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A Short Remark On Energy Functionals Related To Nonlinear Hencky Materials*

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Abstract

We prove interior $C^{1,\alpha}$-regularity of minimizing displacement fields for a class of nonlinear Hencky materials in the $2D$-case.

Let $\Omega \subset \mathbb{R}^2$ denote a bounded open set on which the displacements u of an elastic body are defined. If the case of linear elasticity is considered, then the elastic energy of the deformation is given by

$$J_0[u] = \int_\Omega \Big[\frac{1}{2}\lambda(\mathrm{div} u)^2 + \kappa|\varepsilon(u)|^2\Big]dx, \tag{1}$$

where λ, $\kappa > 0$ denote physical constants and $\varepsilon(u) = \frac{1}{2}(\nabla u + \nabla u^T)$ is the symmetric gradient of u. In order to model a nonlinear material behaviour, in particular the nonlinear Hencky material, see [1], (1) is replaced by the energy

$$J[u] = \int_\Omega \Big[\frac{1}{2}\lambda(\mathrm{div} u)^2 + \varphi\big(|\varepsilon^D(u)|^2\big)\Big]dx \tag{2}$$

for some nonlinear function φ. Here $\varepsilon^D(u)$ is the deviatoric part of $\varepsilon(u)$, i.e. $\varepsilon^D(u) = \varepsilon(u) - \frac{1}{2}(\mathrm{div} u)\mathbf{1}$. The purpose of our short note is to investigate the regularity properties of local minimizers of the functional J under suitable assumptions on the function φ. To be precise and to have more flexibility, we replace the quantity $\varphi(|\varepsilon^D(u)|^2)$ in the expression (2) for the energy by $F(\varepsilon^D(u))$, where F: $\mathbb{S}^2 \to [0,\infty)$ is a function of class C^2 defined on the space $\mathbb{S}^2$ of all symmetric (2×2)-matrices satisfying for some exponent $s \in (1,\infty)$ and with positive constants a, A the ellipticity estimate

$$a\big(1 + |\varepsilon|^2\big)^{\frac{s-2}{2}}|\sigma|^2 \le D^2F(\varepsilon)(\sigma,\sigma) \le A\big(1 + |\varepsilon|^2\big)^{\frac{s-2}{2}}|\sigma|^2 \tag{3}$$

for all ε, $\sigma \in \mathbb{S}^2$.

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DEFINITION 1. A function u from the Sobolev class $W^1_{1,loc}(\Omega;\mathbb{R}^2)$ is called a local minimizer of the functional

$$I[v,\Omega] := \int_\Omega \Big[\frac{\lambda}{2}(\mathrm{div} v)^2 + F(\varepsilon^D(v))\Big]\,dx \tag{4}$$

iff $I[u,\Omega'] < \infty$ and $I[u,\Omega'] \le I[v,\Omega']$ for all $v \in W^1_{1,loc}(\Omega;\mathbb{R}^2)$ such that $\mathrm{spt}(u-v)$ is compactly contained in Ω', Ω' being an arbitrary subdomain with compact closure in Ω, spt denoting the support of a function.

Then we have

THEOREM 1. Let u denote a local minimizer of the functional $I[\cdot,\Omega]$ defined in (4) with F satisfying (3). Then u is in the local Hölder space $C^{1,\alpha}(\Omega;\mathbb{R}^2)$ for any $0<\alpha<1$ provided that $s \in (1,4)$.

REMARK 1. In the case that $s \le 2$ the result from Theorem 1 in principle is a consequence of the work of Frehse and Seregin [2] on plastic materials with logarithmic hardening. They consider the function $F(\varepsilon) = |\varepsilon|\ln(1+|\varepsilon|)$ but it is not hard to show that their arguments actually cover the case of exponents $s \le 2$.

