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Variational Oscillation Criteria for Nonlinear Nonhomogeneous Differential Equations*

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Abstract

Using variational principle and Riccati technique, new oscillation criteria for forced second order nonlinear differential equation are established to handle cases that cannot be dealt with by results in recent papers by Li and Cheng [1] and Cakmak and Tiryaki [3].

1 Introduction

In [1], a question is raised whether Leighton's variational principles for the oscillation of linear second order nonhomogeneous differential equations in [2] can be extended to nonhomogeneous half-linear differential equations of the form

$$\left(p(t)|y'(t)|^{\alpha-1}y'(t)\right)' + q(t)|y(t)|^{\alpha-1}y(t) = e(t),\ t \ge t_0, \tag{1}$$

where α is a positive constant, $p, q, e \in C([t_0, \infty), \mathbb{R})$ with $p(t) > 0$. A result is derived to answer this question [1, Theorem 2].

THEOREM 1.1. Suppose that for any $T \ge t_0$, there exist $T \le s_1 < t_1 \le s_2 < t_2$ such that $e(t) \le 0$ for $t \in [s_1, t_1]$ and $e(t) \ge 0$ for $t \in [s_2, t_2]$. If there exist $H \in D(s_i, t_i) = \{u \in C^1[s_i, t_i] : u(t) \not\equiv 0, u(s_i) = u(t_i) = 0\}$ and a positive, nondecreasing function $\rho \in C^1([t_0, \infty), \mathbb{R})$ such that

$$\int_{s_i}^{t_i} H^2(t)\rho(t)q(t)dt > \left(\frac{1}{\alpha+1}\right)^{\alpha+1} \int_{s_i}^{t_i} \frac{p(t)\rho(t)}{|H(t)|^{\alpha-1}} \left(2|H'(t)| + |H(t)|\frac{\rho'(t)}{\rho(t)}\right)^{\alpha+1} dt \tag{2}$$

for $i = 1, 2$. Then Equation (1) is oscillatory.

Later in [3], Cakmak and Tiryaki consider a more general equation

$$\left(p(t)\Psi(y(t))|y'(t)|^{\alpha-1}y'(t)\right)' + q(t)f(y(t)) = e(t),\ t \ge t_0, \tag{3}$$

where α is a positive constant, $p, q, e \in C([t_0, \infty), \mathbb{R})$ with $p(t) > 0$, $\Psi \in C(\mathbb{R}, (0, \infty))$, $f \in C(\mathbb{R}, \mathbb{R})$ satisfying $uf(u) > 0$ for $u \ne 0$, and obtain a result as follows.

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THEOREM 1.2. Suppose that

$$\frac{f'(u)}{[\Psi(u)|f(u)|^{\alpha-1}]^{1/\alpha}} \geq \gamma > 0 \text{ for } u \neq 0.$$

Suppose further that for any $T \geq t_0$, there exist $T \leq s_1 < t_1 \leq s_2 < t_2$ such that $e(t) \leq 0$ for $t \in [s_1, t_1]$ and $e(t) \geq 0$ for $t \in [s_2, t_2]$. If there exist $H \in D(s_i, t_i) = \{u \in C^1[s_i, t_i] : u(t) \not\equiv 0, u(s_i) = u(t_i) = 0\}$ and a positive function $\rho \in C^1([t_0, \infty), \mathbb{R})$ such that

$$\int_{s_i}^{t_i} H^2(t)\rho(t)q(t)dt > \left(\frac{1}{\alpha+1}\right)^{\alpha+1}\left(\frac{\alpha}{\gamma}\right)^{\alpha} \times \int_{s_i}^{t_i} H^2(t)\rho(t)p(t)\left|\frac{2H'(t)}{H(t)} + \frac{\rho'(t)}{\rho(t)}\right|^{\alpha+1} dt \tag{4}$$

for $i = 1, 2$. Then Equation (3) is oscillatory.

It is not difficult to see that Theorem 1.2 is an extension of Theorem 1.1, and that no restriction is made on the monotonicity of the function ρ.

Unfortunately, neither Theorem 1.1 nor Theorem 1.2 can be applied to the case when $\alpha > 1$, since for $\rho(t) \equiv 1$, the term $|H(t)|^{\alpha-1}$ will appear as a denominator in (2) and (4) so that the requirement $H(s_i) = H(t_i) = 0$ will cause trouble. This certainly calls for investigation of oscillation criteria that can handle such cases.

