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Asymptotic Behaviour Of Solutions Of Matrix Burgers Equation*

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Abstract

In this paper, we study the large time behaviour of solutions of initial value problem for a system of parabolic equations which can be written as Matrix Burgers equation. This work generalizes a result of Hopf (1950) for the Burgers equation.

1 Introduction

In this paper we consider a system of parabolic partial differential equations for the unknown variables $u_1, u_1, u_2, ..., u_N$ of the form

$$(u_j)_t + \frac{1}{2}\sum_{i=1}^{j}(u_i u_{j-i+1})_x = \frac{\epsilon}{2}(u_j)_{xx}, j = 1, 2, ..., N \tag{1}$$

in $-\infty < x < \infty$, $t > 0$, with initial conditions at time $t = 0$,

$$u_j(x, 0) = u_{0j}(x), j = 1, 2, ..., N. \tag{2}$$

Here $\epsilon > 0$ is the viscosity coefficient. The system (1) is called Matrix Burgers equation because it can be written as

$$A_t + \frac{1}{2}(A^2)_x = \frac{\epsilon}{2}A_{xx}. \tag{3}$$

where $A = (A_{i,j})$ is a $(N \times N)$ lower triangular matrix with the i, j-th entry $A_{i,j}$ takes values from $\{u_1, u_2, ..., u_N\}$:

$$A_{i,j} = 0, j > i, A_{i+k,i} = u_{k+1}, k = 1, 2, ..., N - i, i = 1, 2, ..., N.$$

The system (1) has a rich mathematical structure and solutions of special cases of this system were studied previously by many authors in the contexts of vanishing viscosity and large time behaviours. The singularity in the solutions of (1) as ϵ goes to zero increases as N increases.

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When $N = 1$ the equation (1) is the Burgers equation, $u_t + (1/2)(u^2)_x = \frac{\epsilon}{2}u_{xx}$. Hopf [2] used the Hopf-Cole transformation $u = -\epsilon(v_x/v)$ to reduce it into the one dimensional heat equation $v_t = \frac{\epsilon}{2}v_{xx}$ and solved the initial value problem for the Burgers equation explicitly. Letting ϵ go to 0 in the formula, he derived an explicit formula for the entropy solution for the inviscid Burgers equation $u_t + (1/2)(u^2)_x = 0$ with bounded measurable initial data. For $\epsilon = 0$ in (1), the standard theory of Lax [7] does not apply when $N > 1$. Joseph [3] studied the cases $N = 2$ to construct solution for the Riemann problem for the system of conservation laws $(u_1)_t + (1/2)(u_1^2)_x = 0, (u_2)_t + (u_1u_2)_x = 0$, whose solution does not belong to the L^∞ class but may contain δ measures as well. When $N = 3$, the system (1) is more singular than the cases $N = 1, 2$ in the passage to the limit as ϵ goes to zero and was studied in [4] and solutions were constructed in the algebra of generalized functions of Colombeau [1].

Hopf [2], analyzed another aspect of solution of (1) for the case $N = 1$, that is the asymptotic behaviour of solutions of Burgers equation for large time with $\epsilon > 0$. This result was generalized for the case $N = 3$ in [6]. In this paper we extend Hopf's work [2] on the large time behaviour of solutions of the Burgers equation to the solutions of the system (1) with initial data (2). For notational convenience we drop the explicit dependence of ϵ on the solutions.

2 Explicit Solution and Asymptotic Form

An explicit formula for the solution of the initial value problem for (1) and (2) was obtained by Joseph and Vasudeva Murthy [5], using a generalized form of Hopf-Cole transformation. To give this explicit formula we need some notations. Let us consider the polynomial

$$p(x) = a_1x + a_2x^2 + \cdots + a_{N-1}x^{N-1}$$

of degree $N-1$ with real coefficients $a_1, a_2, ..., a_{N-1}$. We define E_j and L_j as coefficients of x^{j-1} in the power series expansion around $x = 0$ of the functions $\exp(p(x))$ and $\log(1 + p(x))$ respectively. More precisely

$$E_{j+1}(a_1, a_2, ..., a_j) = \frac{1}{j!}\frac{d^j}{dx^j}[e^{p(x)}]_{x=0}, j = 0, 1, 2, ... \tag{4}$$

