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Equivalence And New Exact Solutions To The Black Scholes And Diffusion Equations*

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Abstract

In this paper we construct a class of new solutions for both the Black Scholes and the Diffusion Equations using similarities between them. In particular, we construct two important new solutions: one that generalizes both terms of the Black Scholes Classical Solution and another with paradoxical properties. We also establish the equivalence group for the Black Scholes Equation and obtain the largest set of transformations that converts the Black Scholes Equation into the Diffusion Equation.

1 Introduction

Let us consider the Black-Scholes equation (see for example [1])

$$\dot{V} + (\sigma^2 x^2/2)V_{xx} + rxV_x - rV = 0 \tag{1}$$

where $V(x,t) > 0$; $\dot{V} \equiv \partial V/\partial t$; $V_x \equiv \partial V/\partial x$; $x > 0$. The volatility σ ($\sigma > 0$) and risk-free interest rate r are arbitrary constants. The classical solution of the Black Scholes equation (from now on, we will refer to it as "classical solution") for call option has the following structure:

$$V(x,t) = xN(d_1) - Ke^{-r(T-t)}N(d_2); \tag{2}$$

where K and T are positive constants, N is the normal distribution function

$$N(\theta) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\theta} e^{-\frac{u^2}{2}}du,\ N'' + \theta N' = 0, \tag{3}$$

$$d_1 \equiv \frac{\ln x - \ln K}{\tau} + (1-\beta)\tau,\ d_2 \equiv \frac{\ln x - \ln K}{\tau} - \beta\tau,$$

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and

$$\tau \equiv \sigma\sqrt{T-t},\ \beta \equiv (1/2) - (r/\sigma^2). \quad (4)$$

The regular transformation (for instance, Wilmott *et al.* [1])

$$\tau = -\sigma^2 t,\ \xi = \ln x,\ V(t,X) = U(\tau,\xi)\exp\left\{\beta ln x + (\beta-1)^2\sigma^2 t/2\right\} \quad (5)$$

converts the equation (1) into the diffusion equation: $\partial U/\partial\tau = (1/2)\partial^2 U/\partial\xi^2$.

The structure of this paper is the following: in Sec. 2 and Sec. 3 we use the ansatz method to construct two new solutions of the diffusion equation and two new solutions of the Black Scholes equation respectively. In Sec. 4, we review the invariance group of the diffusion equation to show that the new solutions belong to the same class of equivalence. In Sec. 5, we obtain the widest group of equivalence of the Black Scholes equation and apply the same classification for the Black Scholes equation and its new solutions. We also find the largest set of transformations that convert the Black Scholes equation into the diffusion equation.

2 New Solutions to the Diffusion Equation

Let the diffusion equation be:

$$\dot{\Psi} - (1/2)\Psi_{xx} = 0. \quad (6)$$

We impose a solution with the following structure:

$$\Psi = u(x,t)N(\theta(x,t)) \quad (7)$$

where $u(x,t)$ is a particular exact solution of the equation (7) and function $N(\cdot)$ verifies the ordinary differential equation in (3). Let us apply ansatz (7) to equation (6). From this, we obtain a non linear system:

$$\dot{u} - (1/2)u_{xx} = 0, \quad (8)$$

$$\dot{\theta} - \theta_{xx}/2 - (u_x/u)\theta_x + \theta\theta_x^2/2 = 0. \quad (9)$$

Using the parallelism with the Black Scholes classic solution, we suppose that function $\theta(x,t)$ is linear: $\theta(x,t) = \alpha(t)x + \gamma(t)$. Given $\theta_x = \alpha \neq 0$, we get from (9):

$$u(x,t) = \exp\left\{m(t)x^2 + n(t)x + v(t)\right\}, \quad (10)$$

$$m(t) \equiv \left(\dot{\alpha} + \alpha^3/2\right)/2\alpha,\ n(t) \equiv \left(\dot{\gamma} + \gamma\alpha^2/2\right)/\alpha \quad (11)$$

and $v(t)$ is an arbitrary function. The equation (9) gives:

$$\dot{m} = 2m^2,\ \dot{n} = 2mn,\ \dot{v} = m + n^2/2. \quad (12)$$

Afterwards, two possible cases appear.

