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Some Inequalities Of Ostrowski Type And Applications*

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Abstract

Generalizations of Ostrowski type inequality for functions of Lipschitzian type are established. Applications for cumulative distribution functions are given.

1 Introduction

The following Ostrowski inequality ([5] or [4, p.468]) is well known:

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \leq \left[\frac{1}{4} + \frac{(x-(a+b)/2)^2}{(b-a)^2} \right](b-a)M, \quad x \in [a,b], \tag{1}$$

where $f : [a,b] \to \mathbf{R}$ is a differentiable function such that $|f'(x)| \leq M$, for every $x \in [a,b]$.

In Theorem 3.1 of [2], Cheng has generalized the Ostrowski inequality (1) in the following form.

THEOREM 1. Let $f : I \to \mathbf{R}$, where $I \subset \mathbf{R}$ is an interval, be a mapping differentiable in the interior Int$I$ of I, and let $a, b \in \text{Int}I$, $a < b$. If f' is integrable and $\gamma \leq f'(t) \leq \Gamma, \forall t \in [a,b]$ and some constants $\gamma, \Gamma \in \mathbf{R}$, then we have

$$\begin{aligned} &\left| \frac{1}{2}[(x-a)f(a) + (b-a)f(x) + (b-x)f(b)] - \int_a^b f(t)\,dt \right| \\ \leq\ & \frac{1}{4}\left[\left(x - \frac{a+b}{2} \right)^2 + \frac{(b-a)^2}{4} \right](\Gamma - \gamma), \end{aligned} \tag{2}$$

for all $x \in [a,b]$.

From Theorem 2 in [6], we may provide new estimations of the left part of (2) as follows:

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THEOREM 2. Let the assumptions of Theorem 1 hold. Then for all $x \in [a,b]$, we have

$$\begin{aligned} & \left| \frac{1}{2}[(x-a)f(a)+(b-a)f(x)+(b-x)f(b)] - \int_a^b f(t)\,dt \right| \\ \leq\ & \frac{b-a}{2}\left[\frac{b-a}{2} + \left| x - \frac{a+b}{2} \right| \right](S-\gamma) \end{aligned} \tag{3}$$

and

$$\begin{aligned} & \left| \frac{1}{2}[(x-a)f(a)+(b-a)f(x)+(b-x)f(b)] - \int_a^b f(t)\,dt \right| \\ \leq\ & \frac{b-a}{2}\left[\frac{b-a}{2} + \left| x - \frac{a+b}{2} \right| \right](\Gamma - S), \end{aligned} \tag{4}$$

where $S = (f(b)-f(a))/(b-a)$.

In this paper, we shall generalize Theorem 1 and Theorem 2 to functions of some larger classes. For convenience, we define functions of Lipschitzian type as follows:

DEFINITION 1. The function $f : [a,b] \to \mathbf{R}$ is said to be L-Lipschitzian on $[a,b]$ if for some $L > 0$ and all $x, y \in [a,b]$,

$$|f(x)-f(y)| \leq L|x-y|.$$

DEFINITION 2. The function $f : [a,b] \to \mathbf{R}$ is said to be (l,L)-Lipschitzian on $[a,b]$ if

$$l(x_2-x_1) \leq f(x_2)-f(x_1) \leq L(x_2-x_1) \ for\ a \leq x_1 \leq x_2 \leq b,$$

where $l, L \in \mathbf{R}$ with $l < L$.

We will need the following well-known results.

LEMMA 1. (see e.g.(3.3) in [3]) Let $h, g : [a,b] \to \mathbf{R}$ be such that h is Riemann-integrable on $[a,b]$ and g is L-Lipschitzian on $[a,b]$. Then

$$\left| \int_a^b h(t)\,dg(t) \right| \leq L \int_a^b |h(t)|\,dt. \tag{5}$$

LEMMA 2. (see e.g.(2.3) in [3]) Let $h, g : [a,b] \to \mathbf{R}$ be such that h is continuous on $[a,b]$ and g is of bounded variation on $[a,b]$. Then

$$\left| \int_a^b h(t)\,dg(t) \right| \leq \max\nolimits_{t\in[a,b]}|h(t)| V_a^b(g). \tag{6}$$

The purpose of this paper is to generalize Theorem 1 and Theorem 2 to functions which are L-Lipschitzian and (l,L)-Lipschitzian respectively. Applications for cumulative distribution functions are given.

2 Main Results

Our main results are as follows.

