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Solutions Of Shallow-Water Equations In Non-rectangular Cross-section Channels*

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Abstract

Smooth solutions of the shallow-water equations for non-rectangular cross-sections channels are studied. It is found that a complete set of solutions can be classified in four Lie symmetry groups that show distinctive physical features.

1 Introduction

Shallow-water equations have been extensively used to model the hydrodynamic behavior of flows in hydraulics, hydrology, oceanography, meteorology and engineering [1]. Challenging problems in applied sciences have provided new physical models that include additional terms and/or different boundary conditions to the basic shallow-water equations. Thus there has been an increasing need to find out analytical solutions of the model equations to understand the physical phenomena, as well as to parameterize and validate complex numerical codes and analyze their results. These problems also raise a host of interesting mathematical problems.

One of the challenges concerns the density or gravity currents that occur in many natural and industrial situations [2]. These flows are formed by fluid flowing mainly horizontally under the influence of gravity into another fluid of slightly different density. It was found that the experimental results are explained by the similarity solutions of the depth-averaged shallow-water equations, eventually extended to the case of axial geometry [3]. The influence of the cross-section shape on the flow has been recently investigated by Thomas & Marino [4]. A physical model to corroborate the results of laboratory flows evolving in triangular cross-section channels was presented and a particular analytical solution was obtained. However, up today there is not a general study to analyze the complete set of solutions for non-rectangular cross-section shapes to put the found solution into context as well as to look for new ones.

This paper is a first step to analyze possible solutions of shallow-water equations for describing gravity flows in non-rectangular cross-sections channels as those used in

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[4]. Lie symmetry groups are sought in the model equations and it is shown that the solutions can be divided in four different local groups that reveal distinctive features. Examples and physical interpretations are given, focusing in the groups where practical solutions may be found.

2 Model Equations

As mentioned before, the shallow-water equations may be extended for studying gravity currents evolving in a channel of uniform non-rectangular cross-section defined by

$$y = y(z) = \begin{cases} bz^a & \text{for} \quad y \geq 0, \\ -bz^a & \text{for} \quad y < 0, \end{cases} \tag{1}$$

where a, b are constants, and $z \geq 0$, y are the coordinates in the vertical and transversal directions, respectively (cf. [4]). The value $a = 1$ determines a triangular cross-section, $a \leq 1$ indicates a cross-section with a central depression and $a > 1$ provides convex cross-sections. The usual rectangular case is obtained for $a \to \infty$. On the other hand b is related to the geometric parameters of the channel cross-section through $b = w/h_0^a$ where $h_0 > 0$ and $w = y$ $(z = h_0)$ denote the height and width of the channel, respectively. The use of a and b facilitates the analysis of the basic properties of a flow developing in arbitrary cross-section channels. For the case of a fluid layer of density ρ_1 over a layer of density $\rho_2 > \rho_1$ in a horizontal channel in which the frictional effects of the bottom are neglected, the mass and momentum conservation equations in one dimension become the following partial differential equations (PDE) system for the physical variables:

$$\Pi : \begin{cases} \left(h^{a+1}\right)_t + \left(\upsilon h^{a+1}\right)_x &= 0, \\ \upsilon_t + \upsilon\upsilon_x + g' h_x &= 0, \end{cases} \tag{2}$$

where $g' = g\left(\rho_2 - \rho_1\right)/\rho_1$ is the reduced gravity, $h(x,t)$ is the depth of the lower layer and $\upsilon(x,t)$ is the corresponding fluid velocity (cf. [1], §3.2, pp. 80-84). Note how a enters into the first equation of (2) and b is absent in Π. This suggests that w is not a relevant parameter to determine the flow developed, as in rectangular cross-section channels, under the present hypotheses. Scaling the variables with the available parameters g' and $h_0 > 0$, and redefining h/h_0, $\upsilon/\sqrt{g'h_0}$, x/h_0, $t\sqrt{g'/h_0}$ as h, υ, x, t, respectively, Eq. (2) may be expressed in a dimensionless form by

$$\Delta : \begin{cases} (a+1)(h_t + h_x\upsilon) + h\upsilon_x &= 0, \\ \upsilon_t + \upsilon\upsilon_x + h_x &= 0. \end{cases} \tag{3}$$

If $a > -1$ then Δ satisfies the *maximal rank condition.* Now techniques of Lie groups, similarity methods and dimensional analysis are available to solve the PDE system (3) of our interest (cf. [5] [6] [7] [8]).

