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An Effective Method For The Existence Of The Global Attractor Of A Nonlinear Wave Equation*

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Abstract

In this work, the existence of the global attractor for the non-linear wave equation will be established using a method introduced by Ma et al. in [1]. The main advantage of this method is that here, one does not need to obtain estimates in function spaces of higher regularity.

The dynamical systems that arise from problems in mathematical physics are commonly generated by partial differential equations and thus the underlying space is infinite dimensional. The global attractor which is a compact, invariant set and which absorbs every bounded set is an important object to understand the limit behavior of such systems. To show the existence of the global attractor, normally, two things have to be verified: the existence of an absorbing set and the compactness of the solution semigroup. In [1], the authors managed to find an alternative and effective method to prove the existence of the global attractor which is stated below.

THEOREM 1. Let E be a convex Banach space and $\{S(t)\}_{t\geq 0}$ be a semigroup of operators mapping E continuously into itself. Assume that $\{S(t)\}_{t\geq 0}$ possesses a bounded absorbing set in E and satisfies the following condition: **(C)** For any bounded set B of E and for any $\varepsilon > 0$, there exists a $t_* > 0$ and a finite dimensional subspace E_1 of E, such that $\{\|P(S(t)B)\|\}$ is bounded and

$$\|(I-P)S(t)y\| < \varepsilon \quad \forall y \in B,\ \forall t \geq t_*$$

where P is a projection of E into E_1. Then there exists a global attractor for $\{S(t)\}_{t\geq 0}$ in E.

The main advantage of this method is that one does not need to obtain estimates in function spaces of higher regularity. In [1], the authors not only provide the method but also apply it to a semilinear parabolic equation and Navier-Stokes equation in non-smooth domains. They also mention that the method can be applied to the nonlinear wave equation as well. We will demonstrate this in the following discussions.

Let Ω be an open bounded set of R^n with smooth boundary $\partial\Omega$. Consider the following equation with the unknown function $u = u(x,t)$ where $x \in \Omega$ and $t \in R$ (or

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some interval of R):

$$
\begin{aligned}
u_{tt} + \alpha u_t - \Delta u + g(u) &= f \qquad \text{in } \Omega \times R & (1)\\
u &= 0 \qquad \text{on } \partial\Omega \times R & (2)\\
u(x,0) &= u_0(x) \in H_0^1(\Omega) & (3)\\
u_t(x,0) &= p_0(x) \in L^2(\Omega) & (4)
\end{aligned}
$$

with $\alpha > 0$ and $f = f(x) \in L^2(\Omega)$.

Assume that the non-linear function $g : R \to R$ satisfies the following growth condition

$$|g'(s)| \leq \kappa(1 + |s|^\gamma) \tag{5}$$

for some $\kappa > 0$ where

$$
\begin{cases}
0 \leq \gamma < \infty \text{ when } n = 1 \text{ or } n = 2, \\
0 \leq \gamma < 2 \text{ when } n = 3, \\
\gamma = 0 \text{ when } n \geq 4
\end{cases}
$$

Under the above assumption, it is known that g is a bounded operator from $H_0^1(\Omega)$ into $L^2(\Omega)$, see [2].

Let us denote the norm of $L^2(\Omega)$ by $|\cdot|$, the norm of $H_0^1(\Omega)$ by $\|\cdot\|$. We define the inner product on $H_0^1(\Omega)$ by $(u,v)_{H_0^1(\Omega)} = \int_\Omega \nabla u \cdot \nabla v dx$. By E, we denote the energy space $E = \{y = \{u,p\} : u \in H_0^1(\Omega),\ p \in L^2(\Omega)\}$ endowed with the norm $\|y\|_E = (\|u\|^2 + |p|^2)^{1/2}$.

