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Saturation Theorem For The Combinations Of Modified Beta Operators In L_p-Spaces*

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Abstract

In the present paper we obtain a saturation theorem for the linear combination of modified Beta operators.

1 Introduction

For $f \in L_p[0,\infty), p \geq 1$, Gupta and Ahmad [1] introduced an interesting sequence of linear positive operators and termed it as modified Beta operators. The sequence which is the combination of Beta and Baskakov basis functions is defined as

$$B_n(f;x) = \frac{n-1}{n} \sum_{v=0}^{\infty} b_{n,v}(x) \int_0^{\infty} p_{n,v}(t) f(t) dt, x \in [0,\infty) \tag{1}$$

where

$$b_{n,v}(t) = \frac{1}{B(n,v+1)} \frac{x^v}{(1+x)^{n+v+1}}, p_{n,v}(t) = \left(\begin{array}{c} n+v-1 \\ v \end{array} \right) \frac{t^v}{(1+t)^{n+v}} \tag{2}$$

and $B(v+1,n)$ being the Beta function given by $\frac{v!(n-1)!}{(n+v)!}$.

It is easily verified that the operators B_n defined above are linear positive operators and B_n reproduce every constant function. One can see that as such the summation type operators are not L_p-approximation methods. However for obvious reasons the summation type operators are appropriately modified to become L_p-approximation methods. These operators B_n can also be used to approximate Lebesgue integrable functions. In approximation theory related to the linear positive operators lot of contribution is due to Gupta and collaborators, for some examples of similar type of operators we mention the recent papers due to Gupta and Ispir [4] and Srivastava and Gupta [5].

The order of approximation for the operators (1) is at best $O(n^{-1})$ howsoever smooth the function may be, with the aim of bettering the order of approximation, we

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have to slack the positive condition of these operators for which we may take appropriate linear combination of the operators (1). Now we consider the linear combination $B_n(f,k,x)$ of the operators $B_{d_jn}(f,x)$ as

$$B_n(f,k,x) = \sum_{j=0}^{k} C(j,k) B_{d_jn}(f,x) \tag{3}$$

where

$$C(j,k) = \prod_{i=0,\dots,k;i\neq j} \frac{d_j}{d_j - d_i}, C(0,0) = 1. \tag{4}$$

and $d_0, d_1, ..., d_k$ are $k+1$ arbitrary but fixed distinct positive integers. Throughout the present paper we consider $0 < a_1 < a_2 < a_3 < b_3 < b_2 < b_1 < \infty$ and $I_i = [a_i, b_i], i = 1,2,3$, The space $A.C.[a,b]$ denotes the class of absolutely continuous functions on $[a,b]$ for every a, b satisfying $0 < a < b < \infty$.

The present paper is the extension of the previous papers [2] and [3], in which direct and inverse theorems for the linear combinations are established in L_p-norm, only in ordinary approximation. In this paper we state and prove a saturation theorem for the linear combinations (3) of the operators (1) in ordinary approximation.

2 Auxiliary Results

In this section, we give some lemmas, which are essential to prove our main theorem.

LEMMA 1 [1]. Let m-th order central moment $T_{n,m}(x), m \in N$, be defined as

$$T_{n,m}(x) = B_n\big((t-x)^m, x\big) = \frac{n-1}{n} \sum_{v=0}^{\infty} b_{n,v}(x) \int_0^{\infty} p_{n,v}(t)(t-x)^m dt,$$

then $T_{n,0}(x) = 1, T_{n,1} = \frac{3x+1}{n-2}$, and for each $n > m+2$, there holds the recurrence relation

$$\begin{aligned}(n-m-2)T_{n,m+1}(x) &= x(1+x)[T^{(1)}_{n,m}(x) + 2mT_{n,m-1}(x)] \\ &\quad +[(1+2x)(m+1)+x]T_{n,m}(x)\end{aligned}$$

Consequently for each $x \in [0,\infty)$

$$T_{n,m}(x) = O\big(n^{-[(m+1)/2]}\big).$$

LEMMA 2. For $p \in N$ and n sufficiently large, there holds

$$B_n((t-x)^p, k, x) = n^{-(k+1)}\{Q(p,k,x) + o(1)\},$$

where $(Q(p,k,x)$ are certain polynomials in x of degree $p/2$.