`Case` 1. $m = 0$. From (12) $\dot{n} = 0, \dot{v} = n^2/2$,

$$u(x,t) = c\exp\left\{n(x-\lambda) + \frac{n^2}{2}(t-t_0)\right\};\ n, c, \lambda, t_0 = const\ (c \neq 0). \quad (13)$$

This particular solution of the diffusion equation is the one obtained traditionally by separation of variables and used to apply Fourier series. Therefore we conclude:

PROPOSITION 1. The diffusion equation (6) has an exact solution:

$$\Psi(x,t,\lambda,t_0,n,c) = N(\theta(x,t))c\exp\left\{n(x-\lambda)+\frac{n^2}{2}(t-t_0)\right\} \tag{14}$$

with $N(\theta)$ the normal distribution function in (3),

$$\theta(x,t) = \varepsilon\frac{(x-\lambda)+n(t-t_0)}{\sqrt{t-t_0}},\ \varepsilon = \pm 1,$$

and four arbitrary constants c, n, λ, t_0.

`Case` 2. $m \neq 0$. From (12):

$$m = -\frac{1}{2(t-t_0)},\ n = \frac{\lambda}{(t-t_0)},\ v = -\frac{1}{2}\ln(t-t_0)+\lambda^2 m+\ln c,$$

$$u(x,t) = \frac{c}{\sqrt{t-t_0}}\exp\left\{-\frac{(x-\lambda)^2}{2(t-t_0)}\right\};\ c,\ \lambda,\ t_0 = const\ (c\neq 0). \tag{15}$$

In this case, the solution $u(x,t)$ in (15) represents the fundamental solution (or Green function) of the diffusion equation (6) where $t > t_0$, and c, λ, t_0 $(c \neq 0)$ are arbitrary constants. From (11), $\frac{1}{\alpha^2} = a(t-t_0)^2 - (t-t_0)$, $\gamma = \alpha(t-t_0)^2 - (t-t_0)$. Thus, we find a new exact solution.

PROPOSITION 2. The diffusion equation (6) has a solution:

$$\Psi(x,t,\lambda,t_0,c,a,p) = N(\theta(x,t))\frac{c}{\sqrt{t-t_0}}\exp\left\{-\frac{(x-\lambda)^2}{2(t-t_0)}\right\}; c,\lambda,t_0 = const(c\neq 0). \tag{16}$$

with $N(\theta)$ the normal distribution function in (3),

$$\theta(x,t) = \varepsilon\frac{(x-\lambda)+p(t-t_0)}{\sqrt{(t-t_0)[a(t-t_0)-1]}},\ \varepsilon = \pm 1,$$

and five arbitrary constant c, a, λ, t_0, p $(c \neq 0, a > 0)$.

This paradoxical solution has various interesting properties. First, only under the condition $t > t_0 + 1/a$, function $\theta(x,t)$ is real and can have a logical interpretation. This property guarantees that (16) will not share the interpretation difficulty of the Green function (15): the solution (16) does not describe a motion with infinite speed.

We can also state that under conditions $t \to t_0 + 1/a,\ a \to \infty$, solution (16) tends to the Dirac delta function $\delta(x-\lambda)$. Also for $t \to \infty$, $\theta \to \varepsilon p/\sqrt{a}$.

Note that the function (16) can be normalized because $0 \leq N(\theta) \leq 1$ for any value of its argument. Graphically, the presence of $N(\theta)$ creates an asymmetry of the Gauss curve. The selection of ε's sign allows displacement of this deformation to the left or to the right side of the curve. Our idea of interpretation for the paradoxical solution (16) is that it describes the diffusion process superposed by some reactions such as chemical ones (see Sukhomlin and Ortiz [10]).