In this paper, we are concerned with the nonhomogeneous equation (3). By a solution of Equation (3), we mean a function $y \in C^1[T_y, \infty)$, $T_y \geq t_0$, where $T_y \geq t_0$ depends on the particular solution, which has the property $p(t)\Psi(y(t))|y'(t)|^{\alpha-1}y'(t) \in C^1[T_y, \infty)$ and satisfies Equation (3). A nontrivial solution of Equation (3) is called oscillatory if it has arbitrarily large zeros, otherwise, it is said to be non-oscillatory. Equation (3) is said to be oscillatory if all its solutions are oscillatory.

The purpose of this paper is to obtain new oscillation criteria for Equation (3) based on variational principles. Roughly, if the existence of a 'positive' solution of a functional relation implies the 'positivity' of an associated functional over a set of 'admissible' functions, then we say that a variational oscillation principle is valid. For instance, in Theorem 1.1, $H \in D(s_i, t_i)$ is admissible, and the functional is

$$\int_{s_i}^{t_i} \left\{ \left(\frac{1}{\alpha+1}\right)^{\alpha+1} \frac{p(t)\rho(t)}{|H(t)|^{\alpha-1}} \left(2|H'(t)| + |H(t)|\frac{\rho'(t)}{\rho(t)}\right)^{\alpha+1} - H^2(t)\rho(t)q(t) \right\} dt.$$

Our emphasis will be directed towards oscillation criteria that are closely related to the $(\alpha+1)$-degree energy functional for half-linear equations (see [4,5,6,7] for more details on these functionals), which are improvements of Theorem 1.1 and Theorem 1.2 for the case where $\beta = \alpha$, and are generalizations for the case where $\beta > \alpha$. Examples will also be given to illustrate the effectiveness of our main results.

Before going into the main results, let us state three sets of conditions commonly used in the literature which we rely on:

$$\text{(S1)}\quad 0 < \Psi(u) \le M, \text{and } f'(u) \ge K\,|f(u)|^{\frac{\beta-1}{\beta}} > 0 \text{ for } u \ne 0; \tag{5}$$

$$\text{(S2)}\quad \frac{f'(u)}{\left[\Psi(u)|f(u)|^{\beta-1}\right]^{1/\beta}} \ge \gamma > 0 \text{ for } u \ne 0; \tag{6}$$

$$\text{(S3)}\quad 0 < \Psi(u) \le M, \text{and } \frac{f(u)}{|u|^{\beta}\,\mathrm{sgn}u} \ge \delta > 0 \text{ for } u \ne 0. \tag{7}$$

Here, $M, K > 0$, $0 < \alpha \le \beta$ and $\gamma, \delta > 0$ are constants. It is clear that assumption (S1) implies (S2), but not conversely. For example, the function $f(u) = u^3$, $\Psi(u) = u^2$ and $\beta = 1$ do not satisfy (S1), but (S2) holds. In (S1) and (S2), we need f to be differentiable. Clearly, this condition is not required in (S3). These differences force us to study equation (3) under the assumptions (S1), (S2) and (S3) in separate manners.

2 The Case Where $\beta = \alpha$

First we recall a well known inequality, which is a transformation of Young's inequality.

LEMMA 2.1. (see [8]) Suppose X and Y are nonnegative. Then

$$\lambda X Y^{\lambda-1} - X^{\lambda} \le (\lambda - 1)Y^{\lambda}, \quad \lambda > 1, \tag{8}$$

where equality holds if and only if $X = Y$.

For $a, b \in \mathbb{R}$ such that $a < b$, let

$$D(a,b) = \left\{u \in C^1[a,b] : u^{\alpha+1}(t) > 0 \text{ for } t \in (a,b), \text{ and } u(a) = u(b) = 0\right\}.$$

Let $\rho \in C^1([t_0,\infty),\mathbb{R})$ be a positive function. For given $f \in C([t_0,\infty),\mathbb{R})$, as in [3], we define an integral operator A_a^b in terms of $H \in D(a,b)$ and ρ as

$$A_a^b(f;t) = \int_a^b H^{\alpha+1}(t)\rho(t)f(t)dt, \quad a \le t \le b. \tag{9}$$