and are polynomial of $a_1, a_2, ..., a_j, a_j$, of degree j. For example the first four coefficients are given by $E_1 = 1, E_2(a_1) = a_1, E_3(a_1, a_2) = a_2 + \frac{{a_1}^2}{2!}, E_4(a_1, a_2, a_3) = a_3 + a_2a_1 + \frac{{a_1}^3}{3!}, E_5(a_1, a_2, a_3, a_4) = a_4 + a_3a_1 + \frac{a_2{a_1}^2}{2!} + \frac{{a_2}^2}{2} + \frac{{a_1}^4}{4!}$. Similarly

$$L_{j+1}(a_1, a_2, ..., a_j) = \frac{1}{j!}\frac{d^j}{dx^j}[\log(1 + p(x))]_{x=0}, j = 1, 2, ... \tag{5}$$

Here again it follows from (5) that L_{j+1} is a polynomial of $a_1, a_2, ..., a_j$ of degree j. For example the first four coefficients are given by $L_2(a_1) = a_1, L_3(a_1, a_2) = a_2 - \frac{{a_1}^2}{2!}, L_4(a_1, a_2, a_3) = a_3 - a_1a_2 + \frac{{a_1}^3}{3}, L_5(a_1, a_2, a_3, a_4) = a_4 - a_1a_3 + {a_1}^2a_2 - \frac{{a_2}^2}{2} - \frac{{a_1}^4}{4}$.

For $j = 1, 2, ...N$, let us denote

$$w_{0j}(x) = \int_0^x u_{0j}(y)dy, \tag{6}$$

$$E_{0j}(x) = E_j\left(\frac{-w_{02}(y)}{\epsilon}, \frac{-w_{03}(y)}{\epsilon}, ..., \frac{-w_{0j}(y)}{\epsilon}\right) \tag{7}$$

where E_j is defined by (4). Define the functions $v_j, j = 1, 2, ..., N$ by

$$v_j(x,t) = \frac{1}{(2\pi t\epsilon)^{(1/2)}} \int_{R^1} E_{0j}(y) e^{-\frac{1}{\epsilon}[w_{01}(y) + \frac{(x-y)^2}{2t}]} dy. \tag{8}$$

Since $E_{01}(x) = 1$, $v_1(x,t) > 0$.

Assume that $u_{0j}(x)$, $j = 1, 2, \cdots N$ be measurable functions on R^1 with w_{0j} defined by (6) satisfy $|w_{0j}(x)| = o(|x|^2)$, as $|x| \to \infty$. Then it was proved in [5], that u_j defined by

$$u_1(x,t) = -\epsilon \frac{\partial_x v_1(x,t)}{v_1(x,t)} \tag{9}$$

$$u_j(x,t) = -\epsilon \partial_x \left(L_j \left(\frac{v_2(x,t)}{v_1(x,t)}, \frac{v_3(x,t)}{v_1(x,t)}, ... \frac{v_j(x,t)}{v_1(x,t)} \right) \right), j = 2, 3, ..., N \tag{10}$$

is a classical solution to (1) and (2).

The aim of this paper is to study the asymptotic behaviour of this solution as t tends to infinity. We introduce the variable $\xi = x/\sqrt{t\epsilon}$ and a function which appear in the asymptotic form of the solution, namely

$$\begin{aligned} V_j(\xi) &= E_j\left(\frac{-w_{02}(\infty)}{\epsilon}, ..., \frac{-w_{0j}(\infty)}{\epsilon}\right) e^{-\frac{w_{01}(\infty)}{\epsilon}} \int_{-\infty}^{\xi} e^{-y^2/2} dy \\ &+ E_j\left(\frac{-w_{02}(-\infty)}{\epsilon}, ..., \frac{-w_{0j}(-\infty)}{\epsilon}\right) e^{-\frac{w_{01}(-\infty)}{\epsilon}} \int_{\xi}^{\infty} e^{-y^2/2} dy. \end{aligned} \tag{11}$$

Since $E_1 = 1$, $V_1(\xi) > 0$. The main result of the paper is the following.