3 New Solutions to the Black Scholes Equation

We construct new solutions to the Black Scholes equation (1) using the structure of (7) and the same procedure. Considering parallelism with the classical solution (2) we suppose that $\theta(x,t)$ is a linear function of $\ln x$: $\theta(x,t) = \frac{1}{\delta(t)}\ln(x/K) + \gamma(t)$ where functions $\delta(t)$, $\gamma(t)$ are to be determined and K is a positive constant. Substituting ansatz (7) into (1) we get a system similar to (8) and (9):

$$\dot{u} - \frac{\sigma^2}{2}x^2 u_{xx} + rxu_x - ru = 0,$$

$$\dot{\theta} - \frac{\sigma^2}{2}x^2\theta_{xx} - \frac{u_x}{u}\sigma^2 x^2\theta_x - \frac{\sigma^2}{2}x^2\theta\theta_x^2 + rx\theta_x = 0,$$

$$u(x,t) = \exp\left\{\frac{2\delta\dot{\delta}+\sigma^2}{4\sigma^2\delta^2}\ln^2\frac{x}{K} + n(t)\ln\frac{x}{K} + v(t)\right\}$$

with a certain $v(t)$ and $n(t) \equiv -\dot{\gamma}\delta/\sigma^2 + \gamma/2\delta + \beta$. Similar to the diffusion equation, we have two different cases.

`Case` 1. $2\delta\dot{\delta} + \sigma^2 = 0$. $\delta = \varepsilon\sigma\sqrt{T-t}$ ($\varepsilon = \pm 1$) where T ($T > 0$) is an arbitrary constant interpreted as the expiry date for the option. From this we obtain a new exact solution.

PROPOSITION 3. The Black Scholes equation (1) has the following solution:

$$V(x,t,K,T,n,c) = N(\theta(x,t))c\exp\left\{(\beta+n)\ln\frac{x}{K} + \frac{n^2-(\beta-1)^2}{2}\sigma^2(T-t)\right\} \quad (17)$$

with $N(\theta)$ the normal distribution function in (3),

$$\theta(x,t) = \varepsilon\frac{\ln(x/K) + n\sigma^2(T-t)}{\sigma\sqrt{T-t}},\ \varepsilon = \pm 1,$$

and four real arbitrary independent constants K, T, n, c ($K > 0, T > 0, c > 0$) ($\forall\ t \in [0,T]$).

It is important to note that for different values of constant n ($\varepsilon = +1$) the function (17) includes both terms of the classical solution (2): the first term for $n = 1-\beta$ and the second term for $n = -\beta$.

`Case` 2. $2\delta\dot{\delta} + \sigma^2 \neq 0$.

PROPOSITION 4. The Black Scholes Equation (1) has the following exact solution:

$$V(x,t,K,T,a,p,c) = N(\theta(x,t))\frac{c}{\sigma\sqrt{T-t}}\exp\left\{-\frac{[\ln(x/K)-\beta\sigma^2(T-t)]^2}{2\sigma^2(T-t)} - r(T-t)\right\} \quad (18)$$

with $N(\theta)$ the normal distribution function in (3),

$$\theta(x,t) = \varepsilon \frac{\ln(x/K) + p\sigma^2(T-t)}{\sqrt{\sigma^2(T-t)[a(T-t)-1]}}, \ \varepsilon = \pm 1, \tag{19}$$

and five arbitrary constants K, T, a, p, c ($K > 0$, $T > 0$, $a > 1/T$, $c \neq 0$).

Solution (18) has paradoxical properties similar to those of solution (16). First, only under the condition that $0 \leq t \leq T - t_a$ ($t_a \equiv 1/a$) , function (19) becomes real and therefore allows solution (18) to be used in mathematical finance.

On the other hand, it should be noted that solution (18) has properties of a function that converges to the Dirac delta function $\delta(\ln(x/K))$ when $t \to T$ and $a \to \infty$ ($t_a \to 0$). The existence of $a > 0$ is coherent with the market convention that an option cannot be traded when it is almost maturing.

The idea of our interpretation of the new solutions to the Black-Scholes equation is in the superposition of two process: one is the Black-Scholes stochastic process and another is a deterministic process such as a collective stock market behavior. This new compound process will also describe the price formation but in a slightly different way from the one used in the Black-Scholes classical model (see Sukhomlin and Ortiz [10]).

4 Invariance Group for the Diffusion Equation

The very effective approaches have been developed by Shapovalov [2][1] and applied by Sukhomlin to parabolic and ultraparabolic equations [2] - [7]. We use this theory to classify the set of diffusion equation's solutions, in order to show that new solution (14) and our paradoxical solution (16) belong to the same class of equivalence. It is known that the diffusion equation (6) is invariant to the following transformation group (for example, [2]).

