. Let K_n and K be their complements in $\overline{D}$. We say that the sequence ω_n converges in the Hausdorff sense, to ω (or briefly $\omega_n \xrightarrow{H} \omega$) if $\lim_{n \to +\infty} d_H(K_n, K) = 0$.

DEFINITION 3. Let ω_n be a sequence of open subsets of D and ω be an open subset of D. We say that the sequence ω_n converges in the compact sense, to ω (or briefly $\omega_n \xrightarrow{K} \omega$) if

- every compact subset of ω is included in ω_n, for n sufficiently *large*, and
- every compact subset of $\overline{\omega}^c$ is included in $\overline{\omega}_n^c$, for n sufficiently *large.*

DEFINITION 4. Let ω_n be a sequence of open subsets of D and ω be an open subset of D. We say that the sequence ω_n converges in the sense of characteristic functions, to ω (or briefly $\omega_n \xrightarrow{L} \omega$) if χ_{ω_n} converges to χ_ω in $L^p_{loc}(\mathbb{R}^N)$, $p \neq \infty$, (χ_ω is the characteristic function of ω).

LEMMA 1. According to [5], if ω_n is a sequence of open subsets of D, there exists a subsequence (still denoted by ω_n) which converges, in the Hausdorff sense, to some open subset of D.

DEFINITION 5. According to [3], the following holds. Let C be a compact convex set. The bounded domain ω satisfies C-GNP if (i) $\omega \supset int(C)$, (ii) $\partial\omega \setminus C$ is locally

Lipschitz, (iii) for any $c \in \partial C$ there is an outward normal ray Δ_c such that $\Delta_c \cap \omega$ is connected, (iv) for every $x \in \partial\omega \setminus C$ the inward normal ray to ω (if exists) meets C.

REMARK 1. If Ω satisfies the C-GNP and C has a nonempty interior, then Ω is connected.

THEOREM 1. If $\omega_n \in \mathcal{O}_C$, then there exists an open subset $\omega \subset D$ and a subsequence (again labeled ω_n) such that (i) $\omega_n \xrightarrow{H} \omega$, (ii) $\omega_n \xrightarrow{K} \omega$, (iii) χ_{ω_n} converges to χ_ω in $L^1(D)$ and (iv) $\omega \in \mathcal{O}_C$.

For the proof of this theorem, see Theorem 3.1 in [3].

DEFINITION 6. Let C be a convex set. We say that an open subset ω has the C-SP, if (i), (ii), (iii) of Definition 5 are satisfied and if (v) $\forall\, x \in \partial\omega \setminus C \;\; K_x \cap \omega = \emptyset$, where K_x is the closed cone defined by $\left\{y \in \mathbb{R}^N : (y-x)\cdot(z-x) \le 0,\ \forall z \in C\right\}$.

REMARK 2. K_x is the normal cone to the convex hull of C and $\{x\}$.

PROPOSITION 1. ω has the C-GNP if and only if ω satisfies the C-SP.

For the proof of this proposition see Proposition 2.3 in [3].

The aim of the following theorem is to prove the existence of a minimum of J which is of class C^2. This in order to use the shape derivative and so to give a solution to Problem (FL).

THEOREM 2. Let L be a compact subset of $\mathbb{R}^N$. Let f_n be a sequence a functions defined on L. We assume that the functions f_n are of class C^3 and

$$\left|\frac{\partial f_n}{\partial x_i}\right| \le M,\ \left|\frac{\partial^2 f_n}{\partial x_i \partial x_j}\right| \le M,\ \left|\frac{\partial^3 f_n}{\partial x_i \partial x_j \partial x_k}\right| \le M,$$

where M is a strictly positive constant and is independent of n. Define a sequence Ω_n, by $\Omega_n = \{x \in L \,:\, f_n(x) > 0\}$ and suppose there exists $\alpha > 0$ such that $|f_n(x)| + |\nabla f_n(x)| \ge \alpha$ for all x in L. If the domains Ω_n have the C-GNP, then there exists Ω of class C^2 and a subsequence (still denoted by Ω_n) such that Ω_n converges in the compact sense, to Ω and $J(\Omega) = \min_{\omega \in \mathcal{O}_C} J(\omega)$.

The proof of this theorem uses the following lemma which we prove for the convenience of the reader (see [2]).

LEMMA 2. Let L be a compact subset of $\mathbb{R}^N$. Let f_n be a sequence of functions defined as in Theorem 2. Suppose that Ω is an open subset of L such that

$$\Omega = \{x \in L : h(x) > 0\} \text{ and } \partial\Omega = \{x \in L : h(x) = 0\},$$

where h is a continuous function defined in L. If the functions f_n converge uniformly to h in L, then Ω_n converges in the compact sense, to Ω.