THEOREM 3. Let $f : [a,b] \to \mathbf{R}$ be (l,L)-Lipschitzian on $[a,b]$. Then for all $x \in [a,b]$, we have

$$\begin{aligned} &\left|\frac{1}{2}[(x-a)f(a)+(b-a)f(x)+(b-x)f(b)] - \int_a^b f(t)\,dt\right| \\ \leq\ & \frac{1}{4}\left[(x-\frac{a+b}{2})^2 + \frac{(b-a)^2}{4}\right](L-l), \end{aligned} \tag{7}$$

$$\begin{aligned} &\left|\frac{1}{2}[(x-a)f(a)+(b-a)f(x)+(b-x)f(b)] - \int_a^b f(t)\,dt\right| \\ \leq\ & \frac{b-a}{2}\left[\frac{b-a}{2} + \left|x-\frac{a+b}{2}\right|\right](S-l), \end{aligned} \tag{8}$$

and

$$\begin{aligned} &\left|\frac{1}{2}[(x-a)f(a)+(b-a)f(x)+(b-x)f(b)] - \int_a^b f(t)\,dt\right| \\ \leq\ & \frac{b-a}{2}\left[\frac{b-a}{2} + \left|x-\frac{a+b}{2}\right|\right](L-S), \end{aligned} \tag{9}$$

where $S = (f(b)-f(a))/(b-a)$.

PROOF. Let us define the function

$$p(x,t) := \begin{cases} t - \frac{a+x}{2}, & t \in [a,x], \\ t - \frac{x+b}{2}, & t \in (x,b]. \end{cases}$$

Put

$$g(t) := f(t) - \frac{L+l}{2}t. \tag{10}$$

It is easy to find that the function $g : [a,b] \to \mathbf{R}$ is M-Lipschitzian on $[a,b]$ with $M = \frac{L-l}{2}$. So, the Riemann-Stieltjes integral $\int_a^b p(x,t)\,dg(t)$ exists. Using the integration by parts formula for Riemann-Stieltjes integral, we have

$$\begin{aligned} \int_a^b p(x,t)\,dg(t) &= \int_a^x (t-\frac{a+x}{2})\,dg(t) + \int_x^b (t-\frac{x+b}{2})\,dg(t) \\ &= \frac{1}{2}[(x-a)g(a)+(b-a)g(x)+(b-x)g(b)] - \int_a^b g(t)\,dt. \end{aligned} \tag{11}$$

From (5) of the Lemma 1 we have

$$|\frac{1}{2}[(x-a)g(a)+(b-a)g(x)+(b-x)g(b)] - \int_a^b g(t)\,dt| \leq \frac{L-l}{2}\int_a^b |p(x,t)|\,dt. \tag{12}$$

It is not difficult to find that

$$\int_a^b |p(x,t)|\,dt = \frac{(x-a)^2+(b-x)^2}{4} = \frac{1}{2}[(x-\frac{a+b}{2})^2+\frac{(b-a)^2}{4}], \tag{13}$$

and so from (12) and (13) we get

$$\begin{aligned} &\left|\frac{1}{2}[(x-a)g(a)+(b-a)g(x)+(b-x)g(b)] - \int_a^b g(t)\,dt\right| \\ \leq\ & \frac{L-l}{4}\left[(x-\frac{a+b}{2})^2+\frac{(b-a)^2}{4}\right]. \end{aligned} \tag{14}$$

Consequently, the inequality (7) follows from substituting (10) to the left hand side of the inequality (14).

Now we proceed to prove the inequalities (8) and (9). Put

$$g_1(t) := f(t) - lt \text{ and } g_2(t) := f(t) - Lt. \tag{15}$$

It is easy to find that both $g_1, g_2 : [a,b] \to \mathbf{R}$ are functions of bounded variation on $[a,b]$ with

$$V_a^b(g_1) = f(b) - f(a) - l(b-a) \text{ and } V_a^b(g_2) = L(b-a) - [f(b) - f(a)]. \tag{16}$$

So, the Riemann-Stieltjes integrals $\int_a^b p(x,t)\,dg_1(t)$ and $\int_a^b p(x,t)\,dg_2(t)$ exist. Using the integration by parts formula for Riemann-Stieltjes integral, we have

$$\int_a^b p(x,t)\,dg_1(t) = \frac{1}{2}[(x-a)g_1(a)+(b-a)g_1(x)+(b-x)g_1(b)] - \int_a^b g_1(t)\,dt \tag{17}$$

and

$$\int_a^b p(x,t)\,dg_2(t) = \frac{1}{2}[(x-a)g_2(a)+(b-a)g_2(x)+(b-x)g_2(b)] - \int_a^b g_2(t)\,dt. \tag{18}$$

Then by (6) of the Lemma 2 we can deduce that

$$|\frac{1}{2}[(x-a)g_1(a)+(b-a)g_1(x)+(b-x)g_1(b)] - \int_a^b g_1(t)\,dt| \leq \max_{t\in[a,b]}|p(x,t)|V_a^b(g_1)$$

and

$$|\frac{1}{2}[(x-a)g_2(a)+(b-a)g_2(x)+(b-x)g_2(b)] - \int_a^b g_2(t)\,dt| \leq \max_{t\in[a,b]}|p(x,t)|V_a^b(g_2).$$