It is known that (1)-(4) has a unique solution for each given $(u_0, p_0) \in L^2(\Omega) \times H_0^1(\Omega)$, see [2]. This allows us to define the semigroup of operators

$$
\begin{aligned}
S(t) \quad &: \quad E \to E \\
S(t)\{u_0, p_0\} \quad &= \quad y(t) = \{u(t), p(t)\}
\end{aligned}
$$

The following results are well known and may be found in [2].

1. $S(t)$ maps E continuously into itself for each $t \in R$.
2. $\{S(t)\}$ has an absorbing set under the following assumptions:

$$\liminf_{|s|\to\infty} \frac{G(s)}{s^2} \geq 0$$

There exists $c_1 > 0$ such that

$$\liminf_{|s|\to\infty} \frac{sg(s) - c_1 G(s)}{s^2} \geq 0$$

where

$$G(s) = \int_0^s g(r)dr$$

So to prove the existence of the global attractor, there remains to show that the family of operators $\{S(t)\}_{t\in R}$ satisfies the Condition (C) in Theorem 1.

Let $\{e_j\} \subset H_0^1(\Omega)$ be the set of eigenvectors of $-\Delta$ with eigenvalues λ_j. We recall that

$$0 < \lambda_1 \le \lambda_2 \le \cdots$$

and that the eigenvectors $\{e_j\}$ form an orthonormal basis of $H = L^2(\Omega)$. Now let H_1 be the subspace of H spanned by the first N vectors where N will be chosen later and H_2 be the subspace spanned by the remaining vectors so that $H = H_1 \oplus H_2$. Let us denote the projection of H onto H_1 by P_H. For $p \in H$ we define $(I - P_H)p = p_2$.

We know that the vectors $\{\lambda_j^{-1/2} e_j\}$ form an orthonormal basis of $V = H_0^1(\Omega)$. Hence we similarly define the subspaces V_1 and V_2 of V such that $V = V_1 \oplus V_2$ and we denote the canonical projection of V onto V_1 by P_V. For $u \in V$, we define $(I - P_V)u = u_2$. For every u_2, we have

$$|u_2|^2 \le \frac{1}{\lambda_{N+1}} \|u_2\|^2 . \tag{6}$$

Finally, let us define the projection $P : E \to V_1 \times H_1$. For $y \in E$, we define $(I - P)y = y_2 = \{u_2, p_2\} \in V_2 \times H_2$. By orthogonality, we have $\|y\|_E^2 = \|Py\|_E^2 + \|y_2\|_E^2$.

We will now show that the Condition (C) in Theorem 1 is satisfied, i.e.

THEOREM 2. Let $\varepsilon > 0$. Then for any bounded set B in E, there exists a $t_* > 0$ and an $N > 0$ such that $\|y_2(t)\|_E \le \varepsilon$ when $t \ge t_*$ for every $y \in B$.

Let us take an arbitrary bounded set B in E and let the initial condition $y_0 = \{u_0, p_0\}$ be in B. The existence of an absorbing ball guarantees that there exists $t_0(B) > 0$ and $\rho > 0$ such that $S(t)B \subset B(0, \rho)$ when $t > t_0$. So without loss of generality we may assume that $t_0 = 0$ and $y(t) = \{u(t), p(t)\} \in B(0, \rho)$ for all $t \ge 0$. That is $\|u(t)\| < \rho$, $|p(t)| < \rho$ for all $t \ge 0$.

Denoting $\frac{\partial u}{\partial t} = u'$, we restate the equation as

$$u'' + \alpha u' - \Delta u + g(u) = f \tag{7}$$

We set $v = u' + \varepsilon u = \sum_{i=1}^{\infty} \beta_i(t) e_i(x) + \varepsilon \sum_{i=1}^{\infty} \alpha_i(t) e_i(x)$ where $0 < \varepsilon \le \alpha/4$. We rewrite (7) in terms of the new variable v and take its scalar product with $(I - P)v = v_2 = u_2' + \varepsilon u_2$ in L^2 to obtain

$$(v', v_2) + (\alpha - \varepsilon)(v, v_2) - \varepsilon(\alpha - \varepsilon)(u, v_2) - (\Delta u, v_2) + (g(u), v_2) = (f, v_2). \tag{8}$$