PROOF. From Lemma 1, for sufficiently large n, we can write

$$B_n((t-x)^p, x) = \frac{P_0(x)}{n^{[p+1]/2}} + \frac{P_1(x)}{n^{[p+1]/2+1}} + \cdots + \frac{P_{[p/2]}(x)}{n^p},$$

where $P_i's$ are certain polynomials in x of degree at most i. Using the above in (4), we obtain the required result.

LEMMA 3. Let $f \in C^{2k+2}(I_1)$ have a compact support, then there holds

$$B_n(f,k,x) - f(x) = n^{-(k+1)} \left\{ \sum_{i=1}^{2k+2} P(i,k,x) f^{(i)}(x) + o(1) \right\}, n \to \infty, \tag{5}$$

uniformly in $x \in I_1$, where $P(i,k,x)$ is a polynomial in x of degree i and does not vanish for all $i = 1,2,3,...2k+2$ and $C^{2k+2}(I_1)$ denotes the class of $(2k+2)$-times continuously differentiable functions on the interval I_1.

PROOF. The assumed smoothness of f implies that

$$f(t) - f(x) = \sum_{i=1}^{2k+2} \frac{(t-x)^i}{i!} f^{(i)}(x) + \frac{(t-x)^{2k+2}}{(2k+2)!} (f^{(2k+2)}(\xi) - f^{(2k+2)}(x)), \tag{6}$$

for some ξ lying between t and x. For a given $\varepsilon > 0$ such that $|f^{(2k+2)}(x_1) - f^{(2k+2)}(y_1)| < \varepsilon$, whenever $|x_1 - y_1| < \delta, x_1, y_1 \in I_1$ Therefore for all t, x belonging to I_1, we have

$$|(t-x)^{2k+2}(f^{(2k+2)}(\xi) - f^{(2k+2)}(x))| < \varepsilon(t-x)^{2k+2} + \frac{2}{\delta^2} ||f^{(2k+2)}||_{C(I_1)} (t-x)^{2k+4}.$$

Hence by positivity of $B_n(f,x)$ and Lemma 1 it follows that

$$|B_n((t-x)^{2k+2}(f^{(2k+2)}(\xi) - f^{(2k+2)}(x)), x)| < M n^{-(k+1)} (\varepsilon + n^{-1}). \tag{7}$$

Also from Lemma 2, we have

$$B_n((t-x)^i/i!, k, x) = n^{-(k+1)} \{Q(i,k,x) + o(1)\}, n \to \infty, \tag{8}$$

for every $i = 1,2,3,...2k+2$ and the o-term holds uniformly in $x \in I_1$. Applying the operators (3) to (6), we obtain

$$B_n(f,k,x) - f(x) = \sum_{i=1}^{2k+2} \frac{f^{(i)}(x)}{i!} B_n((t-x)^i, k, x)$$

$$+ \frac{1}{(2k+2)!} B_n((t-x)^{2k+2}(f^{(2k+2)}(\xi) - f^{(2k+2)}(x)), k, x). \tag{9}$$

Now since $\varepsilon > 0$ is arbitrary, combining (7), (8) and (9), the required result follows.

LEMMA 4 ([2]). Let $0 < \alpha < 2k+2, f \in L_p[0,\infty), p \geq 1$ and

$$||B_n(f,k,.) - f||_{L_p(I_1)} = O(n^{-\alpha/2}), n \to \infty$$

then

$$\omega_{2k+2}(f, \tau, p, I_2) = O(\tau^\alpha), \tau \to 0.$$

3 Saturation Result

This section deals with the following saturation theorem.