Notice that

$$\max_{t\in[a,b]}|p(x,t)| = \max\left\{\frac{x-a}{2}, \frac{b-x}{2}\right\} = \frac{1}{2}\left[\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|\right]$$

and from (16), we get

$$\left|\frac{1}{2}[(x-a)g_1(a)+(b-a)g_1(x)+(b-x)g_1(b)]-\int_a^b g_1(t)\,dt\right| \leq \frac{b-a}{2}\left[\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|\right](S-l) \quad (19)$$

and

$$\left|\frac{1}{2}[(x-a)g_2(a)+(b-a)g_2(x)+(b-x)g_2(b)]-\int_a^b g_2(t)\,dt\right| \leq \frac{b-a}{2}\left[\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|\right](L-S), \quad (20)$$

where $S=(f(b)-f(a))/(b-a)$.

Consequently, inequalities (8) and (9) follow from substituting (15) to the left hand sides of (19) and (20), respectively.

COROLLARY 1. Under the assumptions of Theorem 3, we get trapezoid inequalities

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \frac{b-a}{8}(L-l),$$

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \frac{b-a}{2}(S-l)$$

and

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \frac{b-a}{2}(L-S).$$

PROOF. We set $x=a$ or $x=b$ in the above theorem.

COROLLARY 2. Under the assumptions of Theorem 3, we get simple three point inequalities (i.e., the average of a mid-point and trapezoid type rules)

$$\left|\frac{f(a)+2f(\frac{a+b}{2})+f(b)}{4}-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \frac{b-a}{16}(L-l),$$

$$\left|\frac{f(a)+2f(\frac{a+b}{2})+f(b)}{4}-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \frac{b-a}{4}(S-l)$$

and

$$\left|\frac{f(a)+2f(\frac{a+b}{2})+f(b)}{4}-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \frac{b-a}{4}(L-S).$$

PROOF. We set $x=\frac{a+b}{2}$ in the above theorem.

REMARK 1. It is clear that Theorem 3 can be regarded as generalization of Theorem 1 and Theorem 2.

THEOREM 4. Let $f:[a,b]\to\mathbf{R}$ be L-Lipschitzian on $[a,b]$. Then for all $x\in[a,b]$, we have

$$\left|\frac{1}{2}[(x-a)f(a)+(b-a)f(x)+(b-x)f(b)]-\int_a^b f(t)\,dt\right| \leq \frac{L}{2}\left[(x-\frac{a+b}{2})^2+\frac{(b-a)^2}{4}\right] \tag{21}$$

and

$$\left|\frac{1}{2}[(x-a)f(a)+(b-a)f(x)+(b-x)f(b)]-\int_a^b f(t)\,dt\right| \leq \frac{b-a}{2}\left[\frac{b-a}{2}+|x-\frac{a+b}{2}|\right](L-|S|), \tag{22}$$

where $S=(f(b)-f(a))/(b-a)$.

PROOF. Inequality (21) is obtained from (7) and $l=-L$. Also, by taking $l=-L$ in (8) we get

$$\left|\frac{1}{2}[(x-a)f(a)+(b-a)f(x)+(b-x)f(b)]-\int_a^b f(t)\,dt\right| \leq \frac{b-a}{2}\left[\frac{b-a}{2}+|x-\frac{a+b}{2}|\right](S+L). \tag{23}$$

Consequently, the inequality (22) follows from (23) and (9) by considering the fact that $\min\{S+L,L-S\}=L-|S|$.

3 Applications

Now we consider some applications for cumulative distribution functions.

Let X be a random variable having the probability density function $f:[a,b]\to\mathbf{R}_+$ and the cumulative distribution function $F(x)=Pr(X\leq x)$, i.e.,

$$F(x)=\int_a^x f(t)\,dt,\quad x\in[a,b].$$

$E(X)$ is the expectation of X. Then we have the following inequality.

THEOREM 5. With the above assumptions and if there exist constants M,m such that $0\leq m\leq f(t)\leq M$ for all $t\in[a,b]$, then we have the inequalities

$$\left|P_r(X\leq x)-\frac{x-E(X)}{b-a}-\frac{b-E(X)}{b-a}\right|\leq\frac{b-a}{2}\left[\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2+\frac{1}{4}\right](M-m), \tag{24}$$

$$\left|P_r(X \le x) - \frac{x - E(X)}{b - a} - \frac{b - E(X)}{b - a}\right| \le \left(\frac{1}{b - a} - m\right)\left[\frac{b - a}{2} + \left|x - \frac{a + b}{2}\right|\right] \quad (25)$$

and

$$\left|P_r(X \le x) - \frac{x - E(X)}{b - a} - \frac{b - E(X)}{b - a}\right| \le \left(M - \frac{1}{b - a}\right)\left[\frac{b - a}{2} + \left|x - \frac{a + b}{2}\right|\right]. \quad (26)$$

