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Uniqueness of Meromorphic Functions Sharing One Value*

Chao Meng†

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Abstract

In this paper, we discuss the problem of meromorphic functions sharing one value and obtain two theorems which improve a result of C.C.Yang and X.H.Hua.

1 Introduction

In this paper, a meromorphic function always mean a function which is meromorphic in the whole complex plane.

DEFINITION 1. Let $f(z)$ and $g(z)$ be nonconstant meromorphic functions, $a \in C \cup \{\infty\}$. We say that f and g share the value a CM if $f-a$ and $g-a$ have the same zeros with the same multiplicities.

DEFINITION 2. Let k be a positive integer or infinity. We denote by $E_{k)}(a,f)$ the set of all a-points of f with multiplicities not exceeding k, where an a-point is counted according to its multiplicity. Particularly if for some $a \in C \cup \{\infty\}$, $E_{\infty)}(a,f) = E_{\infty)}(a,g)$, it is obvious that f and g share a CM.

DEFINITION 3. We denote by $N_{k)}(r,1/(f-a))$ the counting function for zeros of $f-a$ with multiplicity $\le k$, and by $\overline{N}_{k)}(r,1/(f-a))$ the corresponding one for which multiplicity is not counted. Let $N_{(k}(r,1/(f-a))$ be the counting function for zeros of $f-a$ with multiplicity at least k and $\overline{N}_{(k}(r,1/(f-a))$ the corresponding one for which multiplicity is not counted. Set

$$N_k\left(r,\frac{1}{f-a}\right) = \overline{N}\left(r,\frac{1}{f-a}\right) + \overline{N}_{(2}\left(r,\frac{1}{f-a}\right) + ... + \overline{N}_{(k}\left(r,\frac{1}{f-a}\right) \qquad (1)$$

It is assumed that the reader is familiar with the notations of the Nevanlinna theory that can be found, for instance, in [1].

In the 1920's, Nevanlinna [1] proved the following result.

THEOREM A. Let f and g be two nonconstant meromorphic functions. If f and g share four distinct values CM, then f is a fractional transformation of g.

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†Department of Mathematics, Shandong University, Jinan, Shandong 250100, P. R. China

In 1997, Yang and Hua [2] studied meromorphic functions sharing only one value and proved the following result.

THEOREM B. Let f and g be two nonconstant meromorphic functions, $n \geq 11$ an integer and $a \in C - \{0\}$. If $f^n f'$ and $g^n g'$ share the value a CM, then either $f = dg$ for some $(n+1)$th root of unity d or $g(z) = c_1 e^{cz}$ and $f(z) = c_2 e^{-cz}$ where c, c_1 and c_2 are constants and satisfy $(c_1 c_2)^{n+1} c^2 = -a^2$.

Corresponding to entire functions, Xu and Qu [3] proved the following result.

THEOREM C. Let f and g be two nonconstant entire functions, $n \geq 12$ an integer, and $a \in C - \{0\}$. If $f^n f'$ and $g^n g'$ share the value a IM, then either $f = dg$ for some $(n+1)$th root of unity d or $g(z) = c_1 e^{cz}$ and $f(z) = c_2 e^{-cz}$ where c, c_1 and c_2 are constants and satisfy $(c_1 c_2)^{n+1} c^2 = -a^2$.

Recently Lahiri [4] and Banerjee [5] extended Theorem B with the notion of weight sharing respectively. Here we extend Theorem B from a new way.

THEOREM 1. Let f and g be two nonconstant meromorphic functions, $n \geq 11$ an integer and $a \in C - \{0\}$. If $E_{3)}(a, f^n f') = E_{3)}(a, g^n g')$, then either $f = dg$ for some $(n+1)$th root of unity d or $g(z) = c_1 e^{cz}$ and $f(z) = c_2 e^{-cz}$ where c, c_1 and c_2 are constants and satisfy $(c_1 c_2)^{n+1} c^2 = -a^2$.

THEOREM 2. Let f and g be two nonconstant meromorphic functions, $n \geq 13$ an integer and $a \in C - \{0\}$. If $E_{2)}(a, f^n f') = E_{2)}(a, g^n g')$, then either $f = dg$ for some $(n+1)$th root of unity d or $g(z) = c_1 e^{cz}$ and $f(z) = c_2 e^{-cz}$ where c, c_1 and c_2 are constants and satisfy $(c_1 c_2)^{n+1} c^2 = -a^2$.

2 Some Lemmas

We need the following Lemmas in the proof of Theorem 1 and Theorem 2. The first one is in [6].