REMARK 2. For $s \in (1,2]$ the functional $I[\cdot,\Omega]$ also serves as a model for plasticity with power hardening, we refer to [3], [4] and [5]. It is worth remarking that Seregin proved partial regularity in the 3D-case for the above mentioned range of exponents, see e.g. [6], [7].

PROOF OF THEOREM 1. Let $f(\varepsilon) := \frac{\lambda}{2}(\mathrm{tr}\varepsilon)^2 + F(\varepsilon^D), \varepsilon \in \mathbb{S}^2$. Here $\mathrm{tr}\varepsilon$ is the trace of the matrix ε and $\varepsilon^D = \varepsilon - \frac{1}{2}\mathrm{tr}\varepsilon\mathbf{1}$. Clearly $f\colon \mathbb{S}^2 \to [0,\infty)$ is of class C^2 and satisfies for all $\varepsilon, \sigma \in \mathbb{S}^2$

$$D^2f(\varepsilon)(\sigma,\sigma) = \lambda(\mathrm{tr}\sigma)^2 + D^2F(\varepsilon^D)(\sigma^D,\sigma^D). \tag{5}$$

If $s \ge 2$, then (3) and (5) imply with positive constants ν and μ

$$\nu|\sigma|^2 \le D^2f(\varepsilon)(\sigma,\sigma) \le \mu\big(1+|\varepsilon|^2\big)^{\frac{s-2}{2}}|\sigma|^2 \tag{6}$$

for arbitrary matrices ε, $\sigma \in \mathbb{S}^2$. If $1<s<2$, then we observe (see (3) and (5))

$$D^2f(\varepsilon)(\sigma,\sigma) \ge \lambda\big(1+|\varepsilon|^2\big)^{\frac{s-2}{2}}(\mathrm{tr}\sigma)^2 + a\big(1+|\varepsilon|^2\big)^{\frac{s-2}{2}}|\sigma^D|^2$$

which follows from $(1+|\varepsilon|^2)^{(s-2)/2} \le 1$ and $(1+|\varepsilon|^2)^{(s-2)/2} \le (1+|\varepsilon^D|^2)^{(s-2)/2}$. Thus, for a suitable constant $\overline{\nu} > 0$ we find that

$$D^2f(\varepsilon)(\sigma,\sigma) \ge \overline{\nu}\big(1+|\varepsilon|^2\big)^{\frac{s-2}{2}}\Big[|\sigma^D|^2 + (\mathrm{tr}\sigma)^2\Big] \ge \overline{\nu}\big(1+|\varepsilon|^2\big)^{\frac{s-2}{2}}|\sigma|^2,$$

and (see again (3))

$$D^2f(\varepsilon)(\sigma,\sigma) \le \lambda(\mathrm{tr}\sigma)^2 + A|\sigma^D|^2 \le \overline{\mu}|\sigma|^2$$

for some $\overline{\mu} > 0$. Putting together both cases by letting $q := \max\{2,s\}$, $p := \min\{2,s\}$ we deduce from (6) and the calculations following (6) that

$$\overline{\alpha}\big(1+|\varepsilon|^2\big)^{\frac{p-2}{2}}|\sigma|^2 \le D^2f(\varepsilon)(\sigma,\sigma) \le \overline{\beta}\big(1+|\varepsilon|^2\big)^{\frac{q-2}{2}}|\sigma|^2 \tag{7}$$

is true for all ε, $\sigma \in \mathbb{S}^2$ with constants $\overline{\alpha}$, $\overline{\beta} > 0$. But in [8] we showed that any local minimizer $u \in W^1_{1,loc}(\Omega;\mathbb{R}^2)$ of the energy $\int_\Omega f(\varepsilon(v))dx$ with f satisfying (7) is of class $C^{1,\alpha}$ in the interior of Ω provided that the exponents p and q are related through the condition

$$q < \min(2p, p+2). \tag{8}$$

Recalling the definitions of p and q it is immediate that the latter condition on p and q holds for $s \in (1,4)$. The reader should note that in [8] all comparison functions have to satisfy the incompressibility condition $\mathrm{div} v = 0$ but of course the situation now simplifies in comparison to [8] and all results remain valid if this condition is dropped.