PROOF. It is easy to find that the function $F(x) = \int_a^x f(t)\,dt$ is (m, M)-Lipschitzian on $[a, b]$. So, by Theorem 3 we get

$$\begin{aligned} &\left|\frac{1}{2}[(x - a)F(a) + (b - a)F(x) + (b - x)F(b)] - \int_a^b F(t)\,dt\right| \\ \le\ & \frac{(b - a)^2}{4}\left[\left(\frac{x - \frac{a+b}{2}}{b - a}\right)^2 + \frac{1}{4}\right](M - m), \end{aligned}$$

$$\begin{aligned} &\left|\frac{1}{2}[(x - a)F(a) + (b - a)F(x) + (b - x)F(b)] - \int_a^b F(t)\,dt\right| \\ \le\ & \frac{b - a}{2}\left[\frac{b - a}{2} + \left|x - \frac{a + b}{2}\right|\right](S - m) \end{aligned}$$

and

$$\begin{aligned} &\left|\frac{1}{2}[(x - a)F(a) + (b - a)F(x) + (b - x)F(b)] - \int_a^b F(t)\,dt\right| \\ \le\ & \frac{b - a}{2}\left[\frac{b - a}{2} + \left|x - \frac{a + b}{2}\right|\right](M - S), \end{aligned}$$

where $S = \frac{F(b) - F(a)}{b - a}$. As $F(a) = 0, F(b) = 1$, and

$$\int_a^b F(t)\,dt = b - E(X),$$

then we can easily deduce inequalities (24), (25) and (26).

COROLLARY 3. Under the assumptions of Theorem 5, we have

$$\left|E(X) - \frac{a + b}{2}\right| \le \frac{(b - a)^2}{8}(M - m), \quad (27)$$

$$\left|E(X) - \frac{a + b}{2}\right| \le \frac{(b - a)^2}{2}\left(\frac{1}{b - a} - m\right) \quad (28)$$

and

$$\left|E(X) - \frac{a + b}{2}\right| \le \frac{(b - a)^2}{2}\left(M - \frac{1}{b - a}\right). \quad (29)$$

PROOF. We set $x = a$ or $x = b$ in (24)-(26) to get (27)-(29).

REMARK 2. It should be noted that the inequality (27) improves the inequality (5.4) in [1].

COROLLARY 4. Under the assumptions of Theorem 5, we have

$$\left| P_r\left(X \le \frac{a+b}{2}\right) - \frac{1}{2}\right| \le \frac{3(b-a)}{8}(M-m), \tag{30}$$

$$\left| P_r\left(X \le \frac{a+b}{2}\right) - \frac{1}{2}\right| \le \frac{3}{2}[1 - m(b-a)] \tag{31}$$

and

$$\left| P_r\left(X \le \frac{a+b}{2}\right) - \frac{1}{2}\right| \le \frac{3}{2}[M(b-a) - 1]. \tag{32}$$

PROOF. Set $x = \frac{a+b}{2}$ in (24)-(26), we get

$$\left| P_r\left(X \le \frac{a+b}{2}\right) - \frac{2}{b-a}\left[\frac{a+3b}{4} - E(X)\right]\right| \le \frac{b-a}{8}(M-m), \tag{33}$$

$$\left| P_r\left(X \le \frac{a+b}{2}\right) - \frac{2}{b-a}\left[\frac{a+3b}{4} - E(X)\right]\right| \le \frac{b-a}{2}\left(\frac{1}{b-a} - m\right), \tag{34}$$

and

$$\left| P_r\left(X \le \frac{a+b}{2}\right) - \frac{2}{b-a}\left[\frac{a+3b}{4} - E(X)\right]\right| \le \frac{b-a}{2}\left(M - \frac{1}{b-a}\right). \tag{35}$$

Using the triangle inequality, we get

$$\begin{aligned} & \left| P_r\left(X \le \frac{a+b}{2}\right) - \frac{1}{2}\right| \\ = & \left| P_r\left(X \le \frac{a+b}{2}\right) - \frac{1}{2} + \frac{2}{b-a}\left[\frac{a+3b}{4} - E(X)\right] - \frac{2}{b-a}\left[\frac{a+3b}{4} - E(X)\right]\right| \\ \le & \left| P_r\left(X \le \frac{a+b}{2}\right) - \frac{2}{b-a}\left[\frac{a+3b}{4} - E(X)\right]\right| + \frac{2}{b-a}\left|E(X) - \frac{a+b}{2}\right|, \end{aligned}$$

and then inequalities (30)-(32) follow from (27)-(29) and (33)-(35).

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