LEMMA 1. If f, g are nonconstant meromorphic functions and $E_{3)}(1, f) = E_{3)}(1, g)$, then one of the following cases holds: $(1) T(r, f) + T(r, g) \leq 2\{N_2(r, \frac{1}{f}) + N_2(r, f) + N_2(r, \frac{1}{g}) + N_2(r, g)\} + S(r, f) + S(r, g)$, $(2) f \equiv g$, or, $(3) fg \equiv 1$.

LEMMA 2. Let f be a nonconstant meromorphic function and $P(f) = a_0 + a_1 f + a_2 f^2 + ... + a_n f^n$, where $a_0, a_1, a_2, ..., a_n$ are constant and $a_n \neq 0$. Then $T(r, P(f)) = nT(r, f) + S(r, f)$.

The proof of Lemma 2 can be found in [7].

LEMMA 3. Let f be a nonconstant meromorphic function and $F = f^{n+1}/a(n+1)$, n being a positive integer. Then

$$T(r, F) \leq T(r, F') + N\left(r, \frac{1}{f}\right) - N\left(r, \frac{1}{f'}\right) + S(r, f) \tag{2}$$

The proof of Lemma 3 can be found in [5].

LEMMA 4. Let f and g be two nonconstant meromorphic functions, $n \geq 6$. If $f^n f' g^n g' = 1$, then $g = c_1 e^{cz}$, $f = c_2 e^{-cz}$, where $(c_1 c_2)^{n+1} c^2 = -1$.

The proof of Lemma 4 can be found in [2].

LEMMA 5. Let f and g be two nonconstant meromorphic functions and $E_{2)}(1,f) = E_{2)}(1,g)$. Set

$$h = \left(\frac{f''}{f'} - 2\frac{f'}{f-1}\right) - \left(\frac{g''}{g'} - 2\frac{g'}{g-1}\right)$$

If $h \not\equiv 0$, then

$$\begin{aligned} T(r,f) + T(r,g) \quad &\leq \quad 2\left(N_2\left(r,\frac{1}{f}\right) + N_2(r,f) + N_2\left(r,\frac{1}{g}\right) + N_2(r,g)\right) \\ &+\overline{N}_{(3}\left(r,\frac{1}{f-1}\right) + \overline{N}_{(3}\left(r,\frac{1}{g-1}\right) + S(r,f) + S(r,g). \quad (3) \end{aligned}$$

The proof of Lemma 5 can be found in [8].

LEMMA 6. Let f be a nonconstant meromorphic function, k be a positive integer, then

$$N_p\left(r,\frac{1}{f^{(k)}}\right) \leq N_{p+k}\left(r,\frac{1}{f}\right) + k\overline{N}(r,f) + S(r,f), \qquad (4)$$

where $N_p\left(r,\frac{1}{f^{(k)}}\right)$ denotes the counting function of the zeros of $f^{(k)}$ where a zero of multiplicity m is counted m times if $m \leq p$ and p times if $m > p$. Clearly $\overline{N}\left(r,\frac{1}{f^{(k)}}\right) = N_1\left(r,\frac{1}{f^{(k)}}\right)$.

The proof of Lemma 6 can be found in [9].

LEMMA 7. Let h be defined as in Lemma 5, if $h \equiv 0$ and

$$\limsup_{r\to\infty} \frac{\overline{N}\left(r,\frac{1}{f}\right) + \overline{N}(r,f) + \overline{N}\left(r,\frac{1}{g}\right) + \overline{N}(r,g)}{T(r)} < 1, \qquad r \in I \qquad (5)$$

where $T(r) = \max\{T(r,f), T(r,g)\}$, then $f \equiv g$ or $fg \equiv 1$.

The proof of Lemma 7 can be found in [10].

3 Proof of Theorem 1

let $F = f^{n+1}/a(n+1)$ and $G = g^{n+1}/a(n+1)$. Then $F' = f^n f'/a$ and $G' = g^n g'/a$. Since $E_{3)}(a, f^n f') = E_{3)}(a, g^n g')$, it follows that $E_{3)}(1,F') = E_{3)}(1,G')$. Then by Lemma 1, if possible, suppose that

$$\begin{aligned} T(r,F') + T(r,G') \quad &\leq \quad 2\left\{N_2\left(r,\frac{1}{F'}\right) + N_2(r,F') + N_2\left(r,\frac{1}{G'}\right) + N_2(r,G')\right\} \\ &+S(r,F') + S(r,G') \qquad (6) \end{aligned}$$

We see that

$$N_2\left(r,\frac{1}{F'}\right) + N_2(r,F') \leq 2\overline{N}\left(r,\frac{1}{f}\right) + N\left(r,\frac{1}{f'}\right) + 2\overline{N}(r,f), \qquad (7)$$

$$N_2\left(r,\frac{1}{G'}\right)+N_2(r,G')\leq 2\overline{N}\left(r,\frac{1}{g}\right)+N\left(r,\frac{1}{g'}\right)+2\overline{N}(r,g). \tag{8}$$