Using the orthonormality of the base vectors e_i, Green's formula and the fact that $v_2 = 0$ on the boundary, we end up with the equation

$$\frac{1}{2}\frac{d}{dt}(\|u_2\|^2 + |v_2|^2) + \varepsilon \|u_2\|^2 + (\alpha - \varepsilon) |v_2|^2 - \varepsilon(\alpha - \varepsilon)(u_2, v_2) + (g(u), v_2) = (f, v_2).$$

Using $(g(u), v_2) = \varepsilon(g(u), u_2) + \frac{d}{dt}(g(u), u_2) - (g'(u)u', u_2)$, we obtain

$$\begin{aligned} (f, v_2) \quad = \quad & \frac{1}{2}\frac{d}{dt}\Big\{\|u_2\|^2 + |v_2|^2 + 2(g(u), u_2)\Big\} + \varepsilon \|u_2\|^2 + (\alpha - \varepsilon) |v_2|^2 \\ & -\varepsilon(\alpha - \varepsilon)(u_2, v_2) + \varepsilon(g(u), u_2) - (g'(u)u', u_2). \end{aligned} \tag{9}$$

Let us choose N large enough so that

$$\lambda_{N+1} \geq \alpha^2. \tag{10}$$

Now, using (6),

$$-\varepsilon(\alpha-\varepsilon)(u_2, v_2) \geq -\varepsilon(\alpha-\varepsilon)\left|u_2\right|\left|v_2\right| \geq \frac{-\varepsilon(\alpha-\varepsilon)}{\sqrt{\lambda_{N+1}}}\left\|u_2\right\|\left|v_2\right|.$$

Since $\sqrt{\lambda_{N+1}} \geq \alpha-\varepsilon$ by (10), we have

$$-\varepsilon(\alpha-\varepsilon)(u_2, v_2) \geq -\varepsilon(\frac{1}{2}\left\|u_2\right\|^2+\frac{1}{2}\left|v_2\right|^2) \tag{11}$$

Now we will obtain estimates for the term $(g'(u)u', u_2)$ using the estimates introduced in [2]. These estimates will depend on n, the dimension of the space.

1. *The case* $n \geq 4$. In this case $\gamma = 0$ and by (5),

   $$|g'(s)| \leq 2\kappa \qquad ,\forall s \in R.$$

   So, we have

   $$\begin{aligned}(g'(u)u', u_2) &\leq 2\kappa\int_\Omega |u'|\,|u_2|\,dx \leq \frac{2\kappa}{\sqrt{\lambda_{N+1}}}|u'|\,\|u_2\| \\ &\leq \frac{\kappa}{\sqrt{\lambda_{N+1}}}(|u'|^2+\|u_2\|^2) \leq \frac{2\kappa\rho^2}{\sqrt{\lambda_{N+1}}}\end{aligned}$$

   Hence

   $$(g'(u)u', u_2) \leq \frac{C_0}{\sqrt{\lambda_{N+1}}} \tag{12}$$

2. *The case* $n = 3$. We use Hölder's inequality with $p_1 = \frac{6}{\gamma}, p_2 = \frac{6}{3-\gamma}, p_3 = 2$ to obtain

   $$(g'(u)u', u_2) \leq \|g'(u)\|_{L^{\frac{6}{\gamma}}}\|u_2\|_{L^{\frac{6}{3-\gamma}}}|u'|. \tag{13}$$

   Since $0 \leq \gamma < 2$, we have $2 \leq \frac{6}{3-\gamma} < 6$. The case $\gamma = 0$ is handled above so we assume $2 < \frac{6}{3-\gamma} < 6$. By the interpolation inequality in L^p spaces, we know that there exists $\theta \in (0,1)$ such that

   $$\|u_2\|_{L^{\frac{6}{3-\gamma}}} \leq |u_2|^\theta\,\|u_2\|_{L^6}^{(1-\theta)}.$$