PROOF.

1. Let K_1 be a compact subset of Ω. If $\beta_1 = \inf_{K_1} h$, $\beta_1 > 0$ and there exists $n_1 \in \mathbb{N}$ such that for all $n \ge n_1$, $|f_n - h|_{L^\infty(K_1)} < \beta_1$. This implies that for all $x \in K_1$, $f_n(x) > h(x) - \beta_1 \ge 0$ and then K_1 is contained in Ω_n, for $n \ge n_1$.

2. Let K_2 be a compact subset of $\overline{\Omega}^c$. By hypothesis, $\overline{\Omega} = \Omega \cup \partial\Omega = \{x \in L \,:\, h(x) \ge 0\}$.

If $\beta_2 = \max_{K_2} h$, $\beta_2 < 0$ and there exists $n_2 \in \mathbb{N}$ such that for all $n \geq n_2$,

$$|f_n - h|_{L^\infty(K_2)} < -\beta_2.$$

This implies that for all $x \in K_2$, $f_n(x) < h(x) - \beta_1 \leq 0$ and then K_1 is contained in $\overline{\Omega}_n^c$, for $n \geq n_2$ because $\{x \in L : h(x) < 0\} \subset \overline{\Omega}_n^c$.

REMARK 3. The hypothesis in the preceding theorem about the local regularity is not too restrictive because of, for instance, results due to G.M. Lieberman [9].

LEMMA 3. (Hopf's Comparison principle). Let $U \subset \mathbb{R}^N$ be open and bounded, and $v_1,\ v_2 \in C^1\left(\overline{U}\right)$, with $\Delta_p v_1 \leq \Delta_p v_2$. Then the following hold.

1. If $v_1 \geq v_2$ on ∂U, then $v_1 \geq v_2$ in U.
2. Suppose $v_1 > v_2$ in U, $v_1(x) = v_2(x)$ for some $x \in \partial U$, $|\nabla v_2| \geq \gamma$ in U (for some $\gamma > 0$), and U satisfies the interior sphere condition. Then $\frac{\partial v_2}{\partial \nu}(x) > \frac{\partial v_1}{\partial \nu}(x)$, where ν is the unit outward normal vector on ∂U, at x.
3. If $v_1 \geq v_2$ and $v_1 \neq v_2$ in U, $|\nabla v_2| \geq \gamma$ in U (for some $\gamma > 0$), then $v_1 > v_2$ in U.

This lemma is proven in ([12], Lemma 3.2, Proposition 3.4.1, 3.4.2)

3 Main Theorems

In this section we state our main results, which will be proven in Section 6.

THEOREM 3. There exists $\Omega \in \mathcal{O}_C$ which minimizes the functional J on $\mathcal{O}_C$. Ω is of class C^2.

We would like to say that the minimum obtained in Theorem 3 is a solution of the problem (FL). It should be noted that, without any assumptions on f and k, the problem (FL) does not have, in general, a solution. Let us recall two examples of non-existence for the problem (FL) which can be found in [8].

EXAMPLE 1. Put $k = 1$. Let (Ω, u_Ω) be a solution of (FL). Then, integration by parts gives

$$\int f dx = -\int_\Omega \operatorname{div}(|\nabla u_\Omega(x)|^{p-2} \nabla u_\Omega) = -\int_{\partial\Omega} |\nabla u_\Omega(x)|^{p-2} \nabla u_\Omega \cdot \nu = \int_{\partial\Omega} d\sigma. \quad (1)$$

Now let $f = c\chi_U dx$, where $c = N^{(N-1)/N}\left(\frac{c_N}{|U|}\right)^{1/N}$ and c_N is the area of the unit sphere in $\mathbb{R}^N$. Then (1) gives $N^{(N-1)/N}\left(\frac{c_N}{|U|}\right)^{1/N}.|U| = |\partial\Omega|$, and by the (strict) inclusion $U \subset \Omega$,

$$c_N^{1/N}\left(N\,|\Omega|\right)^{(N-1/N)} > c_N^{1/N}\left(N\,|U|\right)^{(N-1/N)} = |\partial\Omega|\,.$$

($|\Omega|$ and $|\partial\Omega|$ are respectively, the volume and the perimeter of Ω). This obviously contradicts the well-known isoperimetric inequality [1]. Therefore for f as above there cannot exist a solution to (FL).

EXAMPLE 2. Let $k = 1$, and suppose (Ω, u_Ω) solves (FL). Set $M = \sup f$, and let $B(x_0, r_\Omega)$ be the smallest ball containing Ω. Then $M r_\Omega > N$.