Recall from [3] that A_a^b has the following properties:

$$A_a^b(\alpha_1 f_1 + \alpha_2 f_2; t) = \alpha_1 A_a^b(f_1;t) + \alpha_2 A_a^b(f_2;t); \tag{10}$$

$$A_a^b(f;t) \ge 0 \qquad \text{whenever} \quad f \ge 0; \tag{11}$$

$$\begin{aligned} A_a^b(g';t) &= -(\alpha+1)A_a^b\left(\left[\frac{H'}{H} + \frac{\rho'}{(\alpha+1)\rho}\right]g;t\right) \\ &\ge -(\alpha+1)A_a^b\left(\left|\frac{H'}{H} + \frac{\rho'}{(\alpha+1)\rho}\right||g|;t\right), \end{aligned} \tag{12}$$

where $f_1, f_2, f \in C([t_0,\infty),\mathbb{R})$, $g \in C^1([t_0,\infty),\mathbb{R})$, and α_1, α_2 are real numbers.

THEOREM 2.2. Assume (S2) holds. Suppose further that for any $T \ge t_0$, there exist $T \le s_1 < t_1 \le s_2 < t_2$ such that $e(t) \le 0$ for $t \in [s_1,t_1]$ and $e(t) \ge 0$ for $t \in [s_2,t_2]$.

If there exist $H \in D(s_i, t_i)$ and a positive function $\rho \in C^1([t_0, \infty), \mathbb{R})$ such that

$$A_{s_i}^{t_i}(q;t) > \left(\frac{\alpha}{\gamma}\right)^{\alpha} A_{s_i}^{t_i}\left(p\left|\frac{H'}{H} + \frac{\rho'}{(\alpha+1)\rho}\right|^{\alpha+1};t\right) \tag{13}$$

for $i = 1, 2$. Then Equation (3) is oscillatory.

PROOF. Suppose to the contrary that there is a non-oscillatory solution y of (3). We may then assume that $y(t) \neq 0$ on $[T_0, \infty)$ for some $T_0 \geq t_0$. Set

$$w(t) = \frac{p(t)\Psi(y(t))|y'(t)|^{\alpha-1}y'(t)}{f(y(t))}, \quad t \geq T_0. \tag{14}$$

Then differentiating (14) and making use of Equation (3), it follows that for all $t \geq T_0$, we have

$$w'(t) = -q(t) + \frac{e(t)}{f(y(t))} - \frac{|w(t)|^{(\alpha+1)/\alpha} f'(y(t))}{[p(t)\Psi(y(t))|f(y(t))|^{\alpha-1}]^{1/\alpha}}. \tag{15}$$

By our assumptions, we can choose $s_i, t_i \geq T_0$ for $i = 1, 2$ so that $e(t) \leq 0$ on the interval $I_1 = [s_1, t_1]$, with $s_1 < t_1$ and $y(t) \geq 0$, or $e(t) \geq 0$ on the interval $I_2 = [s_2, t_2]$, with $s_2 < t_2$ and $y(t) \leq 0$. On the intervals I_1 and I_2, in view of (6) and (15), $w(t)$ satisfies the inequality

$$q(t) \leq -w'(t) - \gamma\frac{|w(t)|^{(\alpha+1)/\alpha}}{p^{1/\alpha}(t)}. \tag{16}$$

Applying the operator $A_{s_i}^{t_i}$ for $i = 1, 2$ to (15), using the fact that $H(s_i) = H(t_i) = 0$ and (12), we obtain

$$\begin{aligned} A_{s_i}^{t_i}(q;t) &\leq A_{s_i}^{t_i}\left(-w' - \gamma\frac{|w|^{(\alpha+1)/\alpha}}{p^{1/\alpha}};t\right) \\ &\leq A_{s_i}^{t_i}\left((\alpha+1)\left|\frac{H'}{H} + \frac{\rho'}{(\alpha+1)\rho}\right||w| - \gamma\frac{|w|^{(\alpha+1)/\alpha}}{p^{1/\alpha}};t\right). \end{aligned} \tag{17}$$

Let

$$X = \frac{\gamma^{\alpha/(\alpha+1)}}{p^{1/(\alpha+1)}}|w|, \qquad \lambda = 1 + \frac{1}{\alpha},$$

and

$$Y = \frac{\alpha^{\alpha} p^{\alpha/(\alpha+1)}}{\gamma^{\alpha^2/(\alpha+1)}}\left|\frac{H'}{H} + \frac{\rho'}{(\alpha+1)\rho}\right|^{\alpha}.$$