3 The Symmetry Groups of Δ

Let M be the subset of points $(x, t, \upsilon, h) \in \mathbb{R}^4$ so that $x > 0$, $t > 0$. By X we denote a vector field on M and let $X^{(1)}$ be the corresponding prolongation of X to the $1-$ jet

$M^{(1)}$. So, if $X \in \mathfrak{X}(M)$ is given as

$$X = \alpha(x,t,\upsilon,h)\frac{\partial}{\partial x} + \beta(x,t,\upsilon,h)\frac{\partial}{\partial t} + \gamma(x,t,\upsilon,h)\frac{\partial}{\partial \upsilon} + \delta(x,t,\upsilon,h)\frac{\partial}{\partial h}$$

then $X^{(1)} \in \mathfrak{X}\left(M^{(1)}\right)$ can be written as

$$X^{(1)} = X + \gamma^x \frac{\partial}{\partial \upsilon_x} + \gamma^t \frac{\partial}{\partial \upsilon_t} + \delta^x \frac{\partial}{\partial h_x} + \delta^t \frac{\partial}{\partial h_t}.$$

By evaluating the first prolongation formula (cf. [7], Th. 2.36, pp. 113) the following relations are obtained:

$$\begin{aligned}\gamma^x &= \gamma_x + (\gamma_\upsilon - \alpha_x)\,\upsilon_x + (\beta_x + \gamma_h)\,h_x + \beta_x \upsilon\upsilon_x - \alpha_\upsilon \upsilon_x^2 + (\beta_\upsilon - \alpha_h)\upsilon_x h_x \\ &\quad + \beta_\upsilon \upsilon\upsilon_x^2 + \beta_h \upsilon\upsilon_x h_x + \beta_h h_x^2,\end{aligned}$$

$$\begin{aligned}\gamma^t &= \gamma_t - \alpha_t \upsilon_x + (\beta_t - \gamma_\upsilon)\,(\upsilon\upsilon_x + h_x) - \upsilon_x h\gamma_h/(a+1) - \gamma_h \upsilon h_x + \alpha_\upsilon \upsilon\upsilon_x^2 \\ &\quad + \alpha_\upsilon \upsilon_x h_x - \alpha_h h\upsilon_x^2/(a+1) - \alpha_h \upsilon\upsilon_x h_x,\end{aligned}$$

$$\begin{aligned}\delta^x &= \delta_x + \delta_\upsilon \upsilon_x + (\delta_h - \alpha_x)\,h_x + \beta_x \upsilon_x h/(a+1) + \beta_x h_x \upsilon - \alpha_\upsilon \upsilon_x h_x - \alpha_h h_x^2 \\ &\quad + \beta_\upsilon h\upsilon_x^2/(a+1) + \beta_\upsilon \upsilon\upsilon_x h_x + \beta_h \upsilon_x h h_x/(a+1) + \beta_h \upsilon h_x^2,\end{aligned}$$

$$\delta^t = \delta_t + \delta_\upsilon \upsilon_t + (\delta_h - \beta_t)h_t - \alpha_t h_x - \alpha_\upsilon \upsilon_t h_x - \alpha_h h_t h_x - \beta_\upsilon \upsilon_t h_t - \beta_h h_t^2.$$

Since $X^{(1)}\Delta(x,t,\upsilon,h) = 0$ (cf. [7], Th. 2.31, pp. 106), it follows that

$$\begin{aligned}(a+1)\left(\gamma h_x + \upsilon\delta^x + \delta^t\right) + \delta\upsilon_x + h\gamma^x &= 0, \qquad (4)\\ \gamma\upsilon_x + \upsilon\gamma^x + \gamma^t + \delta_x &= 0.\end{aligned}$$