This provides us with a test for non-existence. To prove this inequality, one defines

$$v(x) = \left(\frac{p-1}{p}\right) \frac{r_\Omega^{p/(p-1)} - |x - x_0|^{p/(p-1)}}{r_\Omega^{1/(p-1)}}.$$

Then $-\Delta_p v = \frac{N}{r_\Omega}$. Now if $M r_\Omega \leq N$, then

$$\Delta_p u_\Omega = -f \geq -M \geq -\frac{N}{r_\Omega} = \Delta_p v \quad \text{in} \quad \Omega.$$

Since also $u_\Omega = 0 \leq v$ on $\partial\Omega$, one may apply parts 1. and 3. of Lemma 3 to deduce that $v > u_\Omega$ in Ω. (Or at least in some interior neighborhood of $\partial\Omega$) Now let $y \in \partial\Omega$ correspond to largest distance to x_0, i.e. $|y - x_0| = r_\Omega$, and observe that the unit outward normal vector ν at y equals $(y - x_0) / |y - x_0|$ and that $u_\Omega(y) = v(y) = 0$. Invoking part 2. of Lemma 3 one concludes $-1 = \frac{\partial u_\Omega}{\partial \nu}(y) > \frac{\partial v}{\partial \nu}(y) = -1$, which is a contradiction.

The aim, now, is to give a sufficient condition on f and k in order that Ω contains strictly C and that $|\nabla u_\Omega(x)| = k$ on $\partial\Omega$. For that purpose, we will need the Hopf's comparison principle.

THEOREM 4. Suppose that K has a nonempty interior. Let Ω be a minimum of the functional J on $\mathcal{O}_C$ which is of class C^2. Let u_Ω and u_C be, respectively, the solution of Dirichlet problems $P(\Omega, f)$ and $P(int(C), f)$. Suppose that $u_C \in C^1(C)$ and $u_\Omega \in C^1\left(\overline{\Omega}\right)$. If C satisfies the interior sphere condition and if

$$|\nabla u_C| > k \quad \text{on} \quad C \tag{2}$$

then C is strictly contained in Ω and $|\nabla u_\Omega| = k$ on $\partial\Omega$.

4 Continuity With Respect to the Domain

As in the linear case, to obtain a continuity result we can use the compact convergence and the p-stability of the limit domain (we say that an open set Ω is p-stable if for any $u \in H^{1,p}\left(\mathbb{R}^N\right)$ such that $u = 0$ a.e. in $int(\Omega^c)$, we get $u_{|\Omega} \in H_0^{1,p}(\Omega)$). Here, we will use the theorem (see below) obtained by Bucur and Trebeschi where they generalize the Sverak's result [10].

In [4], the authors gave a compactness-continuity result for the solution of a non linear Dirichlet problems (in particular with the p-Laplacian operator) when the domain varies.

DEFINITION 7. (γ_p-convergence) We say that a sequence Ω_n of open subsets of D γ_p-converges to Ω if and only if for any $f \in H^{-1,q}(D)$ ($\frac{1}{p} + \frac{1}{q} = 1$) the solutions u_n of the Dirichlet problems $P(\Omega_n, f)$ converges strongly in $H_0^{1,p}(D)$, as $n \to +\infty$, to the solution u_Ω of $P(\Omega, f)$, (u_n and u_Ω are extended by zero to D).

Set

$$\mathcal{O}_l(D) = \{\omega \subseteq D \mid \sharp\omega^c \leq l\}$$

where $\sharp\omega^c$ denotes the number of connected components of the complement of ω.

THEOREM 5. (Bucur-Trebeschi) Let $N \geq p > N-1$. Consider $\Omega_n \in \mathcal{O}_l(D)$ and assume $\Omega_n \xrightarrow{H} \Omega$, then $\Omega \in \mathcal{O}_l(D)$ and Ω_n γ_p-converges to Ω.

REMARK 4. If $p > N$, any sequence of open sets which converge in the Hausdorff sense is γ_p-convergent.

COROLLARY 1. Assume that the convex C has a nonempty interior. If $\Omega_n \in \mathcal{O}_C$ and $\Omega_n \xrightarrow{H} \Omega$, then Ω_n γ_p-converges to Ω.

PROOF. If the interior of C is nonempty and $\Omega_n \in \mathcal{O}_C$, according to Remark 1, Ω_n is connected. Therefore $\Omega_n \in \mathcal{O}_l(D)$. Now, if $\Omega_n \xrightarrow{H} \Omega$, by the previous theorem Ω_n γ_p-converges to Ω.

