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On A Second-Order Differential Inclusion With Constraints*

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Abstract

We prove the existence of viable solutions to the Cauchy problem $x'' \in F(x,x') + f(t,x,x'), x(0) = x_0, x'(0) = y_0, x(t) \in K$, where $K \subset R^n$ is a closed set, F is a set-valued map contained in the Fréchet subdifferential of a ϕ- convex function of order two and f is a Carathéodory map.

1 Introduction

In this note we consider the second order differential inclusions of the form

$$x'' \in F(x,x') + f(t,x,x'), \quad x(0) = x_0, x'(0) = y_0, \tag{1}$$

where $F(.,.) : D \subset R^n \times R^n \to \mathcal{P}(R^n)$ is a given set-valued map, $f(.,.,.) : D_1 \subset R \times R^n \times R^n \to \mathcal{P}(R^n)$ is a given function and $x_0, y_0 \in R^n$.

Existence of solutions of problem (1.1) that satisfy a constraint of the form $x(t) \in K$, $\forall t$, well known as viable solutions, has been studied by many authors, mainly in the case when the multifunction is convex valued and $f \equiv 0$ ([2], [6], [8], [10] etc.).

Recently in [1], the situation when the multifunction is not convex valued is considered. More exactly, in [1] it is proved the existence of viable solutions of the problem (1) when $F(.,.)$ is an upper semicontinuous, compact valued multifunction contained in the subdifferential of a proper convex function. The result in [1] extends the result in [9] obtained for problems without constraints (i.e., $K = R^n$).

The aim of this note is to prove existence of viable solutions of the problem (1) in the case when the set-valued map $F(.,.)$ is upper semicontinuous compact valued and contained in the Fréchet subdifferential of a ϕ- convex function of order two.

On one hand, since the class of proper convex functions is strictly contained into the class of ϕ- convex functions of order two, our result generalizes the result in [1]. On the other hand, our result may be considered as an extension of our previous viability result for second-order nonconvex differential inclusions in [5] obtained for a problem without perturbations (i.e., $f \equiv 0$).

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The proof of our result follows the general ideas in [1] and [5]. We note that in the proof we pointed out only the differences that appeared with respect to the other approaches.

The paper is organized as follows: in Section 2 we recall some preliminary facts that we need in the sequel and in Section 3 we prove our main result.

2 Preliminaries

We denote by $\mathcal{P}(R^n)$ the set of all subsets of R^n and by R_+ the set of all positive real numbers. For $\epsilon > 0$ we put $B(x,\epsilon) = \{y \in R^n; ||y-x|| < \epsilon\}$ and $\overline{B}(x,\epsilon) = \{y \in R^n; ||y-x|| \leq \epsilon\}$. With B we denote the unit ball in R^n. By $cl(A)$ we denote the closure of the set $A \subset R^n$, by $co(A)$ we denote the convex hull of A and we put $||A|| = \sup\{||a||; a \in A\}$.

Let $\Omega \subset R^n$ be an open set and let $V : \Omega \to R \cup \{+\infty\}$ be a function with domain $D(V) = \{x \in R^n; V(x) < +\infty\}$.

DEFINITION 2.1. The multifunction $\partial_F V : \Omega \to \mathcal{P}(R^n)$, defined as:

$$\partial_F V(x) = \{\alpha \in R^n, \liminf_{y\to x} \frac{V(y) - V(x) - \langle \alpha, y-x\rangle}{||y-x||} \geq 0\} \text{ if } V(x) < +\infty$$

and $\partial_F V(x) = \emptyset$ if $V(x) = +\infty$ is called the *Fréchet subdifferential* of V.

According to [4] the values of $\partial_F V$ are closed and convex.