Let us equate the coefficients of the monomials in the first partial derivatives of υ and h occurring in (3). The equations defining the symmetry groups of Δ are: (i) $\delta = 0$, (ii) $-\gamma_\upsilon = \gamma_t = \gamma_x = 0$, (iii) $-\alpha_t + \gamma = 0$, (iv) $-\alpha_x + \beta_t = 0$, (v) $\beta_x + \gamma_h = 0$, (vi) $\beta_x - \gamma_h = 0$, (vii) $\beta_x = \beta_\upsilon = \beta_h$, (viii) $\alpha_\upsilon = \alpha_h = 0$. Hence, by (ii) is $\gamma = \gamma(h)$ and by (iii) $\gamma = \alpha_t$. Indeed, by (iv) $\alpha_x = \beta_t$. By (v) and (vi) $\beta_x = -\dot{\gamma}(h) = \dot{\gamma}(h)$ and so $\beta_x = 0$ and $\gamma \equiv c_4$ for some constant $c_4 \in \mathbb{R}$. Thus, by (vii) $\beta = \beta(t)$ and by (viii) $\alpha = \alpha(x,t)$. Now, by (iii) we have $\alpha(x,t) = c_4 t + \eta(x)$ for some function $\eta = \eta(x)$ with continuous derivative. Using (iv) we get $\dot{\eta}(x) = \dot{\beta}(t)$ and so $\ddot{\eta}(x) = \ddot{\beta}(t) = 0$, i.e. there are $c_1, c_2, c_3 \in \mathbb{R}$ such that $\eta(x) = c_3 x + c_1$ and $\beta(t) = c_3 t + c_2$. Consequently,

$$\begin{aligned}X &= \alpha(x,t)\frac{\partial}{\partial x} + \beta(t)\frac{\partial}{\partial t} + c_4\frac{\partial}{\partial \upsilon}\\ &= (c_4 t + c_3 x + c_1)\frac{\partial}{\partial x} + (c_3 t + c_2)\frac{\partial}{\partial t} + c_4\frac{\partial}{\partial \upsilon}\\ &= c_1\frac{\partial}{\partial x} + c_2\frac{\partial}{\partial t} + c_3\left(x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t}\right) + c_4\left(t\frac{\partial}{\partial x} + \frac{\partial}{\partial \upsilon}\right).\end{aligned}$$

Thus the Lie algebra of infinitesimal symmetries of Δ is spanned by the four vector fields

$$X_1 = \partial_x, \quad X_2 = \partial_t, \quad X_3 = x\partial_x + t\partial_t, \quad X_4 = t\partial_x + \partial_\upsilon. \tag{5}$$

The commutation relationships among these vector fields are given by the following table, where the entry in row i and column j represents $[X_i, X_j]$:

	$\mathbf{X}_1$	$\mathbf{X}_2$	$\mathbf{X}_3$	$\mathbf{X}_4$
$\mathbf{X}_1$	0	0	0	X_1
$\mathbf{X}_2$	0	0	$X_1 + X_2$	X_2
$\mathbf{X}_3$	0	$-X_1 - X_2$	0	0
$\mathbf{X}_4$	$-X_1$	$-X_2$	0	0

The corresponding local parameter groups are the following:

$$\begin{aligned} G_1 : &\quad (x,t,\upsilon,h) \underset{\longrightarrow}{\sigma_s^1} (x+s,t,\upsilon,h), \\ G_2 : &\quad (x,t,\upsilon,h) \underset{\longrightarrow}{\sigma_s^2} (x,t+s,\upsilon,h), \\ G_3 : &\quad (x,t,\upsilon,h) \underset{\longrightarrow}{\sigma_s^3} (x\exp s, t\exp s, \upsilon, h), \\ G_4 : &\quad (x,t,\upsilon,h) \underset{\longrightarrow}{\sigma_s^4} (x+st, t, \upsilon+s, h), \end{aligned} \tag{6}$$