5 Optimality Condition

As it is mentioned in the introduction of this paper, we are going to use the standard tool of the domain derivative to write down the optimality condition. Let us recall the definition of the domain derivative, see for instance [11]. We assume that the minimum Ω of the functional J is of class C^2. Let us consider a deformation field $V \in C^2(\mathbb{R}^N; \mathbb{R}^N)$ and set $\Omega_t = \{x + tV(x),\ x \in \Omega\}$, $t > 0$. The application $Id + tV$ is a perturbation of the identity which is a Lipschitz diffeomorphism for t small enough. By definition, the derivative of J at Ω in the direction V is

$$dJ(\Omega, V) = \lim_{t\to 0} \frac{J(\Omega_t) - J(\Omega)}{t}.$$

As the functional J depends on the domain Ω through the solution of the Dirichlet problem $P(\Omega, f)$, we need to define also the domain derivative of u_Ω. If u'_Ω denotes the domain derivative of u_Ω, then $u'_\Omega = \lim_{t\to 0} \frac{u_{\Omega_t} - u_\Omega}{t}$. Now, if $J(\Omega) = \int_\Omega h(u_\Omega)dx$, by the Hadamard formula

$$dJ(\Omega, V) = \int_\Omega h'(u_\Omega)\, u'_\Omega dx + \int_{\partial\Omega} h(u_\Omega) V \cdot n \, d\sigma.$$

Furthermore, we can prove (see [11]) that u'_Ω is a solution of some linear Dirichlet problem with $u'_\Omega = -\frac{\partial u_\Omega}{\partial n} V \cdot n$ on $\partial\Omega$. This, together with $u_\Omega = 0$ on $\partial\Omega$ and the Hadamard formula implies

$$dJ(\Omega; V) = \frac{1}{p} \int_{\partial\Omega} \left(k^p - |\nabla u_\Omega(x)|^p\right) V.n \, d\sigma. \tag{3}$$

where n is the outward normal vector to $\partial\Omega$.

Now since Ω is the minimum for the functional J, $dJ(\Omega; V) \geq 0$ for every admissible direction V. Therefore $\int_{\partial\Omega} \left(k^p - |\nabla u_\Omega(x)|^p\right) V.n \, d\sigma \geq 0$ for every admissible direction V. We mean by admissible displacement the one which allows us to keep the C-GNP or the C-SP (according to Proposition 1 above). Since Ω has the C-GNP, it satisfies the C-SP. Then

$$\forall x \in \partial\Omega \setminus C \;\; K_x \cap \Omega = \emptyset.$$

For t sufficiently small, let $\Omega_t = \Omega + tV(\Omega)$ be the deformation of Ω in the direction V. Let $x_t \in \partial\Omega_t$. There exists $x \in \partial\Omega$ s.t $x_t = x + tV(x)$. Using the definition of K_{x_t} and the equality above, it is obvious to get (for t small enough and for every displacement $V : \forall\, x_t \in \partial\Omega_t \setminus C \;\; K_{x_t} \cap \Omega_t = \emptyset$, which means that Ω_t satisfies the C-SP (and so the C-GNP) for every displacement V when t is sufficiently small. Then, using V and $-V$, and the fact that the set of the functions $V \cdot \nu$ is dense in $L^2(\partial\Omega)$, we deduce

$$|\nabla u_\Omega(x)| = k \text{ on } \partial\Omega \setminus C. \tag{4}$$

On the other hand, the admissible directions V on $\partial\Omega \cap \partial C$ must satisfy $V(x) \cdot n(x) \geq 0$, and one gets

$$|\nabla u_\Omega(x)| \leq k \text{ on } \partial\Omega \cap \partial C. \tag{5}$$

6 Proofs of the Main Theorems

6.1 Proof of Theorem 3

PROOF. Using the variational formulation of the Dirichlet problem $P(\omega, f)$, we get $\int_\omega |\nabla u_\omega(x)|^p dx = \int_\omega f u_\omega$. If u_D denotes the solution of the Dirichlet problem $P(D, f)$, by the Hopf's comparison principle (see Lemma 3 part 1.), $0 \leq u_\omega \leq u_D$ so

$$J(\omega) = -\frac{p-1}{p} \int_\omega f u_\omega + \frac{k^p}{p} \int_\omega dx \geq -\frac{p-1}{p} \int_D f u_D$$

and inf J exists. Let Ω_n be a minimizing sequence in $\mathcal{O}_C$ as in Theorem 2.