THEOREM. Assume that the initial data $u_{0j}(x), j = 1, 2, ..., N$ are integrable, then the solution $u_j(x,t), j = 1, 2, ..., N$ of (1) and (2) given by (9) and (10) has the following asymptotic behaviour as t tends to infinity:

$$\lim_{t\to\infty} \sqrt{t/\epsilon}.u_1(x,t) = -\frac{d}{d\xi} \log(V_1(\xi)) \tag{12}$$

$$\lim_{t\to\infty} \sqrt{t/\epsilon}.u_j(x,t) = -\frac{d}{d\xi}\left(L_j\left(\frac{V_2(\xi)}{V_1(\xi)}, \frac{V_3(\xi)}{V_1(\xi)}, ..., \frac{V_j(\xi)}{V_1(\xi)}\right)\right), j = 2, 3, ..., N, \tag{13}$$

uniformly with respect to the variable $\xi = \frac{x}{\sqrt{t\epsilon}}$.

PROOF. To prove this theorem, we write the formula (10) for the solution in a convenient way, namely

$$u_j(x,t) = \sum_{k=1}^{j-1} \partial_k L_j\left(\frac{v_2}{v_1}, ..., \frac{v_j}{v_1}\right)\left(\frac{\partial_x v_{k+1}}{v_1} - \frac{v_{k+1}}{v_1}\frac{\partial_x v_1}{v_1}\right) \tag{14}$$

where $\partial_k L_j$ denotes derivative of $L_j(a_1, ..., a_j)$ with respect to the k-th variable a_k. Note that typical terms in these expressions are of the form

$$v_j(x,t) = \frac{1}{\sqrt{2\pi t\epsilon}} \int_{-\infty}^{\infty} E_{0j}(y) e^{-\frac{1}{\epsilon}[w_{01}(y) + \frac{(x-y)^2}{2t}]} dy \tag{15}$$

and its partial derivative with respect to x, where $E_{0j}(x)$ and $w_{01}(x)$ has finite limit as $|x|$ tends to infinity because of the integrability assumption on u_{0j}.

Keeping the variable $\xi = x/\sqrt{\epsilon t}$ fixed, we make a change of variable $z = \frac{\sqrt{\epsilon t}\xi - y}{\sqrt{\epsilon t}}$ and renaming z as y, we get

$$v_j(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} E_{0j}(\sqrt{\epsilon t}(\xi - y) e^{-[\frac{w_{01}(\sqrt{\epsilon t}(\xi-y)}{\epsilon} + y^2/2]} dy. \tag{16}$$

Now we split the integral in (16) in the following fashion.

$$\begin{aligned} \sqrt{(2\pi)} v_j(x,t) &= \textstyle\int_{-\infty}^{\xi-\delta} E_{0j}(\sqrt{\epsilon t}(\xi - y)) e^{-[\frac{1}{\epsilon} w_{01}(\sqrt{\epsilon t}(\xi-y) + y^2/2]} dy \\ &+ \textstyle\int_{\xi+\delta}^{\infty} E_{0j}(\sqrt{\epsilon t}(\xi - y)) e^{-[\frac{1}{\epsilon} w_{01}(\sqrt{\epsilon t}(\xi-y) + y^2/2]} dy \\ &+ \textstyle\int_{\xi-\delta}^{\xi+\delta} E_{0j}(\sqrt{\epsilon t}(\xi - y)) e^{-[\frac{1}{\epsilon} w_{01}(\sqrt{\epsilon t}(\xi-y) + y^2/2]} dy. \end{aligned} \tag{17}$$

Now we fix $\delta > 0$ and study each of these integrals as t tends to infinity, we get

$$\lim_{t\to\infty} \int_{-\infty}^{\xi-\delta} E_{0j}(\sqrt{\epsilon t}(\xi - y) e^{-[\frac{1}{\epsilon} w_{0j}(\sqrt{\epsilon t}(\xi-y) + y^2/2]} dy = e^{-\frac{w_{0j}(\infty)}{\epsilon}} E_j(\infty) \int_{-\infty}^{\xi-\delta} e^{-y^2/2} dy,$$

$$\lim_{t\to\infty} \int_{\xi+\delta}^{\infty} E_{0j}(\sqrt{\epsilon t}(\xi - y) e^{-[\frac{1}{\epsilon} w_{0j}(\sqrt{\epsilon t}(\xi-y) + y^2/2]} dy = e^{-\frac{w_{0j}(-\infty)}{\epsilon}} E_j(-\infty) \int_{\xi+\delta}^{\infty} e^{-y^2/2} dy,$$