where $\sigma_s^j = \exp(sX_j)$, $1 \le j \le 4$. Since each local Lie group G_j is a symmetry group, the transformations σ_s^j suggest that the solutions $\upsilon^j = \upsilon^j(x,t)$ and $h^j = h^j(x,t)$ of Δ are

$$\begin{aligned} \upsilon^1(x,t) &= \upsilon(x-s,t) &\text{and}\quad h^1(x,t) &= h(x-s,t), \\ \upsilon^2(x,t) &= \upsilon(x,t-s) &\text{and}\quad h^2(x,t) &= h(x,t-s), \\ \upsilon^3(x,t) &= \upsilon(x-st,t)+s &\text{and}\quad h^3(x,t) &= h(x-st,t), \\ \upsilon^4(x,t) &= \upsilon(x\exp(-s), t\exp(-s)) &\text{and}\quad h^4(x,t) &= h(x\exp(-s), t\exp(-s)). \end{aligned} \tag{7}$$

4 Invariance Analysis of Δ

From (4) it is inferred that the *projectably action* of the local groups G_1, G_2, G_3 and G_4 on M, i.e. the changes of the independent variables x and t do not depend on the independent variables υ and h. Those groups induce *semi-regular actions* (i.e. all orbits have the same dimension) with one dimensional orbit. Moreover, those actions are *regular* because any point of M has a neighborhood that intersects any orbit in an arcwise connected set. Consequently (cf. [7], Th. 2.17, pp. 88) we can determine a complete set of *functionally independent invariants* related to the vector fields (5) evaluated in Section 3. Under these assumptions, locally associated to each field, there exists a single invariant $y = y(x,t)$ of the projected group action on the half-plane $x > 0, t > 0$. Furthermore, there are two additional invariants $z^1 = z^1(x,t,\upsilon,h)$, $z^2 = z^2(x,t,\upsilon,h)$ on M so that y, z^1, z^2 provide the required complete set. By invoking the implicit function theorem we can do the following analysis of each vector field of (5):

(i) Clearly $X_1(t) = X_1(\upsilon) = X_1(h) = 0$. If $(x,t,\upsilon,h) \in M$ we write $y(x,t) = t$, $z^1(x,t,\upsilon,h) = \upsilon$, $z^2(x,t,\upsilon,h) = h$. Then $\upsilon = z^1 = z^1(y) = z^1(t)$ and $h = z^2 =$

$z^2(y) = z^2(t)$. So,

$$\upsilon_x = h_x = 0, \upsilon_t = z_t^1,\ h_t = z_t^2. \tag{8}$$

Replacing (8) in Δ we obtain $\upsilon_t = h_t = 0$, i.e. υ and h must be constants, thus constituting a trivial solution of Δ.

(ii) Here $X_2(x) = X_2(\upsilon) = X_2(h) = 0$. If $(x, t, \upsilon, h) \in M$ let

$$y(x,t) = x,\ z^1(x,t,\upsilon,h) = \upsilon,\ z^2(x,t,\upsilon,h) = h.$$

Then $\upsilon = z^1 = z^1(y) = z^1(x)$ and $h = z^2 = z^2(y) = z^2(x)$. So,

$$\Delta : \begin{cases} (a+1)h_x\upsilon + h\upsilon_x &= 0, \\ h_x + \upsilon\upsilon_x &= 0. \end{cases} \tag{9}$$

Since $\upsilon = \upsilon\,(x)$ and $h = h(x)$, $h + \upsilon^2/2 = c$ for some $c \in \mathbb{R}$ by the second equation of Δ. Then the first equation of Δ gives $\dot{\upsilon}\,\left(c - (a+3/2)\,\upsilon^2\right) = 0$. In such a case $d/dx\left[\left(c - (a+3/2)\,\upsilon^2\right)^2\right] = 0$ and υ and h must be constants.