PROPOSITION 5. The largest invariance group admitted by the diffusion equation (6) is $\Gamma_{dif} = G_{dif} \otimes N_{dif}$, where G_{dif} is the Galilean subgroup and N_{dif} is a discrete subgroup defined by:

A) Galilean subgroup G_{dif} : $\tau = s^2(t - t_0)$, $\xi = s[(x - x_0) + n(t - t_0)]$, $\Psi(x,t) = a(x,t)U(\xi,\tau)$, $a(x,t) \equiv \exp\left\{ n(x - x_0) + n^2(t - t_0)/2 \right\}$, $s = const \neq 0$; t_0, x_0, $n = const$.

B) discrete subgroup is a cyclic group: $N_{dif} \equiv [\nu, \nu^2, \nu^3, \nu^4]$ where ν is the following transformation: $\tau = -1/t$, $\xi = x/t$, $\Psi(x,t) = a(x,t)U(\xi,\tau)$, $a(x,t) \equiv [1/\sqrt{t}] \exp\left\{-x^2/2t\right\}$.

For each subgroup: Galilean and discrete, equation (6) transforms into:

$$\frac{\partial \Psi}{\partial t} - \frac{1}{2}\frac{\partial^2 \Psi}{\partial x^2} = a(x,t)\dot{\tau}\left\{\frac{\partial U}{\partial \tau} - \frac{1}{2}\frac{\partial^2 U}{\partial \xi^2}\right\} = 0.$$

The group Γ_{dif} contains four parameters: s ($s \neq 0$), t_0, x_0, n. In mechanics, the discrete subgroup N_{dif} is interpreted as the rotation of $\pi/2$ on the phase plane.

[1] The Shapovalov's non Lie approaches in symmetry are very constructive also. Using it Sukhomlin [11] - [12] built the *Conservation Law of Strike Price*, resolved exactly the market calibration problem to the Black Scholes model and to similar model with non-constant volatility.

PROPOSITION 6. The solutions (14) and (16) belong to the same equivalent class.

Indeed, solution (14) is equal to $g'_G[g_N(g_G(\Psi))]$, where Ψ is the solution in (16), $g_G, g'_G \in G_{dif}$; and $g_N \in N_{dif}$.

5 Equivalence Group for the Black Scholes Equation

In this section we construct the widest equivalence group of the Black Scholes equation and therefore generalize Sukhomlin [4]. Using this group we apply the same classification to the Black Scholes equation's solutions and particularly to new exact solutions. We also find the largest set of transformations that convert the Black Scholes equation into the diffusion equation.

Let the Black Scholes Equation be (1). We consider the regular transformation of variables:

$$\tau = \tau(t),\ \xi = \xi(x,t),\ V(x,t) = a(x,t)\ Q(\xi,\tau) \tag{20}$$

which does not change the structure of (1), though the new equation with new variables can be multiplied by a non null function:

$$f(x,t)\ \left\{\frac{\partial Q}{\partial \tau} + \frac{1}{2}(\sigma_0)^2\xi^2\frac{\partial^2 Q}{\partial \xi^2} + r_0\xi\frac{\partial Q}{\partial \xi} - r_0 Q\right\} = 0. \tag{21}$$

The set of all those transformations constitutes a group (see Proposition 7 below). If we impose $\sigma_0 = \sigma$, $r_0 = r$ we obtain the *invariance group* of the Black Scholes equation. If we assume that constants σ_0, r_0 are not equal to σ, r respectively, the result represents the *equivalence group* for the Black Scholes equation. Then we state two facts.

Consider the Black Scholes Equation (1). The transformation:

$$\tau = T_0 - \frac{\alpha^2\sigma^2}{(\sigma_0)^2}(T-t), \forall t \in [0,T],\ \xi = K_0\left(\frac{x}{K}\right)^{\alpha}\exp\left\{-\alpha n\sigma^2(T-t)\right\}(n = const),$$

$$a(x,t) = \exp\left\{[\beta - \alpha(\beta_0+n)]\ln\frac{x}{K} + \frac{1}{2}[\alpha^2(\beta_0+n-1)^2 - (\beta-1)^2]\sigma^2(T-t)\right\} \tag{22}$$

converts (1) into (21) with the exterior factor $f(x,t) = a(x,t)\ \dot{\tau}\ \neq 0$ and with four real arbitrary independent constants: α, T, K, n ($\alpha \neq 0$, $T > 0$, $K > 0$). K_0, T_0 are dimensional constants. Here constant β is defined by (4) and $\beta_0 \equiv (1/2) - (r_0/\sigma_0{}^2)$.