$$\limsup_{t\to\infty} | \int_{\xi-\delta}^{\xi+\delta} E_{0j}(\sqrt{\epsilon t}(\xi - y) e^{-\frac{1}{\epsilon}[w_{0j}(\sqrt{\epsilon t}(\xi-y) + y^2/2]} dy| = O(\delta),$$

uniformly with respect to ξ. Now first let t tends to infinity and then δ tends to 0, in (17) we get

$$\lim_{t\to\infty} \sqrt{2\pi}.v_j(x,t) = e^{\frac{-w_{01}(\infty)}{\epsilon}} E_{0j}(\infty) \int_{-\infty}^{\xi} e^{-y^2/2} dy + e^{\frac{-w_{01}(-\infty)}{\epsilon}} E_{0j}(-\infty) \int_{\xi}^{\infty} e^{-y^2/2} dy. \tag{18}$$

This limit is valid uniformly for $\xi \in R^1$ and for the x-derivative we have

$$\lim_{t\to\infty} \sqrt{2\pi\epsilon t}.\partial_x v_j(x,t) = \left(E_{0j}(\infty) e^{\frac{-w_{01}(\infty)}{\epsilon}} - E_{0j}(-\infty) e^{\frac{-w_{01}(-\infty)}{\epsilon}} \right) e^{-\xi^2/2}. \tag{19}$$

In this analysis, we have used the fact that $u_{0j}, j = 1, 2, ..., N$ are integrable and hence $w_{0j}(-\infty), w_{0j}(\infty), E_{0j}(-\infty), E_{0j}(\infty)$ exists. Indeed

$$w_{0j}(-\infty) = \int_0^{-\infty} u_{0j}(y) dy, \;\; w_{0j}(\infty) = \int_0^{\infty} u_{0j}(y) dy \tag{20}$$

$E_{01} = 1$, and for $j = 2, 3, ..., N$

$$\begin{aligned} E_{0j}(-\infty) &= E_j \left(\frac{-w_{02}(-\infty)}{\epsilon}, ..., \frac{-w_{0j}(-\infty)}{\epsilon} \right) \\ E_{0j}(\infty) &= E_j \left(\frac{-w_{02}(\infty)}{\epsilon}, ..., \frac{-w_{0j}(\infty)}{\epsilon} \right) \end{aligned} \tag{21}$$

Since $V_1(\xi) > 0$ from (11),(18),(19),(20) and (21) we get, for $j = 1, 2, ..., N$

$$\lim_{t\to\infty} \frac{v_j(x,t)}{v_1(x,t)} = \frac{V_j(\xi)}{V_1(\xi)}, \tag{22}$$

$$\lim_{t\to\infty} \sqrt{\epsilon.t}\frac{\partial_x v_j(x,t)}{v_1(x,t)} = \frac{V_j(\xi)'}{V_1(\xi)}, \tag{23}$$

uniformly with respect to ξ. Here $V_j(\xi)'$ means derivative of $V_1(\xi)$ with respect to ξ. From (9), (14), (22) and (23), we get

$$\lim_{t\to\infty} \sqrt{t/\epsilon}u_1(x,t) = -\frac{V_1(\xi)'}{V_1(\xi)},$$

$$\begin{aligned} \lim_{t\to\infty} \sqrt{t/\epsilon}.u_j(x,t) &= \sum_{k=1}^{j-1}(\partial_k L_j)\left(\frac{V_2(\xi)}{V_1(\xi)}, \frac{V_3(\xi)}{V_1(\xi)}, ..., \frac{V_j(\xi)}{V_1(\xi)}\right)\left(\frac{V'_{k+1}(\xi)}{V_1(\xi)} - \frac{V_{k+1}(\xi)}{V_1(\xi)}.\frac{V'_1(\xi)}{V_1(\xi)}\right) \\ &= -\frac{d}{d\xi}\left(L_j\left(\frac{V_2(\xi)}{V_1(\xi)}, \frac{V_3(\xi)}{V_1(\xi)}, ..., \frac{V_j(\xi)}{V_1(\xi)}\right)\right) \end{aligned}$$

The proof of the theorem is complete.

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