Also by Lemma 2, we have

$$T(r,F')\leq 2T(r,F)+S(r,F)=2(n+1)T(r,f)+S(r,f), \tag{9}$$

$$T(r,G')\leq 2T(r,G)+S(r,G)=2(n+1)T(r,g)+S(r,g). \tag{10}$$

So $S(r,F')=S(r,f)$ and $S(r,G')=S(r,g)$. From (7), (8) we get

$$\begin{aligned}T(r,F')+T(r,G') \leq\ & 4\overline{N}\left(r,\frac{1}{f}\right)+2N\left(r,\frac{1}{f'}\right)+4\overline{N}(r,f)+4\overline{N}\left(r,\frac{1}{g}\right)\\ &+2N\left(r,\frac{1}{g'}\right)+4\overline{N}(r,g)+S(r,f)+S(r,g).\end{aligned} \tag{11}$$

By Lemma 3 and (11), we have

$$\begin{aligned}&T(r,F)+T(r,G)\\ \leq\ & T(r,F')+N\left(r,\frac{1}{f}\right)-N\left(r,\frac{1}{f'}\right)+T(r,G')\\ &+N\left(r,\frac{1}{g}\right)-N\left(r,\frac{1}{g'}\right)+S(r,f)+S(r,g)\\ \leq\ & 4\overline{N}\left(r,\frac{1}{f}\right)+4\overline{N}(r,f)+N\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{f}\right)+\overline{N}(r,f)+4\overline{N}\left(r,\frac{1}{g}\right)\\ &+4\overline{N}(r,g)+N\left(r,\frac{1}{g}\right)+N\left(r,\frac{1}{g}\right)+\overline{N}(r,g)+S(r,f)+S(r,g).\end{aligned} \tag{12}$$

So we get

$$(n-10)T(r,f)+(n-10)T(r,g)\leq S(r,f)+S(r,g), \tag{13}$$

which is a contradiction. Hence by Lemma 1 either $F'\equiv G'$ or $F'G'\equiv 1$.

If $F'\equiv G'$. Then $F=G+c$, where c is a constant. If possible, let $c\neq 0$. Then by the second fundamental theorem, we get

$$\begin{aligned}(n+1)T(r,f) \leq\ & \overline{N}(r,F)+\overline{N}\left(r,\frac{1}{F}\right)+\overline{N}\left(r,\frac{1}{F-c}\right)+S(r,F)\\ =\ & \overline{N}(r,f)+\overline{N}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{g}\right)+S(r,f)\\ \leq\ & 2T(r,f)+T(r,g)+S(r,f)\\ \leq\ & 3T(r)+S(r)\end{aligned} \tag{14}$$

where $T(r)=max\{T(r,f),T(r,g)\}$. In a similar manner, we get

$$(n+1)T(r,g)\leq 3T(r)+S(r). \tag{15}$$

This shows that

$$(n-2)T(r)\leq S(r), \tag{16}$$

which is contrary to the assumption. So $c=0$. That is $F=G$ or $f=dg$, where d is some $(n+1)$th root of unity.

If $F'G'\equiv 1$. Then $f^nf'g^ng'=a^2$. Set $f_1=a^{\frac{-1}{n+1}}f$ and $g_1=a^{\frac{-1}{n+1}}g$. So using Lemma 4, we get $g(z)=c_1e^{cz}$ and $f(z)=c_2e^{-cz}$ where c, c_1 and c_2 are constants and satisfy $(c_1c_2)^{n+1}c^2=-a^2$. This completes the proof of the theorem.

4 Proof of Theorem 2

let $F=f^{n+1}/a(n+1)$ and $G=g^{n+1}/a(n+1)$. Then $F'=f^nf'/a$ and $G'=g^ng'/a$. Since $E_{2)}(a,f^nf')=E_{2)}(a,g^ng')$, it follows that $E_{2)}(1,F')=E_{2)}(1,G')$. Set

$$H=\left(\frac{F'''}{F''}-2\frac{F''}{F'-1}\right)-\left(\frac{G'''}{G''}-2\frac{G''}{G'-1}\right) \tag{17}$$

Suppose that $H\not\equiv 0$. Then by Lemma 5, we obtain

$$\begin{aligned} & T(r,F')+T(r,G') \\ \leq\ & 2\left(N_2\left(r,\frac{1}{F'}\right)+N_2(r,F')+N_2\left(r,\frac{1}{G'}\right)+N_2(r,G')\right)+\overline{N}_{(3}\left(r,\frac{1}{F'-1}\right) \\ & +\overline{N}_{(3}\left(r,\frac{1}{G'-1}\right)+S(r,F')+S(r,G'). \end{aligned} \tag{18}$$