REMARK 3. If Ω is a domain in $\mathbb{R}^3$ and if F satisfies (3), then local minimizers u of the functional $I[\cdot,\Omega]$ are of class $C^{1,\alpha}$ on an open subset Ω_0 of Ω such that $\Omega - \Omega_0$ is of Lebesgue measure zero, provided $s < 10/3$. This follows from the results in [9] if again the incompressibility condition is neglected.

REMARK 4. If we replace the term $\frac{\lambda}{2}(\mathrm{div} u)^2$ in the functional (4) by an expression like $g(\mathrm{div} u)$ with function g of growth rate $r \in (1,\infty)$, then a regularity result like Theorem 1 follows along the lines of [8] if we require (8) to hold with the choices $q := \max\{s,r\}$, $p := \min\{s,r\}$.

Let us now partially remove the restriction that $s < 4$.

THEOREM 2. Suppose that $u \in W^1_{1,loc}(\Omega;\mathbb{R}^2)$ is a local minimizer of the energy $I[\cdot,\Omega]$ defined in (4), where F satisfies (3) for some exponent $s \geq 4$. Then the first derivatives of u are α-continuous functions in the interior of Ω for any $\alpha \in (0,1)$, provided we assume that $\mathrm{div} u \in L^s_{loc}(\Omega)$.

PROOF. We define the class

$$V := \left\{ v \in L^2(\Omega;\mathbb{R}^2) : \mathrm{div} v \in L^2(\Omega), \quad \varepsilon^D(v) \in L^s(\Omega;\mathbb{S}^2) \right\}$$

being the subspace of $W^1_{1,loc}(\Omega;\mathbb{R}^2)$ on which the functional I is well defined. If $v \in V$ is compactly supported in Ω, then it follows from

$$I[u, \mathrm{spt} v] \leq I[u + tv, \mathrm{spt} v], \ t \in \mathbb{R},$$

that

$$\int_\Omega \Big[\lambda \mathrm{div} u \, \mathrm{div} v + DF(\varepsilon^D(u)) : \varepsilon^D(v) \Big] dx = 0. \tag{9}$$

For $h \in \mathbb{R} - \{0\}$ and $k \in \{1,2\}$ let

$$\Delta_h w(x) := \frac{1}{h}[w(x + he_k) - w(x)]$$

denote the difference quotient of the function w. For $\varphi \in C_0^\infty(\Omega)$ it is easy to check that $v := \Delta_{-h}(\varphi^2 \Delta_h(u - Px))$ is admissible in (9) for any constant matrix P. Following for example the calculations carried out in [9], it is not hard to show that after passing to the limit $h \to 0$ the next inequality can be deduced (from now on summation w.r.t. k)

$$\begin{aligned} &\int_\Omega D^2W(\varepsilon(u))(\partial_k \varepsilon(u), \partial_k \varepsilon(u)) \varphi^2 dx \\ &\quad \leq \ -\int_\Omega D^2W(\varepsilon(u))(\partial_k \varepsilon(u), \nabla \varphi^2 \odot \partial_k[u - Px]) dx, \end{aligned} \tag{10}$$

where

$$W(\sigma) := \frac{\lambda}{2}(\mathrm{tr}\sigma)^2 + F(\sigma^D),\ \sigma \in \mathbb{S}^2,$$

