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Some Remarks On The Convergence Of Picard Iteration To A Fixed Point For A Continuous Mapping*

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Abstract

Some examples are given to illustarte that the characterization in [8] for the convergence of Picard iteration to a fixed point for a continuous mapping, is false.

Throughout this paper, let T be a mapping from a nonempty subset D of a metric space X into X and let $F(T)$ be a fixed point set of T. A mapping $T : D \to X$ is called *nonexpansive* if for each $x, y \in D$, $d(T(x), T(y)) \le d(x, y)$. T is said to be *quasi-nonexpansive* if for each $x \in D$ and for every $p \in F(T)$, $d(T(x), p) \le d(x, p)$. T is *conditionally quasi-nonexpansive* if it is quasi-nonexpansive whenever $F(T) \neq \emptyset$.

Since convergence theorems of iterations to a fixed point for nonexpansive mapping are discussed in [1-2], many results on the convergence of some iterations to fixed points for nonexpansive, quasi-nonexpansive and generalized types of quasi-nonexpansive mappings in metric and Banach spaces have appeared (for example, [3-8]).

Following Ghosh and Debnath [4], if D is a convex subset of a normed space X and $T : D \to D$, Ishikawa introduced the following iteration

$$x_0 \in D,\ x_n = T^n_{\lambda,\mu}(x_0),\ T_{\lambda,\mu} = (1-\lambda)I + \lambda T[(1-\mu)I + \mu T],$$

for each $n \in N$ (the set of all positive integers), where $\lambda \in (0, 1)$ and $\mu \in [0, 1)$. When $\mu = 0$, it yields that $T_{\lambda,\mu} = T_\lambda$ and the iteration becomes

$$x_0 \in D,\ x_n = T^n_\lambda(x_0),\ T_\lambda = (1-\lambda)I + \lambda T.$$

This iteration is called Mann iteration. If $T_\mu = (1-\mu)I + \mu T$, $T_{\lambda,\mu}$ may be written in the form

$$T_{\lambda,\mu} = (1-\lambda)I + \lambda T T_\mu.$$

Recall a mapping T is *asymptotically regular* at $x_0 \in D$ if

$$\lim_{n\to\infty} d(T^n(x_0), T^{n+1}(x_0)) = 0.$$

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T is said to be asymptotically regular on D if T is asymptotically regular at every point in D.

In 1973, Petryshyn and Williamson [3] proved the following theorem.

THEOREM 1. Let D be a closed subset of a Banach space X. Let T be a conditionally quasi-nonexpansive mapping of D into X. Suppose that the sequence $\{T^n(x_0)\}$ is contained in D, for some $x_0 \in D$. Then $\{T^n(x_0)\}$ converges to a fixed point of T in D if and only if

(i) T is asymptotically regular at x_0, and

(ii) there exists a compact set K such that $\lim_{n\to\infty} d(T^n(x_0), K) = 0$.

Recently, Ahmed [8] generalized and improved Theorem 1 and proved the following result.

THEOREM 2. (See [8, Theorem 2.1]) Let T be a continuous mapping from a subset D of a metric space X into X. Then the sequence $\{T^n(x_0)\}$, for some $x_0 \in D$, converges to a unique fixed point of T if and only if T is asymptotically regular at x_0 and $\lim_{n\to\infty} T^n(x_0)$ exists.

REMARK 3. (See [8, Remark 2.1]) Theorem 2 improves Theorem 1:

(1) the closedness of D is superfluous;

(2) T needs not to be conditionally quasi-nonexpansive; and

(3) $\{T^n(x_0)\}$ needs not to be contained in D.

However, we note Theorem 2 requires the existence of $\lim_{n\to\infty} T^n(x_0)$. In order to prove the existence of $\lim_{n\to\infty} T^n(x_0)$, the author gave the following proposition.

PROPOSITION 4. (See [8, Proposition 2.1]) Let T be a mapping from a subset D of a complete metric space X into X. If there exists a nonempty subset K of X such that $\lim_{n\to\infty} d(T^n(x_0), K) = 0$, then $\lim_{n\to\infty} T^n(x_0)$ exists.

Using the above proposition, the author obtained the following remark.

REMARK 5. (See [8, Remark 2.1]) If condition (ii) of Theorem 2 holds and X is complete, then we have from Proposition that $\lim_{n\to\infty} T^n(x_0)$ exists.

It is our purpose in this note to show that Proposition 4 and Remark 5 are false. It is clear from the proof of Proposition 4 that for all $l, k \geq n_0$,

$$d(T^l(x_0), T^k(x_0)) \leq d(T^l(x_0), y_0) + d(T^k(x_0), y_0), \quad \text{whenever} \quad y_0 \in K.$$

However, if we take the infimum over $y_0 \in K$, we do not get that

$$d(T^l(x_0), T^k(x_0)) \leq d(T^l(x_0), K) + d(T^k(x_0), K).$$

Indeed, we take a nonempty subset K of X such that $\{T^n(x_0), n \in N\} \subset K$ and let $l \neq k$, then

$$0 < d(T^l(x_0), T^k(x_0)) \leq d(T^l(x_0), K) + d(T^k(x_0), K) = 0,$$

and this is a contradiction.

We may take the following example which is contradictory with Proposition 4.

EXAMPLE 6. Let $X = K = R, D = [1, 3]$, be endowed with the Euclidean metric d. We define the mapping $T : D \to X$ by $Tx = 2x^2$ for each $x \in D$. For a given $x_0 = 2 \in D$, we have that $\lim_{n\to\infty} d(T^n(x_0), K) = 0$, but $\lim_{n\to\infty} T^n(x_0) = \infty$.

REMARK 7. As for a mapping T from a subset D of a complete metric space X into X with $F(T) \neq \emptyset$, even if we assume T satisfies $\lim_{n\to\infty} d(T^n(x_0), F(T)) = 0$, we can not pledge that $\lim_{n\to\infty} T^n(x_0)$ exists.

EXAMPLE 8. Let $X = K = [0, 1]$, be endowed with the Euclidean metric d. We define the mapping $T : K \to K$ by $T(0) = 0, T(1) = 1$ and

$$Tx = \begin{cases} 1 - x, \quad x \in [\frac{1}{2}, 1), \qquad 1 - \frac{x}{10}, \quad x \in (0, \frac{1}{2}), \\ \\ 1 - \frac{x}{10}, \quad x \in (0, \frac{1}{2}), \end{cases}$$

then we know that $F(T) = \{0, \frac{1}{2}, 1\}$. Take $x_0 = 0.1$, then we have $\{T^n(x_0), n \geq 0\} = \{0.1, 0.99, 0.01, 0.999, 0.001, 0.9999, \cdots, \}$ which implies that $\lim_{n\to\infty} d(T^n(x_0), F(T)) = 0$, but $\lim_{n\to\infty} T^n(x_0)$ does not exist.

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