DEFINITION 2.2. Let $V : \Omega \to R \cup \{+\infty\}$ be a lower semicontinuous function. We say that V is a *ϕ-convex of order* 2 if there exists a continuous map $\phi_V : (D(V))^2 \times R^2 \to R_+$ such that for every $x, y \in D(\partial_F V)$ and every $\alpha \in \partial_F V(x)$ we have

$$V(y) \geq V(x) + \langle \alpha, x-y\rangle - \phi_V(x,y,V(x),V(y))(1+||\alpha||^2)||x-y||^2. \tag{2}$$

In [4], [7] there are several examples and properties of such maps. For example, according to [4], if $M \subset R^2$ is a closed and bounded domain, whose boundary is a C^2 regular Jordan curve, the indicator function of M

$$V(x) = I_M(x) = \begin{cases} 0, & \text{if} \quad x \in M \\ +\infty, & \text{otherwise} \end{cases}$$

is ϕ- convex of order 2.

In what follows we assume the next assumptions.

HYPOTHESIS 2.3. i) $\Omega = K \times 0$, where $K \subset R^n$ is a closed set and $O \subset R^n$ is a nonempty open set.

ii) $F(.,.) : \Omega \to \mathcal{P}(R^n)$ is upper semicontinuous (i.e., $\forall z \in \Omega, \forall \epsilon > 0$ there exists $\delta > 0$ such that $||z - z'|| < \delta$ implies $F(z') \subset F(z) + \epsilon B$) with compact values.

iii) $f(.,.,.) : R \times \Omega \to R^n$ is a Carathèodory function, i.e., $\forall (x,y) \in \Omega$, $t \to f(t,x,y)$ is measurable, for all $t \in R$ $f(t,.)$ is continuous and there exists $m(.) \in L^2(R, R_+)$ such that $||f(t,x,y)|| \leq m(t)$ $\forall (t,x,y) \in R \times \Omega$.

iv) For all $(t,x,v) \in R \times \Omega$, there exists $w \in F(x,v)$ such that

$$\liminf_{h\to 0} \frac{1}{h^2} d(x+hv+\frac{h^2}{2}w+\int_t^{t+h} f(s,x,v)ds, K) = 0.$$

v) There exists a proper lower semicontinuous ϕ- convex function of order two $V : R^n \to R \cup \{+\infty\}$ such that

$$F(x,y) \subset \partial_F V(y), \quad \forall (x,y) \in \Omega.$$

3 The Mmain Result

In order to prove our result we need the following lemmas.

LEMMA 3.1 ([1]). Assume that Hypotheses 2.3 i)-iv) are satisfied. Consider $(x_0,y_0) \in \Omega$, $r > 0$ such that $B(x_0,r) \subset O$, $M := \sup\{||F(t,x)||; (t,x) \in \Omega_0 := [K \times \overline{B}(y_0,r)] \cap \overline{B}((x_0,y_0),r)\}$, $T_1 > 0$ such that $\int_0^{T_1}(m(s)+M+1) < \frac{r}{3}$, $T_2 = \min\{\frac{r}{3(M+1)}, \frac{2r}{3(||y_0||+r)}\}$ and $T \in (0, \min\{T_1,T_2\})$. Then for every $\epsilon > 0$ there exists $\eta \in (0,\epsilon)$ and $p \geq 1$ such that for all $i = 1,...,p-1$ there exists $(h_i,(x_i,y_i),w_i) \in [\eta,\epsilon] \times \Omega_0 \times R^n$ with the following properties

$$x_i = x_{i-1} + h_{i-1}y_{i-1} + \frac{h_{i-1}^2}{2}w_{i-1} + \int_{h_{i-2}}^{h_{i-2}+h_{i-1}} f(s,x_{i-1},y_{i-1})ds \in K,$$

$$y_i = y_{i-1} + h_{i-1}w_{i-1}, \quad w_i \in F(x_i,y_i) + \frac{\epsilon}{T}B,$$

and

$$(x_i,y_i) \in \Omega_0, \quad \sum_{i=0}^{p-1} h_i < T \leq \sum_{i=0}^{p} h_i.$$

Moreover, for $\epsilon > 0$ sufficiently small we have $\sum_{i=0}^{p-1} \frac{h_i^2}{2} \leq \sum_{i=0}^{p-1} h_i < T$.