By Lemma 2.1, we obtain

$$(\alpha+1)\left|\frac{H'}{H} + \frac{\rho'}{(\alpha+1)\rho}\right||w| - \gamma\frac{|w|^{(\alpha+1)/\alpha}}{p^{1/\alpha}} \leq \left(\frac{\alpha}{\gamma}\right)^{\alpha} p\left|\frac{H'}{H} + \frac{\rho'}{(\alpha+1)\rho}\right|^{\alpha+1}. \tag{18}$$

Then in view of (17), (18) and the properties (10) as well as (11), we see that

$$A_{s_i}^{t_i}(q;t) \leq \left(\frac{\alpha}{\gamma}\right)^{\alpha} A_{s_i}^{t_i}\left(p\left|\frac{H'}{H} + \frac{\rho'}{(\alpha+1)\rho}\right|^{\alpha+1};t\right)$$

for $i = 1, 2$, which is contrary to (13). This completes our proof.

We remark that the assumption that $e(t) \le 0$ for $t \in [s_1, t_1]$ and $e(t) \ge 0$ for $t \in [s_2, t_2]$ can be replaced by $e(t) \ge 0$ for $t \in [s_1, t_1]$ and $e(t) \le 0$ for $t \in [s_2, t_2]$. Similar remarks hold for the other results that follow.

COROLLARY 2.3. If $\rho(t) \equiv 1$ in Theorem 2, and (13) is replaced by

$$Q_i(H) := \int_{s_i}^{t_i} \left[q(t) H^{\alpha+1}(t) - \left(\frac{\alpha}{\gamma}\right)^{\alpha} p(t) |H'(t)|^{\alpha+1} \right] dt > 0 \tag{19}$$

for $i = 1, 2$, then Equation (3) is oscillatory. If $\rho(t) \equiv 1$ in Theorem 2, and (13) is replaced by

$$\bar{Q}_i(H) := \int_{s_i}^{t_i} \left[q(t) H^{\alpha+1}(t) - p(t) |H'(t)|^{\alpha+1} \right] dt > 0 \tag{20}$$

for $i = 1, 2$, then Equation (1) is oscillatory.

We remark that Corollary 2 is closely related to the $(\alpha + 1)$-degree functional for half-linear differential equations. Furthermore, in Theorem 2.2, there is no additional restriction on the positive constant α.

THEOREM 2.4. Assume (S3) holds. Suppose further that for any $T \ge t_0$, there exist $T \le s_1 < t_1 \le s_2 < t_2$ such that $e(t) \le 0$ for $t \in [s_1, t_1]$ and $e(t) \ge 0$ for $t \in [s_2, t_2]$. If there exist $H \in D(s_i, t_i)$ and a positive function $\rho \in C^1([t_0, \infty), \mathbb{R})$ such that

$$A_{s_i}^{t_i}(\delta q; t) > M \cdot A_{s_i}^{t_i}\left(p \left| \frac{H'}{H} + \frac{\rho'}{(\alpha+1)\rho} \right|^{\alpha+1}; t \right) \tag{21}$$

for $i = 1, 2$. Then Equation (3) is oscillatory.

PROOF. Suppose to the contrary that there is a non-oscillatory solution y. We assume that $y(t) \neq 0$ on $[T_0, \infty)$ for some $T_0 \ge t_0$. Set

$$w(t) = \frac{p(t)\Psi(y(t))|y'(t)|^{\alpha-1}y'(t)}{|y(t)|^{\alpha-1}y(t)}, \quad t \ge T_0. \tag{22}$$

Then differentiating (14) and making use of Equation (3) and (S3), we see that for all $t \ge T_0$, we have

$$\begin{aligned} w'(t) =& - q(t) \frac{f(y(t))}{|y(t)|^{\alpha-1}y(t)} + \frac{e(t)}{|y(t)|^{\alpha-1}y(t)} - \alpha \frac{|w(t)|^{(\alpha+1)/\alpha}}{[p(t)\Psi(y(t))]^{1/\alpha}} \\ \le& - \delta q(t) + \frac{e(t)}{|y(t)|^{\alpha-1}y(t)} - \frac{\alpha}{M^{1/\alpha}} \frac{|w(t)|^{(\alpha+1)/\alpha}}{p^{1/\alpha}(t)}. \end{aligned} \tag{23}$$