$\odot$ being the symmetric product of tensors. In particular, all the derivatives of u occurring in (10) exist in the weak sense. Let

$$H := \big(D^2W(\varepsilon(u))(\partial_k\varepsilon(u), \partial_k\varepsilon(u))\big)^{\frac{1}{2}}$$

and observe that $H^2 \in L^1_{loc}(\Omega)$ which can be justified in strength via difference quotient technique but can be made plausible as follows: since we assume $\mathrm{div}u \in L^s_{loc}(\Omega)$, we have

$$\varepsilon(u) = \varepsilon^D(u) + \frac{1}{2}(\mathrm{div}u)\mathbf{1} \in L^s_{loc}(\Omega;\mathbb{S}^2)\,,$$

hence $\nabla u \in L^s_{loc}(\Omega;\mathbb{R}^{2\times 2})$ by Korn's inequality. If we apply the Cauchy-Schwarz inequality to the bilinear form $D^2W(\varepsilon(u))$, we get

$$\begin{aligned} &\Big|D^2W(\varepsilon(u))(\partial_k\varepsilon(u), \nabla\varphi^2 \odot \partial_k u)\Big| \\ &\quad\le\ 2\big[D^2W(\varepsilon(u))(\partial_k\varepsilon(u), \partial_k\varepsilon(u))\varphi^2\big]^{\frac{1}{2}}\big[D^2W(\varepsilon(u))(\nabla\varphi \odot \partial_k u, \nabla\varphi \odot \partial_k u)\big]^{\frac{1}{2}}. \end{aligned}$$

Applying this estimate on the r.h.s. of (10) (choosing $P = 0$ for the moment) and using Young's inequality we obtain

$$\begin{aligned} \int_\Omega \varphi^2 H^2 dx &= \int_\Omega \varphi^2 D^2W(\varepsilon(u))(\partial_k\varepsilon(u), \partial_k\varepsilon(u))dx \\ &\le c\int_\Omega |D^2W(\varepsilon(u))||\nabla\varphi|^2|\nabla u|^2 dx \\ &\le c\int_\Omega |\nabla\varphi|^2(1+|\nabla u|^2)^{\frac{s}{2}}dx < \infty \end{aligned}$$

on account of our assumption. This "shows" the local integrability of the function H^2, and a similar argument gives that the integral on the r.h.s. of (10) is well defined. Note that $H^2 \in L^1_{loc}(\Omega)$ is equivalent to

$$\mathrm{div}\partial_k u \in L^2_{loc}(\Omega), \quad \big(1+|\varepsilon^D(u)|^2\big)^{\frac{s-2}{4}}|\partial_k\varepsilon^D(u)| \in L^2_{loc}(\Omega). \tag{11}$$

In fact, the definition of W implies for any ε, τ, $\sigma \in \mathbb{S}^2$

$$D^2W(\varepsilon)(\tau,\sigma) = \lambda(\mathrm{tr}\tau)(\mathrm{tr}\sigma) + D^2F(\varepsilon^D)(\tau^D,\sigma^D),$$

thus

$$H^2 = \lambda|\nabla\mathrm{div}u|^2 + D^2F(\varepsilon^D(u))(\partial_k\varepsilon^D(u), \partial_k\varepsilon^D(u))$$

and therefore (11) is immediate. Moreover, we have

$$\begin{aligned}
&\int_\Omega \big|D^2W(\varepsilon(u))(\partial_k\varepsilon(u),\nabla\varphi^2\odot\partial_k[u-Px])\big|dx\\
&\leq\ c\int_\Omega\Big[|\nabla \mathrm{div} u||\nabla\varphi||\nabla u-P|+\big(1+|\varepsilon^D(u)|^2\big)^{\frac{s-2}{2}}|\nabla\varepsilon^D(u)||\nabla\varphi||\nabla u-P\Big]dx\\
&\leq\ c\int_\Omega\big(1+|\varepsilon^D(u)|^2\big)^{\frac{s-2}{4}}|\nabla u-P||\nabla\varphi|\big[|\nabla \mathrm{div} u|+\big(1+|\varepsilon^D(u)|^2\big)^{\frac{s-2}{4}}|\nabla\varepsilon^D(u)|\big]dx\\
&\leq\ c\int_\Omega\big(1+|\varepsilon^D(u)|^2\big)^{\frac{s-2}{4}}|\nabla u-P|H|\nabla\varphi|dx.
\end{aligned}$$