   By the Sobolev embedding theorem and the fact that $\|u_2\| \leq \|u\| \leq \rho$, there exists C such that

   $$\|u_2\|_{L^6}^{(1-\theta)} \leq C\,\|u_2\|^{(1-\theta)} \leq C\rho^{(1-\theta)} = C_1.$$

   Using (6) and the above estimate,

   $$\|u_2\|_{L^{\frac{6}{3-\gamma}}} \leq C_1|u_2|^\theta \leq C_1\frac{\|u_2\|^\theta}{(\lambda_{N+1})^{\theta/2}} \leq \frac{C_1\rho^\theta}{(\lambda_{N+1})^{\theta/2}} = \frac{C_2}{(\lambda_{N+1})^{\theta/2}}.$$

The condition (5) guarantees that $\|g'(u)\|_{L^{\frac{6}{\gamma}}} \leq C_4$ whenever $\|u\| \leq \rho$:

$$\|g'(u)\|_{L^{\frac{6}{\gamma}}}^{\frac{6}{\gamma}} \leq \int_\Omega [\kappa(1+|u|^\gamma)]^{6/\gamma} dx \leq \kappa c(\gamma)[|\Omega| + \|u\|_{L^6}^6] \leq \kappa c(\gamma)[|\Omega| + \rho^6] \leq C_3.$$

Hence (13) becomes

$$(g'(u)u', u_2) \leq C_4 \frac{C_2}{(\lambda_{N+1})^{\theta/2}} |u'| = \frac{C_5}{(\lambda_{N+1})^{\theta/2}}. \tag{14}$$

3. *The case* $n = 1, 2$. For this case, we again apply Hölder's inequality with exponents $p_1 = 4, p_2 = 4, p_3 = 2$,

$$(g'(u)u', u_2) \leq \|g'(u)\|_{L^4} \|u_2\|_{L^4} |u'|. \tag{15}$$

Using the interpolation inequality and the embedding as above

$$\|u_2\|_{L^4} \leq |u_2|^{1/4} \|u_2\|_{L^6}^{3/4} \leq \frac{C_6}{(\lambda_{N+1})^{1/8}}.$$

After similar calculations as in the case $n = 3$, it can be shown that

$$\|g'(u)\|_{L^4} \leq C_7 \int_\Omega (1 + |u|^{4\gamma}) dx \leq C_8$$

since $H_0^1(\Omega) \hookrightarrow L^{4\gamma}(\Omega)$. Using $|u'| < \rho$, (15) becomes

$$(g'(u)u', u_2) \leq \frac{C_6 C_8 \rho}{(\lambda_{N+1})^{1/8}} = \frac{C_9}{(\lambda_{N+1})^{1/8}}. \tag{16}$$

By the results (12), (14) and (16), there exists $N_0 > 0$ such that whenever

$$N > N_0 \tag{17}$$

λ_{N+1} is large enough to guarantee

$$(g'(u)u', u_2) \leq \frac{\varepsilon^3}{32}. \tag{18}$$

Now let $f(x) = \sum_{i=1}^{\infty} a_i e_i(x)$ and $f_m(x) = \sum_{i=1}^{N} a_i e_i(x)$ be the partial sums. Then there exists $N_1 > 0$ such that whenever

$$N \geq N_1 \tag{19}$$

we have $|f - f_N| < \frac{\varepsilon^4}{16}$ so that,

$$(f, v_2) = (f - f_N, v_2) \leq |f - f_N| |v_2| \leq (\frac{1}{2\varepsilon} |f - f_N|^2 + \frac{\varepsilon}{2} |v_2|^2)$$

That is

$$(f, v_2) \leq \frac{\varepsilon^3}{32} + \frac{\varepsilon}{2} |v_2|^2. \tag{20}$$