THEOREM 1. Let $1 \leq p < \infty, f \in L_p[0,\infty)$ and $0 < a_1 < a_2 < a_3 < b_3 < b_2 < b_1 < \infty$, then in the following statements, the implications $(i) \Rightarrow (ii) \Rightarrow (iii)$ and $(iv) \Rightarrow (v) \Rightarrow (vi)$ hold

(i)$||B_n(f,k,.) - f||_{L_p(I_1)} = O(n^{-(k+1)}), n \to \infty$;

(ii) f coincides a.e. with a function F on I_2 having $(2k+2)$-th derivative such that (a) when $p > 1, F^{(2k+1)} \in A.C.(I_2)$ and $F^{(2k+2)} \in L_p(I_2)$, and (b) when $p = 1, F^{(2k)} \in A.C.(I_2)$ and $F^{(2k+1)} \in L_p(I_2)$;

(iii) $||B_n(f,k,.) - f||_{L_p(I_2)} = O(n^{-(k+1)}), n \to \infty$;

(iv) $||B_n(f,k,.) - f||_{L_p(I_1)} = o(n^{-(k+1)}), n \to \infty$;

(v) f coincides a.e. with a function F on I_2, where F is $(2k+2)$ times continuously differentiable on I_2 and satisfies $\sum_{i=1}^{2k+2} P(i,k,x)F^{(i)}(x) = 0$ where $P(i,k,x)$ are polynomial occurring in (5);

(vi)$||B_n(f,k,.) - f||_{L_p(I_2)} = o(n^{-(k+1)}), n \to \infty$.

PROOF. We choose pairs of points x_1, x_2 and y_1, y_2 such that $a_1 < x_1 < x_2 < a_2 < b_2 < y_2 < y_1 < b_1$, it follows from Lemma 4 that f coincides a.e. on (x_1, y_1) with a function F such that $F^{(2k)}$ is absolutely continuous and $F^{(2k+1)} \in L_p[x_1, y_1]$. We choose a function $q \in C_0^{2k+2}$ with supp $q \subset (a_1, b_1)$ such that $q = 1$ on the closed interval $[x_1, y_1]$. We denote $\overline{f} = Fq$. Now

$$||B_n(\overline{f},k,.) - \overline{f}||_{L_p[x_2,y_2]} \leq ||B_n(f,k,.) - f||_{L_p[x_2,y_2]} + ||B_n(\overline{f} - f,k,.)||_{L_p[x_2,y_2]}$$

Since $\overline{f} = f$ on $[x_2, y_2]$, the contribution of second term can be made arbitrarily small. Hence, using (i), it follows that

$$||B_n(\overline{f},k,.) - \overline{f}||_{L_p[x_2,y_2]} = O(n^{-(k+1)}), n \to \infty$$

Now if $p > 1$, it follows by Alaoglu's theorem that there exists a function $h(x) \in L_p[x_2, y_2]$ such that for some subsequence $\{n_j\}$ and for every $g \in C_0^{2k+2}$ with supp $g \subset (a_1, b_1)$, we have

$$\lim_{n_j \to \infty} n_j^{k+1} \langle B_{n_j}(\overline{f},k,.) - \overline{f}, g \rangle = \langle h, g \rangle. \tag{10}$$

When $p = 1$, the function $\phi_n(x)$ defined by $\phi_n(x) = \int_{x_1}^{x_2} n^{k+1}(B_n(\overline{f},k,x) - \overline{f}(x))dx$ are uniformly bounded and of bounded variation. Making use of Alaoglu's theorem, it follows that there exists a function $\phi_0(x)$ of bounded variation such that

$$\lim_{n_j \to \infty} n_j^{k+1} \langle B_{n_j}(\overline{f},k,.) - \overline{f}, g \rangle = -\langle \phi_0, g' \rangle. \tag{11}$$

