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A Common Fixed Point Theorem Satisfying Integral Type Implicit Relations*

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Abstract

In this note, a general common fixed point theorem of integral type for two pairs of weakly compatible mappings satisfying integral type implicit relations is obtained in symmetric spaces by using the notion of a pair of mappings satisfying property (E.A). Our main result improves and extends several known results.

1 Introduction

Commutativity of two mappings was weakened by Sessa [13] with weakly commuting mappings. Later on, Jungck [7] enlarged the class of non-commuting mappings by compatible mappings which asserts that a pair of self mappings S and T of a metric space (X, d) is compatible if, $\lim_{n\to\infty} d(TSx_n, STx_n) = 0$, whenever $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$, for some $t \in X$. The concept of compatible mappings was further improved by Jungck and Rhoades [9] with the notion of coincidentally commuting (or weakly compatible) mappings which merely commute at their coincidence points.

In recent years, several common fixed point theorems for contractive type mappings have been established by several authors (see, for instance, Jachymski [6], Jungck et al. [8] and Pant [10]). Using the concept of reciprocal continuity which is a weaker form of continuity of mappings (See Pant [10]), Popa ([11], [12]) proved some fixed point theorems satisfying certain implicit relation. Recently, Aliouche et al. [3] established a general common fixed point theorem for a pair of reciprocally continuous mappings satisfying an implicit relation. Hicks and Rhoades [4] established some common fixed point theorems in symmetric spaces using the fact that some of the properties of metric are not required in the proofs of certain metric theorems. Recall that a symmetric on a set X is a nonnegative real valued function d on $X \times X$ such that

(i) $d(x, y) = 0$ if and only if $x = y$,
(ii) $d(x, y) = d(y, x)$.

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Let d be a symmetric on a set X and for $r > 0$ and any $x \in X$, let $B(x,r) = \{y \in X : d(x,y) < r\}$. A topology $t(d)$ on X is given by $U \in t(d)$ if and only if for each $x \in U, B(x,r) \subset U$ for some $r > 0$. A symmetric d is a semi-metric if for each $x \in X$ and each $r > 0$, $B(x,r)$ is a neighborhood of x in the topology $t(d)$. Note that $\lim_{n\to\infty} d(x_n,x) = 0$ if and only if $x_n \to x$ in the topology $t(d)$.

Amri and Moutawakil [1] proved some common fixed point theorems under strict contractive conditions on a metric space for mappings satisfying the property (E.A).

Recall that the pair (S,T) satisfies property (E.A) if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$.

The following two axioms were appeared in Wilson [14].

(W. 3) Given $\{x_n\}$, x and y in X, $\lim_{n\to\infty} d(x_n,x) = 0$ and $\lim_{n\to\infty} d(x_n,y) = 0$ imply $x = y$.

(W. 4) Given $\{x_n\}$, y_n and x in X, $lim_{n\to\infty} d(x_n,x) = 0$ and $\lim_{n\to\infty} d(x_n,y_n) = 0$ imply that $lim_{n\to\infty} d(y_n,x) = 0$.

Recently, Aliouche [2] introduced the definition of (HE) as follows :

DEFINITION 1.1 ([2]). Let (X,d) be a symmetric space. We say that (X,d) satisfies property (HE) if given $\{x_n\}$, $\{y_n\}$ and x in X, the conditions $\lim_{n\to\infty} d(x_n,x) = 0$ and $\lim_{n\to\infty} d(y_n,x) = 0$ imply $\lim_{n\to\infty} d(x_n,y_n) = 0$.

Let $\mathbb{R}_+$ denote the set of nonnegative real numbers.

In [2], the following common fixed points theorem for self mappings in a symmetric space under a contractive condition of integral type was proved.