For $k \geq 1$ and $q = 1,...,p$ denote by h_q^k the real number associated to $\epsilon = \frac{1}{k}$ and $(t,x,y) = (h_{q-1}^k, x_q, y_q)$ given by Lemma 3.1. Define $t_k^0 = 0, t_k^p = T$, $t_p^q = h_0^k + ... + h_{q-1}^k$ and consider the sequence $x_k(.) : [t_k^{q-1}, t_k^q] \to R^n$, $k \geq 1$ defined by

$$x_k(0) = x_0,$$

$$x_k(t) = x_{q-1} + (t-t_k^{q-1})y_{q-1} + \frac{1}{2}(t-t_k^{q-1})^2 w_{q-1} + \int_{t_k^{q-1}}^t (t-s)f(s,x_{q-1},y_{q-1})ds.$$

LEMMA 3.2 ([1]). Assume that Hypotheses 2.3 i)-iv) are satisfied and consider $x_k(.)$ the sequence constructed above. Then there exists a subsequence, still denoted by $x_k(.)$ and an absolutely continuous function $x(.) : [0,T] \to R^n$ such that

i) $x_k(.)$ converges uniformly to $x(.)$,

ii) $x'_k(.)$ converges uniformly to $x'(.)$,

iii) $x''_k(.)$ converges weakly in $L^2([0,T],R^n)$ to $x''(.)$,

iv) The sequence $\left(\sum_{q=1}^{p}\int_{t_k^{q-1}}^{t_k^q} < x_k''(s), f(s, x_k(t_k^{q-1}), x_k'(t_k^{q-1})) > ds\right)_k$ converges to $\int_0^T < x''(s), f(s, x(s), x'(s)) > ds$,

v) For every $t \in (0, T)$ there exists $q \in \{1, ..., p\}$ such that

$$\lim_{k\to\infty} d((x_k(t), x_k'(t), x_k''(t) - f(t, x_k(t_k^{q-1}), x_k'(t_k^{q-1}))), graph(F)) = 0,$$

vi) $x(.)$ is a solution of the convexified problem

$$x'' \in coF(x, x') + f(t, x, x'), \quad x(0) = x_0, x'(0) = v_0.$$

We are now able to prove our result.

THEOREM 3.3. Assume that Hypothesis 2.3 is satisfied. Then, for every $(x_0, y_0) \in \Omega$ there exist $T > 0$ and $x(.) : [0, T] \to R^n$ a solution of problem (1) that satisfies $x(t) \in K$ $\forall t \in [0, T]$.

PROOF. Let $(x_0, y_0) \in \Omega$ and consider $r > 0$, $T > 0$ as in Lemma 3.1 and $x_k(.) : [0, T] \to R^n$, $x(.) : [0, T] \to R^n$ as in Lemma 3.2. Let ϕ_V the continuous function appearing in Definition 2.2. Since $V(.)$ is continuous on $D(V)$ (e.g. [7]), by possibly decreasing r one can assume that for all $y \in B(y_0, r) \cap D(V)$

$$|V(y) - V(y_0)| \leq 1.$$

Set

$$S := \sup\{\phi_v(y_1, y_2, z_1, z_2); y_i \in \overline{B}(y_0, r), z_i \in [V(y_0) - 1, V(y_0) + 1], i = 1, 2\}.$$

From the statement vi) in Lemma 3.2 and Hypothesis 2.3 v) it follows that for almost all $t \in [0, T]$,

$$x''(t) - f(t, x(t), x'(t)) \in \partial_F V(x'(t)). \tag{3}$$

Since the mapping $x(.)$ is absolutely continuous, from (3) and Theorem 2.2 in [4] we deduce that there exists $T_3 > 0$ such that the mapping $t \to V(x'(t))$ is absolutely continuous on $[0, \min\{T, T_3\}]$ and

$$(V(x'(t)))' = \langle x''(t), x''(t) - f(t, x(t), x'(t))\rangle \quad a.e.\ [0, \min\{T, T_3\}]. \tag{4}$$