Putting the results (11), (18) and (20) in (9), we find that

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt}(\|u_2\|^2 + |v_2|^2 + 2(g(u),u_2)) + \left(\varepsilon - \frac{\varepsilon}{2}\right)\|u_2\|^2 \\ & + \left(\alpha - \varepsilon - \frac{\varepsilon}{2} - \frac{\varepsilon}{2}\right)|v_2|^2 + \varepsilon(g(u),u_2) \\ \leq \quad & \frac{\varepsilon^3}{32} + \frac{\varepsilon^3}{32}. \end{aligned}$$

Since $\varepsilon < \alpha/4$, we have $\alpha - 2\varepsilon > \varepsilon$, so that

$$\frac{d}{dt}(\|u_2\|^2 + |v_2|^2 + 2(g(u),u_2)) + \varepsilon\|u_2\|^2 + \varepsilon|v_2|^2 + 2\varepsilon(g(u),u_2) \leq \frac{\varepsilon^3}{8}$$

We put $\Phi(t) = \|u_2(t)\|^2 + |v_2(t)|^2 + 2(g(u),u_2)$ to obtain

$$\frac{d}{dt}\Phi + \varepsilon\Phi \leq \frac{\varepsilon^3}{8}.$$

By Gronwall's inequality,

$$\Phi(t) \leq \Phi(t_0)\exp(\varepsilon(t_0 - t)) + \frac{\varepsilon^2}{8}\left(1 - \exp(\varepsilon(t_0 - t))\right). \tag{21}$$

Since g is a bounded operator from V into H, there exists $R > 0$ such that $|g(u)| \leq R$ whenever $\|u\| \leq \rho$. Hence we can obtain the estimate

$$2(g(u),u_2) \leq 2|g(u)||u_2| \leq \frac{2R}{\sqrt{\lambda_{N+1}}}\|u_2\| \leq \frac{2R\rho}{\sqrt{\lambda_{N+1}}} \leq \frac{\varepsilon^2}{8}$$

by choosing N large enough to guarantee that

$$\lambda_{N+1} \geq \left(\frac{8R\rho}{\varepsilon^2}\right)^2. \tag{22}$$

So (21) becomes

$$\|u_2(t)\|^2 + |v_2(t)|^2 \leq \frac{\varepsilon^2}{8} + \Phi(t_0)\exp(\varepsilon(t_0 - t)) + \frac{\varepsilon^2}{8}\left(1 - \exp(\varepsilon(t_0 - t))\right) \tag{23}$$

and we have

$$\begin{aligned} \|u_2\|^2 + |u_2'|^2 & \leq \|u_2\|^2 + |(u_2' + \varepsilon u_2) - \varepsilon u_2|^2 \\ & \leq \|u_2\|^2 + 2|v_2|^2 + 2\varepsilon^2|u_2|^2 \\ & \leq \|u_2\|^2 + 2|v_2|^2 + \frac{2\varepsilon^2}{\lambda_{N+1}}\|u_2\|^2 \\ & \leq 2(\|u_2\|^2 + |v_2|^2) \end{aligned}$$

since (10) implies

$$\lambda_{N+1} \geq 2\varepsilon^2.$$

Hence (23) becomes

$$\|y_2\|_E^2 = \|u\|^2 + |u'|^2 \leq \frac{\varepsilon^2}{4} + 2\Phi(t_0)\exp(\varepsilon(t_0 - t)) + \frac{\varepsilon^2}{4}\left(1 - \exp(\varepsilon(t_0 - t))\right). \quad (24)$$

We want to find a $t_* > 0$ such that the right hand side of (24) becomes less than ε^2. For this we choose,

$$t_* = t_0 + \frac{1}{\varepsilon}\ln\left(\frac{8\Phi(t_0) - \varepsilon^2}{2\varepsilon^2}\right)$$

And this proves the theorem, that is for any given ε if one chooses N so that (10), (17), (19) and (22) are satisfied then

$$\|y_2(t)\|_E < \varepsilon \qquad \forall t \geq t_*.$$

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