THEOREM 1.1 ([2]). Let d be a symmetric for X which satisfies (W.3), (W.4) and (HE). Let A, B, S and T be self mappings of X such that

$$A(X) \subset T(X) \text{ and } B(X) \subset S(X), \tag{1}$$

$$\int_0^{d(Ax,By)} \varphi(t)dt \leq \phi\Big(\int_0^{\max\{d(Sx,Ty),d(Sx,By),d(By,Ty)\}} \varphi(t)dt\Big), \tag{2}$$

for all $x, y \in X$, where $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a Lebesgue-integrable mapping which is summable, non-negative and such that

$$\int_0^{\epsilon} \varphi(t)dt > 0 \text{ for all } \epsilon > 0. \tag{3}$$

Suppose that (A,S) or (B,T) satisfies property (E.A) and (A,S) and (B,T) are weakly compatible. If one of the subspace AX, BX, SX and TX of X is complete, then A, B, S and T have a unique common fixed point of X.

We now introduce the definitions of properties (CE.1) and (CE.2) as follows:

DEFINITION 1.2. Let (X,d) be a symmetric space. We say that (X,d) satisfies the property

(CE.1) Given $\{x_n\}$, x and y in X, $\lim_{n\to\infty} d(x_n,x) = 0$ implies $\lim_{n\to\infty} d(x_n,y) = d(x,y)$.

(CE.2) Given $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ in X, the condition $\lim_{n\to\infty} d(x_n,y_n) = 0$ implies $\limsup_{n\to\infty} d(z_n,y_n) = \limsup_{n\to\infty} d(z_n,x_n)$.

2 Implicit Relations

Let $\mathcal{F}_6$ be the set of all continuous functions $F(t_1, \ldots, t_6) : \mathbb{R}_+^6 \to \mathbb{R}_+$ satisfying the following conditions:

(F_a) $\int_0^{F(u,0,u,0,u,0)} \varphi(t)\,dt \leq 0$ implies $u = 0$;

(F_b) $\int_0^{F(u,0,0,u,0,u)} \varphi(t)\,dt \leq 0$ implies $u = 0$.

The function $F(t_1, \ldots, t_6) : \mathbb{R}_+^6 \to \mathbb{R}_+$ satisfies the condition (F_1) if

(F_1) $\int_0^{F(u,u,0,0,u,u)} \varphi(t)\,dt > 0$; for all $u > 0$ and $\varphi : \mathbb{R}_+ \to \mathbb{R}$ is a Lebesgue-integrable mapping which is summable.

EXAMPLE 2.1. Let $F(t_1, \ldots, t_6) := t_1 - \ c \max\{t_2, t_3, t_4 t_5, t_6\}$, where $0 < c < 1$ and $\varphi(t) = \frac{3\pi}{4(1+t)^2} cos \frac{3\pi t}{4(1+t)}$ for all t in $\mathbb{R}_+$. Then

(F_a) $\int_0^{F(u,0,u,0,u,0)} \frac{3\pi}{4(1+t)^2} cos \frac{3\pi t}{4(1+t)}\ dt \leq 0$; i.e., $\int_0^{(1-c)u} \frac{3\pi}{4(1+t)^2} cos \frac{3\pi t}{4(1+t)}\ dt \leq 0$, so $sin \frac{3\pi(1-c)u}{4\{1+(1-c)u\}} \leq 0$, which implies $u = 0$.

Similarly,

(F_b) $\int_0^{F(u,0,0,u,0,u)} \frac{3\pi}{4(1+t)^2} cos \frac{3\pi t}{4(1+t)}\ dt \leq 0$; so $sin \frac{3\pi(1-c)u}{4\{1+(1-c)u\}} \leq 0$, which implies $u = 0$.

Further,

(F_1) $\int_0^{F(u,u,0,0,u,u)} \frac{3\pi}{4(1+t)^2} cos \frac{3\pi t}{4(1+t)}\ dt > 0$; so $sin \frac{3\pi(1-c)u}{4\{1+(1-c)u\}} > 0$, for all $u > 0$.

REMARK 2.1. In Example 2.1 above, we observe that $\varphi(t) = \frac{3\pi}{4(1+t)^2} cos \frac{3\pi t}{4(1+t)}$ is negative for $t \in (2, \infty)$, positive for $t \in [0, 2)$ and vanishes at $t = 2$.

The main object of this note is to prove a common fixed point theorem for a quadruple of mappings satisfying certain integral type implicit relations, which are viable, productive and powerful tool in finding the existence of common fixed point for two pairs of weakly compatible mappings satisfying a contractive condition in symmetric spaces. The requisition of conditions (W.3), (W.4) and property (HE) of Theorem 1.1 are retained, while condition (3) is dropped and certain restrictions on implicit relations which were used in [5] and [11] are softened.