Without loss of generality we may assume that $T = \min\{T, T_3\}$. From (4) we have

$$V(x'(T)) - V(y_0) = \int_0^T ||x''(s)||^2 ds - \int_0^T \langle x''(s), f(s, x(s), x'(s)\rangle\, ds. \tag{5}$$

On the other hand, for $q = 1, ..., p$ and $t \in [t_k^{q-1}, t_k^q)$

$$x_k''(t) - f(t, x_k(t_k^{q-1}), x_k'(t_k^{q-1})) \in F(x_k(t_k^{q-1}), x_k'(t_k^{q-1})) + \frac{1}{kT}B$$

and therefore

$$x_k''(t) - f(t, x_k(t_k^{q-1}), x_k'(t_k^{q-1})) \in \partial_F V(x_k'(t_k^{q-1})) + \frac{1}{kT}B.$$

We deduce the existence of $b_k^q \in B$ such that

$$x_k''(t) - f(t, x_k(t_k^{q-1}), x_k'(t_k^{q-1})) - \frac{b_k^q}{kT} \in \partial_F V(x_k'(t_k^{q-1})).$$

Taking into account Definition 2.2 we obtain

$$\begin{aligned} V(x_k'(t_k^q)) - V(x_k'(t_k^{q-1})) \quad \geq \quad & \left\langle x_k''(t) - f(t, x_k(t_k^{q-1}), x_k'(t_k^{q-1})) - \frac{b_k^q}{kT}, \int_{t_k^{q-1}}^{t_k^q} x_k''(s) ds \right\rangle \\ & -\phi_V \left(x_k'(t_k^q), x_k'(t_k^{q-1}), V(x_k'(t_k^q)), V(x_k'(t_k^{q-1})) \right) \\ & \times \left(1 + \left\| x_k''(t) - f(t, x_k(t_k^{q-1}), x_k'(t_k^{q-1})) - \frac{b_k^q}{kT} \right\|^2 \right) \\ & \times \left\| x_k'(t_k^q) - x_k'(t_k^{q-1}) \right\|^2 . \end{aligned}$$

Using the fact that $x_k''(.)$ is constant on $[t_k^{q-1}, t_k^q]$ one may write

$$\begin{aligned} V(x_k'(t_k^q)) - V(x_k'(t_k^{q-1})) \quad \geq \quad & \int_{t_k^{q-1}}^{t_k^q} \langle x_k''(s), x_k''(s) \rangle \, ds - \int_{t_k^{q-1}}^{t_k^q} \left\langle x_k''(s), \frac{b_k^q}{kT} \right\rangle ds \\ & - \int_{t_k^{q-1}}^{t_k^q} \left\langle x_k''(s), f(s, x_k(t_k^{q-1}), x_k'(t_k^{q-1})) \right\rangle ds \\ & -\phi_V \left(x_k'(t_k^q), x_k'(t_k^{q-1}), V(x_k'(t_k^q)), V(x_k'(t_k^{q-1})) \right) \\ & \times \left(1 + \left\| x_k''(t) - f(t, x_k(t_k^{q-1}), x_k'(t_k^{q-1})) - \frac{b_k^q}{kT} \right\|^2 \right) \\ & \times \left\| x_k'(t_k^q) - x_k'(t_k^{q-1}) \right\|^2 . \end{aligned}$$

By adding on q the last inequalities we get

$$\begin{aligned} V(x_k'(T)) - V(y_0) \quad \geq \quad & \int_0^T ||x_k''(s)||^2 ds + a(k) + b(k) \\ & - \sum_{q=1}^{p} \int_{t_k^{q-1}}^{t_k^q} \left\langle x_k''(s), f(s, x_k(t_k^{q-1}), x_k'(t_k^{q-1})) \right\rangle ds, \end{aligned} \tag{6}$$

where

$$a(k) = -\sum_{q=1}^{p} \frac{1}{kT} \int_{t_k^{q-1}}^{t_k^q} \langle x_k''(s), b_k^q \rangle \, ds,$$

$$\begin{aligned} b(k) \quad = \quad & -\sum_{q=1}^{p} \phi_V \left(x_k'(t_k^q), x_k'(t_k^{q-1}), V(x_k'(t_k^q))), V(x_k'(t_k^{q-1}) \right) \\ & \times \left(1 + \left\| x_k''(t) - f(t, x_k(t_k^{q-1}), x_k'(t_k^{q-1})) - \frac{b_k^q}{kT} \right\|^2 \right) \left\| x_k'(t_k^q) - x_k'(t_k^{q-1}) \right\|^2 . \end{aligned}$$