(iii) We have $X_3\,(\upsilon) = X_3(h) = 0$. To seek a third invariant let us consider the characteristic equation $dx/x = dt/t$. Since x/t is constant on any characteristic curve we put

$$y(x,t) = \frac{x}{t},\ z^1(x,t,\upsilon,h) = \upsilon,\ z^2(x,t,\upsilon,h) = h. \tag{10}$$

Since $\upsilon = z^1 = z^1(y) = z^1(x/t)$ and $h = z^2 = z^2(y) = z^1(x/t)$ we have

$$\upsilon_x = \frac{1}{t}\frac{dz^1}{dy},\ \upsilon_t = -\frac{x}{t^2}\frac{dz^1}{dy},\ h_x = \frac{1}{t}\frac{dz^2}{dy},\ h_t = -\frac{x}{t^2}\frac{dz^2}{dy}.$$

Therefore

$$\Delta : \begin{cases} (a+1)\left(z^1 - y\right)dz^2/dy + z^2 dz^1/dy &= 0, \\ \left(z^1 - y\right)dz^1/dy + dz^2/dy &= 0. \end{cases}$$

Hence, if $dz^1/dy \neq 0$ then (locally)

$$z^2 = (a+1)\,(z^1 - y)^2. \tag{11}$$

There are several alternatives to work with Eq. (11). In particular, by replacing z^2 in the first equation of Δ and then integrating we get

$$(2a+3)\,z^1 = 2(a+1)y + c,$$

where $c \in \mathbb{R}$. Let us impose the condition $z^1 = 0$ if $z^2 = 1$. If $a > -1$ and we write

$$z_c^1 = \frac{2\,(a+1)\,y + c}{2a+3},\quad z_c^2 = (a+1)\left(\frac{c-y}{2a+3}\right)^2,$$

then $z_c^2 = 1$ if and only if $y = c \mp (2a+3)/\sqrt{a+1}$. So, $z_c^1 = 0$ if and only if $c = \pm 2\sqrt{a+1}$. Consequently,

$$\begin{array}{l} z^1 = z^1_{\pm}\,(x,t) = \frac{2\sqrt{a+1}\left[(x/t)\sqrt{a+1}\pm 1\right]}{2a+3}, \\ z^2 = z^2_{\pm}\,(x,t) = \frac{(a+1)\left[2\sqrt{a+1}\mp x/t\right]^2}{(2a+3)^2}. \end{array} \tag{12}$$

(iv) Finally, $X_4(t) = X_4(h) = 0$. As the function $x/t - \upsilon$ is constant in any solution of the characteristic equation $dx/t = d\upsilon$, it results

$$y(x,t) = t,\ z^1(x,t,\upsilon,h) = h,\ z^2(x,t,\upsilon,h) = \frac{x}{t} - \upsilon.$$

Then $h = z^1 = z^1(y) = z^1(t)$ and $\upsilon = x/t - z^2 = x/t - z^2(y) = x/t - z^2(t)$, i.e.

$$\upsilon_x = \frac{1}{t},\ \upsilon_t = -\frac{dz^2}{dt} - \frac{x}{t^2},\ h_x = 0,\ h_t = \frac{dz^1}{dt}.$$

Therefore

$$\Delta : \begin{cases} (a+1)dz^1/dt + z^1/t &= 0, \\ dz^2/dt + z^2/t &= 0. \end{cases}$$

The solutions of Δ are $z^1(t) = ct^{-1/(a+1)}$ and $z^2 = d/t$, where $c, d \in \mathbb{R}$. Consequently

$$h(x,t) = ct^{-1/(\alpha+1)},\ \upsilon(x,t) = \frac{x-d}{t}. \tag{13}$$

5 Physical Interpretation

The set of solutions of the first two symmetry groups are invariant under the actions of a spatial or a temporal shift of magnitude s. Thus the form of the solution does not depend on the origin of the spatial or temporal coordinate axis, respectively. A uniform flow is an example of these groups as seen in Section 4 (i) and (ii).