3 Main Results

Now we state and prove our main result.

THEOREM 3.1. Let d be a symmetric for X which satisfies (W.3), (W.4) and properties (HE), (CE.1) and (CE.2), and let A, B, S and T be self mappings of X. Suppose there exists $F \in \mathcal{F}_6$ such that

$$AX \subset TX,\ BX \subset SX \tag{4}$$

$$\int_0^{F\left(d(Ax,By),d(Sx,Ty),d(Ax,Sx),d(By,Ty),d(Ax,Ty),d(Sx,By)\right)} \varphi(t)\,dt \leq 0, \tag{5}$$

for all $x, y \in X$, where F satisfies properties (F_a), (F_b) and (F_1) and $\varphi : \mathbb{R}_+ \to \mathbb{R}$ is a Lebesgue-integrable mapping which is summable. Suppose that (A, S) or (B, T)

satisfies property (E.A) and (A,S) and (B,T) are weakly compatible. If one of the subspace AX, BX, SX and TX of X is complete, then A, B, S and T have a unique common fixed point of X.

PROOF. Since (B,T) satisfies property (E.A), there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} d(Bx_n, z) = \lim_{n\to\infty} d(Tx_n, z) = 0 \quad \text{for some} \quad z \in X.$$

By property (HE), we have

$$\lim_{n\to\infty} d(Bx_n, Tx_n) = 0. \tag{6}$$

Since $BX \subset SX$, there exists a sequence $\{y_n\}$ in X such that $Bx_n = Sy_n$ for all $n \in \mathbb{N}$. By (6) and property (CE.2) , we get

$$\limsup_{n\to\infty} d(Ay_n, Tx_n) = \limsup_{n\to\infty} d(Ay_n, Bx_n) \text{ and } \lim_{n\to\infty} d(Bx_n, Sy_n) = 0. \tag{7}$$

Now we show that $\lim_{n\to\infty} Ay_n = z$. To show this, suppose $\limsup_{n\to\infty} d(Ay_n, Bx_n) = \varepsilon \neq 0$. Using (5) with $x = y_n$ and $y = x_n$, we have

$$\int_0^{F\left(d(Ay_n,Bx_n),d(Sy_n,Tx_n),d(Ay_n,Sy_n),d(Bx_n,Tx_n),d(Ay_n,Tx_n),d(Sy_n,Bx_n)\right)} \varphi(t)dt \leq 0;$$

i.e.,

$$\int_0^{F\left(d(Ay_n,Bx_n),d(Bx_n,Tx_n),d(Ay_n,Bx_n),d(Bx_n,Tx_n),d(Ay_n,Bx_n),0\right)} \varphi(t)dt \leq 0.$$

Taking *lim sup* on both sides of the above inequality, we get

$$\int_0^{F\left(\varepsilon,0,\varepsilon,0,\varepsilon,0\right)} \varphi(t)\, dt \leq 0,$$

which implies, by using the condition (F_a), $\varepsilon = 0$; i.e., $\limsup_{n\to\infty} d(Ay_n, Bx_n) = 0$. Thus, $\lim_{n\to\infty} d(Ay_n, Bx_n) = 0$. By axiom (W.4), we get $\lim_{n\to\infty} d(Ay_n, z) = 0$; i.e., $\lim_{n\to\infty} Ay_n = \lim_{n\to\infty} Bx_n = z$.

Suppose SX is complete then $z = Su$ for some $u \in X$. Consequently, we have

$$\lim_{n\to\infty} Ay_n = \lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sy_n = Su.$$

If $Au \neq z$ using (5), we have

$$\int_0^{F\left(d(Au,Bx_n),d(Su,Tx_n),d(Au,Su),d(Bx_n,Tx_n),d(Au,Tx_n),d(Su,Bx_n)\right)} \varphi(t)\, dt \leq 0.$$

Taking limit as $n \to \infty$ and using property (CE.1), we get

$$\int_0^{F\left(d(Au,Su),0,d(Au,Su),0,d(Au,Su),0\right)} \varphi(t)dt \leq 0,$$

which implies $d(Au, Su) = 0$ by using the condition (F_a). Thus, $Au = Su = z$.