Transformation (22) represents a continuous group with four parameters: constant α corresponds to the change on the time scale with a simultaneous scale variation of variable $\ln x$. Constants T, T_0 represent the liberty of selecting the exercise date for the option. Constants K, $K_0 > 0$ are related with liberty of selecting the strike price. Constants α, K, K_0 define the units for coordinates $\{x,t\}$ and $\{\xi,\tau\}$. The constant n defines the liberty to select the actualization rate.

We also call this group the Galilean group of Black Scholes equation G_{BS}. Its invariance subgroup G_0 appears using (22) along with the conditions $\sigma_0 = \sigma$, $r_0 = r$:

$$\tau = T_0 - \alpha^2(T-t),\ \xi = K_0\left(\frac{x}{K}\right)^{\alpha}\exp\left\{-\alpha n\sigma^2(T-t)\right\}(n = const),$$

$$a(x,t) = \exp\left\{[\beta - \alpha(\beta+n)]\ln\frac{x}{K} + \frac{1}{2}[\alpha^2(\beta+n-1)^2 - (\beta-1)^2]\sigma^2(T-t)\right\} \quad (23)$$

Besides this group, there is also another special transformation that does not change the structure of equation (1). The following transformation, which we denote by ν:

$$x' = \exp\{-(\ln x)/t\}\ , t' = -1/t. \quad (24)$$

converts the Black Scholes equation (1) into equation (21). The collection $[\nu, \nu^2, \nu^3, \nu^4]$ represents a discrete group, which we denote by N_{BS}.

PROPOSITION 7. Consider the Black Scholes equation (1) with constants $\sigma \neq 0$, r.

1) The largest *equivalence group* admitted by equation (1) is $\Gamma_{BS} = G_{BS} \otimes N_{BS}$, where G_{BS} is the Galilean group and N_{BS} represents the discrete group defined by (22) and (24) respectively.

2) The *invariance group* for equation (1) is $\Gamma_{0\ BS} = G_0 \otimes N_{BS}$ where G_0 ($G_0 \subset G_{BS}$) is the invariance subgroup (23).

It is easy to verify this Proposition by applying transformation (20) to equation (1) and imposing the equation structure on (21). As in the diffusion equation case, the equivalence between the paradoxical solution (18) and the new solution (17) can be established. The proof is the same as in Section 4.

To generalize these results and to explain the correspondence between these two equations, we denote the transformation (5) by $g_{BS\to dif}$.

PROPOSITION 8. Consider the Black Scholes equation (1) and the diffusion equation (6). Let Γ_{dif} and Γ_{BS} be the invariance group for the diffusion equation and the equivalence group for the Black Scholes equation which are defined in Propositions 5 and 7 respectively. The largest set of transformations of the equation (1) into (6) has the following structure: $\Gamma_{BS\to dif} = \Gamma_{BS} \otimes g_{BS\to dif} \otimes \Gamma_{dif}$.

Gazizov and Ibraguimov [8] constructed two particular transformations of this set. Pooe *et al.* [9] used these transformations.

6 Conclusions

Actually the ansatz approach has a wide variety of applications when searching for the exact solutions of non linear equations. We illustrate that this method gives new interesting results for known linear equations. Using the ansatz (7) we obtain two new particular solutions for the Black Scholes equation and the diffusion equation. One solution has paradoxical properties. It shows the Dirac delta function's behavior under certain conditions but it does not share the interpretation difficulty of the Green function: it does not describe motion with infinite speed.

We obtained the largest equivalence group of the Black Scholes equation and we found an equivalence class that includes our new solutions. Also, the invariance group of the diffusion equation and the equivalence group of the Black Scholes equation allow the construction of the largest set of transformations between these equations.

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