We see that

$$N_2\left(r,\frac{1}{F'}\right)+N_2(r,F')\leq 2\overline{N}\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{f'}\right)+2\overline{N}(r,f), \tag{19}$$

$$N_2\left(r,\frac{1}{G'}\right)+N_2(r,G')\leq 2\overline{N}\left(r,\frac{1}{g}\right)+N\left(r,\frac{1}{g'}\right)+2\overline{N}(r,g). \tag{20}$$

As in the proof of Theorem 1, we have $S(r,F')=S(r,f)$ and $S(r,G')=S(r,g)$. By Lemma 6, we have

$$\begin{aligned} \overline{N}_{(3}\left(r,\frac{1}{F'-1}\right) &\leq \frac{1}{2}N\left(r,\frac{F'}{F''}\right)=\frac{1}{2}N\left(r,\frac{F''}{F'}\right)+S(r,f) \\ &\leq \frac{1}{2}\overline{N}(r,F)+\frac{1}{2}\overline{N}\left(r,\frac{1}{F'}\right)+S(r,f) \\ &\leq \frac{1}{2}\overline{N}(r,F)+\frac{1}{2}\left(N_2\left(r,\frac{1}{F}\right)+\overline{N}(r,F)\right)+S(r,f) \\ &\leq \frac{1}{2}\overline{N}(r,f)+\frac{1}{2}\left(2\overline{N}\left(r,\frac{1}{f}\right)+\overline{N}(r,f)\right)+S(r,f) \\ &\leq 2T(r,f)+S(r,f). \end{aligned} \tag{21}$$

By Lemma 3, we have

$$\begin{aligned} T(r,F)+T(r,G) \leq\ & T(r,F')+N\left(r,\frac{1}{f}\right)-N\left(r,\frac{1}{f'}\right)+T(r,G') \\ & +N\left(r,\frac{1}{g}\right)-N\left(r,\frac{1}{g'}\right)+S(r,f)+S(r,g). \end{aligned} \tag{22}$$

Substitute (18), (19), (20) and (21) into (22), we get

$$(n-12)T(r,f)+(n-12)T(r,g)\le S(r,f)+S(r,g), \tag{23}$$

which contradicts the assumption. Thus $H\equiv 0$. Since

$$\overline{N}\left(r,\frac{1}{f'}\right)\le T(r,f')-m\left(r,\frac{1}{f'}\right)\le 2T(r,f)-m\left(r,\frac{1}{f'}\right)+S(r,f), \tag{24}$$

we see that

$$\begin{aligned}
&\overline{N}\left(r,\frac{1}{F'}\right)+\overline{N}(r,F')+\overline{N}\left(r,\frac{1}{G'}\right)+\overline{N}(r,G')\\
\le\;& \overline{N}\left(r,\frac{1}{f}\right)+\overline{N}(r,f)+\overline{N}\left(r,\frac{1}{g}\right)+\overline{N}(r,g)+\overline{N}\left(r,\frac{1}{f'}\right)+\overline{N}\left(r,\frac{1}{g'}\right)\\
\le\;& 4T(r,f)+4T(r,g)-m\left(r,\frac{1}{f'}\right)-m\left(r,\frac{1}{g'}\right)+S(r)\\
\le\;& 8T(r)-m\left(r,\frac{1}{f'}\right)-m\left(r,\frac{1}{g'}\right)+S(r).
\end{aligned} \tag{25}$$

Using Lemma 2, we get

$$\begin{aligned}
T(r,F')+m\left(r,\frac{1}{f'}\right) &= m\left(r,\frac{f^nf'}{a}\right)+m\left(r,\frac{1}{f'}\right)+N\left(r,\frac{f^nf'}{a}\right)\\
&\ge m\left(r,\frac{f^n}{a}\right)+N(r,f^n)=T(r,f^n)+O(1).
\end{aligned} \tag{26}$$

Similarly we have

$$T(r,G')+m\left(r,\frac{1}{g'}\right)\ge nT(r,g)+O(1). \tag{27}$$

From (26) and (27), we get

$$\max\{T(r,F'),T(r,G')\}\ge nT(r)-m\left(r,\frac{1}{f'}\right)-m\left(r,\frac{1}{g'}\right)+O(1). \tag{28}$$

By (25) and (28), applying Lemma 7, we get either $F'\equiv G'$ or $F'G'\equiv 1$. Proceeding as in the proof of Theorem 1, we get either $f=dg$ for some $(n+1)$th root of unity d or $g(z)=c_1e^{cz}$ and $f(z)=c_2e^{-cz}$ where c, c_1 and c_2 are constants and satisfy $(c_1c_2)^{n+1}c^2=-a^2$. This completes the proof of Theorem 2.

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