The spatial and time variables of the local group G_3 are linked by means of a velocity s suggesting that the set of solutions have wave features. In particular, the self-similar solution (12) is represented in the figure below. By replacing $\upsilon = z^1$, $h = z^2$ and $y = const = s$ in Eq. (11) it is obtained $h = (a+1)\,(\upsilon - s)^2$. Here s is a characteristic velocity that in the physical variables reads

$$s = \sqrt{\frac{g'h_0}{a+1}}. \tag{14}$$

This relationship generalizes the well-known wave speed $s = \sqrt{g'h_0}$ for rectangular cross-section channels. The corresponding solutions may be obtained from the PDEs (2) looking for solutions where h and υ are function of $(x - st)$. In particular, for the case of a small perturbation $h^*(x,t) \ll h_0$ and $\upsilon^*(x,t) \ll s$ of a steady state $h = h_0$ and $\upsilon = 0$, it is found that system (2) becomes the well-known wave equations whose solutions are any derivable functions $h^*(x - st)$ and $\upsilon^*(x - st)$ where s is given by (14).

Finally, the local group G_4 refers to solutions related to the transformations $h(x,t) \to h(xe^{-s}, t)$ and $\upsilon(x,t) \to \upsilon\,(x, te^{-s})$. This transformation may be considered as a change of scale in both independent variables x and t in the form $x \to s^*x$ and $t \to s^*t$, where

$s^* = e^{-s}$. This change of scale is additional to that verified in the PDE system Δ given by Eq. (3) that is independent of the physical scales as shown in Section 2 . The solution (13) suggests a particular form of discharge of the channel in which depth is maintained strictly uniform, that is h does not depend on x but only on t. The fluid velocity is proportional to the position x indicating that the output of the fluid is at infinity.

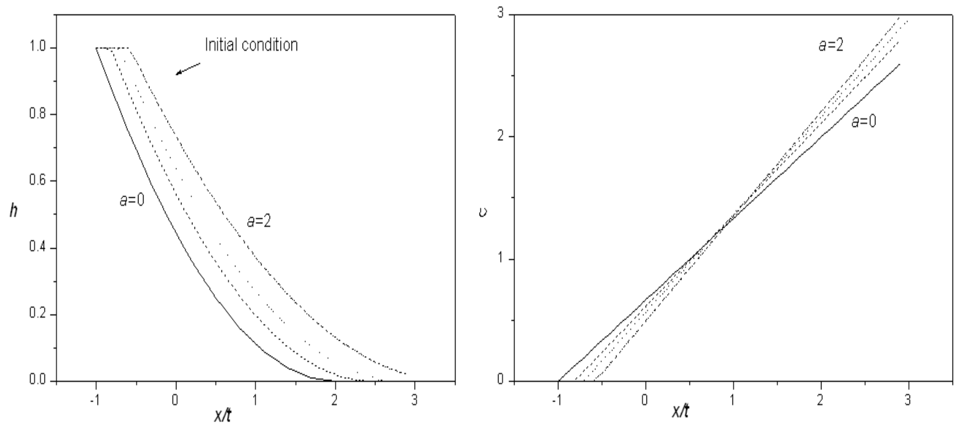

Figure 1. Height and velocity distributions provided by the self-similar solution for underflows in channels with $a = 0, 0.5, 1, 2$.

6 Conclusions

The smooth solutions of the PDE system defined by (3) were determined by means of the invariance analysis. The fields X_1 and X_2 give constant solutions, and fields X_3 and X_4 provide the non-trivial solutions as indicated by (12) and (13). Hence the action of the four Lie groups described by (6) provides genuine solutions that include the self-similar solutions previously obtained [4] and traveling waves for the new forms of the cross-sectional section of a uniform channel. The existence of some of these solutions is certainly direct from (2), but here they are found by means of a general and exhaustive method. Therefore, the present study may be considered as a reliable beginning for studying the solutions of systems of PDE analogous to (2) but including additional terms giving account of more complex physical phenomena.

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