On the other hand, one has

$$\begin{aligned} |a(k)| &\leq \frac{1}{kT}\sum_{q=1}^{p}||b_k^q||.\int_{t_k^{q-1}}^{t_k^q}||x_k''(s)||ds \\ &\leq \frac{1}{kT}\int_0^T||x_k''(s)||ds \leq \frac{1}{kT}\int_0^T[M+\frac{1}{T}+m(s)]ds \end{aligned}$$

and

$$\begin{aligned} |b(k)| &\leq \sum_{q=1}^{p} S(1+M^2)||\int_{t_k^{q-1}}^{t_k^q} x_k''(s)ds||^2 \\ &\leq S(1+M^2)\sum_{q=1}^{p}\frac{1}{k}\int_{t_k^{p-1}}^{t_k^p}||x_k''(s)||^2ds \leq S(1+M^2)\frac{1}{k}\int_0^T||x_k''(s)||^2ds \\ &\leq \frac{1}{k}S(1+M^2)\int_0^T[M+\frac{1}{T}+m(s)]^2ds. \end{aligned}$$

We infer that

$$\lim_{k\to\infty} a(k) = \lim_{k\to\infty} b(k) = 0.$$

Hence using also statement iv) in Lemma 3.2 and the continuity of the function $V(.)$ by passing to the limit as $k \to \infty$ in (6) we obtain

$$V(x'(T)) - V(y_0) \geq \limsup_{k\to\infty}\int_0^T ||x_k''(s)||^2ds - \int_0^T \left\langle x''(s), f(s, x_(s), x'(s))\right\rangle ds. \quad (7)$$

Using (4) we infer that

$$\limsup_{k\to\infty}\int_0^T ||x_k'(t)||^2dt \leq \int_0^T ||x''(t)||^2dt$$

and, since $x_k''(.)$ converges weakly in $L^2([0,T], R^n)$ to $x''(.)$, by the lower semicontinuity of the norm in $L^2([0,T], R^n)$ (e.g. Prop. III.30 in [3]) we obtain that

$$\lim_{k\to\infty}\int_0^T ||x_k''(t)||^2dt = \int_0^T ||x''(t)||^2dt$$

i.e., $x_k''(.)$ converges strongly in $L^2([0,T], R^n)$. Hence, there exists a subsequence (still denoted) $x_k''(.)$ that converges pointwise to $x''(.)$. From the statement v) in Lemma 3.2 it follows that

$$d((x(t), x'(t), x''(t) - f(t, x(t), x'(t))), graph(F)) = 0 \quad a.e.\ [0,T].$$

and since by Hypothesis 2.4 $graph(F)$ is closed we obtain

$$x''(t) \in F(x(t), x'(t)) + f(t, x(t), x'(t)) \quad a.e.\ [0,T].$$

In order to prove the viability constraint satisfied by $x(.)$ fix $t \in [0,T]$. There exists a sequence $(t_k^q)_k$ such that $t = \lim_{k\to\infty} t_k^q$. But $\lim_{k\to\infty} ||x(t) - x_k(t_k^q)|| = 0$ and $x_k(t_k^q) \in K$. So the fact that K is closed gives $x(t) \in K$ and the proof is complete.

REMARK 3.4. If $V(.) : R^n \to R$ is a proper lower semicontinuous convex function then (e.g. [7]) $\partial_F V(x) = \partial V(x)$, where $\partial V(.)$ is the subdifferential in the sense of convex analysis of $V(.)$, and Theorem 3.3 yields the result in [1]. At the same time if in Theorem 3.3 $f \equiv 0$ then Theorem 3.3 yields the result in [5].

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