Also it is obvious to show that $\langle B_n(\overline{f},k,.) - \overline{f}, g \rangle = \langle B_n(g,k,.) - g, \overline{f} \rangle$. Hence using Lemma 3, we have

$$\lim_{n_j \to \infty} n_j^{k+1} \langle B_{n_j}(\overline{f},k,.) - \overline{f}, g \rangle = \lim_{n_j \to \infty} n_j^{k+1} \langle \lim_{n_j \to \infty} n_j^{k+1} \langle B_{n_j}(g,k,.) - g, \overline{f} \rangle$$

$$= \langle \sum_{i=1}^{2k+2} P(i,k,x) D^i \overline{f}, g \rangle = \langle g, \sum_{i=1}^{2k+2} P^*(i,k,.) D^i \overline{f} \rangle, (D = \frac{\partial}{\partial x}) \tag{12}$$

where $P^*_{2k+2}(D) = \sum_{i=1}^{2k+2} P^*(i,k,.)D^i$ denotes the differentiable operator adjoint to $P_{2k+2}(D) = \sum_{i=1}^{2k+2} P(i,k,.)D^i$. Comparing (10) and (12), we get

$$h = P^*_{2k+2}(D)\overline{f} \tag{13}$$

as generalized functions. Now following Lemma 3, we have $P(2k+2,k,x) \neq 0$. Hence regarding (13) as generalized first order linear differentiable equation for $\overline{f}^{(2k+1)}$ with the non homogeneous terms linearly depending on $\overline{f}^{(i)}, 0 \leq i \leq 2k$ and h with polynomial coefficients, as $\overline{f}^{(i)} \in C[x_2,y_2], 0 \leq i \leq 2k$ and $h \in L_p[x_2,y_2]$, we conclude that $\overline{f}^{(2k+1)} \in A.C.[x_2,y_2]$ and therefore $\overline{f}^{(2k+2)} \in L_p[x_2,y_2]$. Since $\overline{f}$ coincides with F on $[x_1,y_1]$, it follows that $F^{(2k+1)} \in A.C.(I_2)$ and that $F^{(2k+2)} \in L_p(I_2)$. When $p = 1$, proceeding in the similar way as in the case $p > 1$ with (10) replaced by (11), we find that $F^{(2k)} \in A.C.(I_2)$ and $F^{(2k+1)} \in B.V.(I_2)$. This completes the proof of implication $(i) \Rightarrow (ii)$. The implication $(ii) \Rightarrow (ii)$ follow from Theorems 1 and 2 of [3], for the case $p > 1$ and $p = 1$ respectively. Assuming (iv), since $n^{k+1} \left\| B_n(\overline{f},k,.) - \overline{f} \right\|_{L_p(I_1)} \to 0$ as $n \to \infty$ proceeding as in the proof of $(i) \Rightarrow (ii)$ it follows that, $n^{k+1} \left\| B_n(\overline{f},k,.) - \overline{f} \right\|_{L_p[x_2,y_2]} \to 0, n \to \infty$ and hence we find that $h(x)$ and $\phi_0(x)$ are zero functions. Thus $P^*_{2k+2}(D)\overline{f}(x) = 0$. This implies that $\overline{f}$ is $(2k+2)$ times continuously differentiable function. Now applying Lemma 3 for the function $\overline{f}$, we get

$$\lim_{n_j \to \infty} n_j^{k+1} \langle B_{n_j}(\overline{f},k,.) - \overline{f}, g \rangle = \langle P_{2k+2}(D)\overline{f}, g \rangle \tag{14}$$

Comparing (12) and (14), we have $P_{2k+2}(D)\overline{f}(x) = 0$. Hence for I_2, F is $(2k+2)$ times continuously differentiable function and $P_{2k+2}(D)F(x) = 0$. Finally, $(v) \Rightarrow (vi)$ follows from Lemma 3. This completes the proof of theorem.

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