Letting $h:=(1+|\varepsilon^D(u)|^2)^{(s-2)/4}$ and returning to (10) it is shown that

$$\int_{B_R}H^2dx\leq\frac{c}{R}\int_{B_{2R}}hH|\nabla u-P|dx, \tag{12}$$

provided we choose $\varphi\in C_0^\infty(B_{2R})$, $\varphi\equiv 1$ on B_R, $|\nabla\varphi|\leq c/R$, where B_{2R} is any open disc with compact closure in Ω. Now (12) exactly corresponds to (2.4) in [8], and with $\gamma:=4/3$ we end up with (2.5) of [8], i.e. we deduce from (12) (choosing $P=\fint_{B_{2R}}\nabla u dx$)

$$\fint_{B_R}H^2dx\leq c\left[\fint_{B_{2R}}(Hh)^\gamma dx\right]^{\frac{1}{\gamma}}\left[\fint_{B_{2R}}|\nabla^2u|^\gamma dx\right]^{\frac{1}{\gamma}}. \tag{13}$$

Noting that

$$|\nabla^2u|\leq c|\nabla\varepsilon(u)|\leq c(|\nabla \mathrm{div} u|+|\nabla\varepsilon^D(u)|)\leq cHh,$$

(13) turns into

$$\left[\fint_{B_R}H^2dx\right]^{\frac{\gamma}{2}}\leq c\fint_{B_R}(hH)^\gamma dx. \tag{14}$$

But (14) is the starting inequality for applying Lemma 1.2 of [8], provided we let $d:=2/\gamma$, $f:=H^\gamma$, $g:=h^\gamma$ in the lemma. Note that as in Section 2 of [8] all the assumptions of the lemma are satisfied since by (11)

$$\Phi:=\big(1+|\varepsilon^D(u)|^2\big)^{\frac{s}{4}}\in W^1_{2,loc}(\Omega)$$

and therefore

$$\int_{B_\rho}\exp(\beta h^2)dx\leq c(\rho,\beta)<\infty$$

follows as outlined after (2.7) in [8]. We conclude from the lemma that (with a suitable constant c_0)

$$\int_{B_\rho}H^2\log^{c_0\beta}(e+H)dx\leq c(\beta,\rho)<\infty \tag{15}$$

is valid for any $\beta>0$ and all $\rho<2R$. Finally, let $\sigma:=DW(\varepsilon(u))$. Then

$$|\nabla\sigma|\leq c\big[|\nabla \mathrm{div} u|+\big(1+|\varepsilon^D(u)|^2\big)^{\frac{s-2}{2}}|\nabla\varepsilon^D(u)|\big]\leq chH,$$

and we may proceed as in [8] (see the calculations after (2.11)) to combine the latter inequality with (15) in order to get

$$\int_{B_R} |\nabla\sigma|^2 \log^\alpha(e + |\nabla\sigma|)dx \leq c(R, \alpha) \tag{16}$$

for any $\alpha > 0$ (compare (2.11) of [8]). By results outlined in [10] (in particular Example 5.3) (16) shows the continuity of σ. Since $\varepsilon(u) = (DW)^{-1}(\sigma)$, the continuity of $\varepsilon(u)$ follows as well. If $v := \partial_k u$, $k = 1, 2$, then

$$0 = \int_\Omega D^2W(\varepsilon(u))(\varepsilon(v), \varepsilon(\varphi))dx \tag{17}$$

for all $\varphi \in C_0^\infty(\Omega; \mathbb{R}^2)$, and (17) can be seen as an elliptic system with continuous coefficients. Then we use Campanato-type estimates for systems with constant coefficients (see, e.g. [11]) combined with a freezing argument to deduce $v \in C^{0,\alpha}(\Omega; \mathbb{R}^2)$, $0 < \alpha < 1$. A detailed proof can be found for example in [12], Corollary 5.1, where of course the incompressibility condition can be dropped.

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