Since $AX \subset TX$, there exists $v \in X$ such that $z = Au = Tv$. If $Bv \neq z$, using (5) again, we have

$$\int_0^{F\left(d(Au,Bv),d(Su,Tv),d(Au,Su),d(Bv,Tv),d(Au,Tv),d(Su,Bv)\right)} \varphi(t)\,dt \leq 0,$$

i.e.,

$$\int_0^{F\left(d(z,Bv),0,0,d(z,Bv),0,d(z,Bv)\right)} \varphi(t)\,dt \leq 0,$$

which implies $Bv = Tv = z$ by using the condition (F_b).

Since the pair (A, S) is weakly compatible, it follows that $ASu = SAu$; i.e., $Az = Sz$.

If $z \neq Az$, using (5) we have

$$\int_0^{F\left(d(Az,Bv),d(Sz,Tv),d(Az,Sz),d(Bv,Tv),d(Az,Tv),d(Sz,Bv)\right)} \varphi(t)\,dt \leq 0,$$

i.e.,

$$\int_0^{F\left(d(Az,z),d(Az,z),0,0,d(Az,z),d(Az,z)\right)} \varphi(t)\,dt \leq 0,$$

which contradicts (F_1). Hence, $z = Az = Sz$.

Similarly, the weak compatibility of B and T with (5) yields $z = Bz = Tz$. Thus, z is a common fixed point of A, B, S and T.

When TX is assumed to be complete subspace of X, then the proof is similar. On the other hand the cases in which AX or BX is a complete subspaces of X are, respectively, similar to the cases in which TX or SX is complete.

For the uniqueness of common fixed point z, let $w \neq z$ be another common fixed point of A, B, S and T. Then using (5), we obtain

$$\int_0^{F\left(d(Az,Bw),d(Sz,Tw),d(Az,Sz),d(Bw,Tw),d(Az,Tw),d(Sz,Bw)\right)} \varphi(t)\,dt \leq 0,$$

i.e.,

$$\int_0^{F\left(d(z,w),d(z,w),0,0,d(z,w),d(z,w)\right)} \varphi(t)\,dt \leq 0,$$

which is a contradiction of (F_1). Therefore $z = w$.

For $\varphi(t) = 1$ in Theorem 3.1, we obtain the following corollary, which improves Theorem 2 of Popa [10] and Theorem 2.1 of Imdad et al. [5].

COROLLARY 3.1. Let d be a symmetric for X which satisfies (W.3), (W.4) and properties (HE), (CE.1) and (CE.2), and let A, B, S and T be self mappings of X which satisfy (3.1). Suppose there exists $F \in \mathcal{F}_6$ such that

$$F\big(d(Ax,By),d(Sx,Ty),d(Ax,Sx),d(By,Ty),d(Ax,Ty),d(Sx,By)\big) \leq 0 \tag{8}$$

for all $x, y \in X$, where F satisfies properties (F_a), (F_b) and (F_1). Suppose that (A, S) or (B, T) satisfies property (E.A) and (A, S) and (B, T) are weakly compatible mappings. If one of the subspace AX, BX, SX and TX of X is complete, then A, B, S and T have a unique common fixed point of X.

If we define $F(t_1, \ldots, t_6) := t_1 - \ c \max\{t_2, t_3, t_4 t_5, t_6\}$ in Corollary 3.1, where $0 < c < 1$, then we have the following corollary.

COROLLARY 3.2. Let d be a symmetric for X which satisfies (W.3), (W.4) and properties (HE), (CE.1) and (CE.2), and let A, B, S and T be self mappings of X which satisfy (4) and

$$d(Ax, By) \leq c \ \max\{d(Sx, Ty), d(Ax, Sx), d(By, Ty), d(Ax, Ty), d(Sx, By)\} \tag{9}$$

for all $x, y \in X$, where $0 < c < 1$. Suppose that (A, S) or (B, T) satisfies property (E.A) and (A, S) and (B, T) are weakly compatible. If one of the subspace AX, BX, SX and TX of X is complete, then A, B, S and T have a unique common fixed point of X.

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