Since $int(C) \subset \Omega_n \subset D$, according to (i) of Theorem 1 and the continuity of the inclusion for the Hausdorff topology, there exist an open set Ω, and a subsequence of Ω_n (still denoted by Ω_n) such that $\Omega_n \xrightarrow{H} \Omega$ and $int(C) \subset \Omega \subset D$. (ii) of Theorem 1 together with Theorem 2 implies that Ω is of class C^2. Now by (iii) of Theorem 1, $\int_{\Omega_n} dx$ converges to $\int_\Omega dx$, and by Corollary 1, $\int_D f u_n \chi_{\Omega_n}$ converges to $\int_D f u_\Omega \chi_\Omega = \int_\Omega |\nabla u_\Omega(x)|^p dx$. Hence $J(\Omega) \leq \liminf_{n \to +\infty} J(\Omega_n)$. According to (iv) of Theorem 1, $\Omega \in \mathcal{O}_C$, therefore $J(\Omega) = \min_{\omega \in \mathcal{O}_C} J(\omega)$.

6.2 Proof of Theorem 4

PROOF. Since the minimum Ω is of class C^2, one can use the shape derivative for the functional J and obtain (4) and (5). We must have $\partial\Omega \neq \partial C$, otherwise $\Omega = int(C)$ and $u_\Omega = u_C$. But (5) gives $|\nabla u_C| = |\nabla u_\Omega| \leq k$ on $\partial\Omega$, which contradicts (2). Now, suppose that $\partial\Omega \cap \partial C \neq \emptyset$. Since u_Ω and u_C are in $C^1(C)$,

$$\Delta_p u_\Omega = -f = \Delta_p u_C \text{ in } int(C) \text{ and } u_\Omega \geq 0 = u_C \text{ on } \partial C,$$

part 1. of Lemma 3 implies that $u_\Omega \geq u_C$ in $int(C)$. But $u_\Omega \neq u_C$ in $int(C)$, then $u_\Omega > u_C$ in $int(C)$. Now, since C satisfies the interior sphere condition, $|\nabla u_C| > k$ on

$int(C)$ and $u_\Omega = u_C$ on $\partial\Omega \cap \partial C$, part 2. of Lemma 3, gives $\frac{\partial u_\Omega}{\partial n} < \frac{\partial u_C}{\partial n}$ on $\partial\Omega \cap \partial C$. Now since u_Ω vanishes on $\partial\Omega$, $|\nabla u_\Omega| = -\dfrac{\partial u_\Omega}{\partial n}$, the previous inequality becomes $|\nabla u_C| < |\nabla u_\Omega|$ on $\partial\Omega \cap \partial C$. This together with (5) implies $|\nabla u_C| < k$ *on* $\partial\Omega \cap \partial C$, which contradicts (2). It then follows that C is strictly contained in Ω and thus, by (4)

$$|\nabla u_\Omega| = k \text{ on } \partial\Omega.$$

References

[1] C. Bandle, Isoperimetric Inequalities and applications, Pitman, 1980.

[2] M. Barkatou, D. Seck and I. Ly, An existence result for a quadrature surface free boundary problem, Cent. Eur. J. Math., 3(1)(2005), 39–57

[3] M. Barkatou, Some geometric properties for a class of non Lipschitz-domains, New York J. Math., (8)(2002), 189–213.

[4] D. Bucur and P. Trebeschi, Shape optimization problems governed by nonlinear state equations, Proc. Roy. Soc. Edinburgh, 128A(1998), 945–963

[5] D. Bucur and J. P. Zolesio, N-dimensional shape optimization under capacitary constraints, J. Diff. Eq., (123-2)(1995), 504–522.

[6] G. Gidas, W. M. Ni and L. Nirenberg, Symmetry and related properties via the maximum principle. Comm. Math. Phys., (68)(1979), 209–300.

[7] B. Gustafsson and H. Shahgholian, Existence and geometric properties of solutions of a free boundary problem in potential theory, J. für die Reine und Ang. Math., 473(1996), 137–179.

[8] H. Hosseinzadeh and H. Shahgholian, Some qualitative aspects of a free boundary problem for the p-Laplacian, Ann. Acad. Scient. Fenn. Math., 24(1999), 109–121.

[9] G. M. Liberman, Boundary regularity for solutions of degenerate elliptic equations, Nonlinear Analysis, 12(1988), 1203–1219.

[10] V. Šverak, On optimal shape design, J. Math. Pures Appl., 72(6)(1993), 537–551.

[11] J. Sokolowski and J. P. Zolesio, Introduction to shape optimization : shape sensitity analysis, Springer Series in Computational Mathematics, 16, Springer, Berlin, 1992.

[12] P. Tolksdorf, On the Dirichlet problem for quasilinear equations in domains with conical boundary points, Comm. Partial Diff. Eq., 8(7)(1983), 773–817.