By our assumptions, we can choose $s_i, t_i \ge T_0$ for $i = 1, 2$ so that $e(t) \le 0$ on the interval $I_1 = [s_1, t_1]$, with $s_1 < t_1$ and $y(t) \ge 0$, or $e(t) \ge 0$ on the interval $I_2 = [s_2, t_2]$, with $s_2 < t_2$ and $y(t) \le 0$. On the intervals I_1 and I_2, (23) implies that $w(t)$ satisfies the inequality

$$\delta q(t) \le -w'(t) - \frac{\alpha}{M^{1/\alpha}} \frac{|w(t)|^{(\alpha+1)/\alpha}}{p^{1/\alpha}(t)}. \tag{24}$$

The remaining proof is similar to that of Theorem 2.2. The proof is complete.

EXAMPLE 2.5. Consider the following forced half-linear differential equation

$$\left(t^{\lambda}|y'(t)|^{\alpha-1}y'(t)\right)' + Kt^{\lambda}|y(t)|^{\alpha-1}y(t) = -\sin t, \tag{25}$$

for $t \geq 1$, where $K, \lambda > 0$ are constants and $\alpha = 5/3 > 1$, so neither Theorem 1.1 nor Theorem 1.2 can be applied to this case. However, we may show that Equation (25) is oscillatory for $K > \frac{3}{4}(1 + \frac{3}{8}\lambda)^{8/3}\pi$. Indeed, since the zeros of the forcing term $-\sin t$ are $n\pi$, the constant γ in (6) is α, i.e., $\gamma = \alpha$. Let $H(t) = \sin t$ and $\rho(t) = t^{-\lambda}$. For any $T \geq 1$, choose n sufficiently large so that $n\pi = 2k\pi \geq T$ and $s_1 = 2k\pi$ and $t_1 = (2k+1)\pi$. It is easy to verify that

$$\begin{aligned} A_{s_1}^{t_1}(q;t) =& K\int_{2k\pi}^{(2k+1)\pi} \sin^{8/3} t dt \\ >& K\int_{2k\pi}^{(2k+1)\pi} \sin^3 t dt = K\int_0^{\pi} \sin^3 t dt = \frac{4}{3}K, \end{aligned}$$

$$\begin{aligned} A_{s_1}^{t_1}\left(p\left|\frac{H'}{H} + \frac{\rho'}{(\alpha+1)\rho}\right|^{\alpha+1};t\right) &= \int_{2k\pi}^{(2k+1)\pi} \left|\cos t - \frac{3\lambda \sin t}{8t}\right|^{8/3} dt \\ &< \int_{2k\pi}^{(2k+1)\pi} (1 + \frac{3}{8}\lambda)^{8/3} dt = (1 + \frac{3}{8}\lambda)^{8/3}\pi. \end{aligned}$$

So (13) is true for $i = 1$. Similarly, for $s_2 = (2k+1)\pi$ and $t_2 = (2k+2)\pi$, we can show (13) is true for $i = 2$. So Equation (25) is oscillatory for $K > \frac{3}{4}(1 + \frac{3}{8}\lambda)^{8/3}\pi$ by Theorem 2.2.

EXAMPLE 2.6. Consider the following forced half-linear differential equation

$$\left[(2 + \cos t)|y'(t)|^{\alpha-1}y'(t)\right]' + K|y(t)|^{\alpha-1}y(t) = \sin t, \quad t \geq 1, \tag{26}$$

where $\alpha = 1/3$, with $H(t) = \sin t$ and $\rho(t) \equiv 1$, Li and Cheng in [1] obtain oscillation for Equation (26) when $K \geq 5(\frac{3}{2})^{4/3} \doteq 8.585$. Using Corollary 2.3, we obtain the oscillation of Equation (26) when $K > 2$. In fact, for any $T \geq 1$, choose n sufficiently large so that $n\pi = 2k\pi \geq T$ and $s_1 = 2k\pi$ and $t_1 = (2k+1)\pi$. It is easy to verify that

$$\begin{aligned} \bar{Q}_1(H) &= \int_{s_1}^{t_1} [q(t)H^{\alpha+1}(t) - p(t)|H'(t)|^{\alpha+1}]dt \\ &= \int_{2k\pi}^{(2k+1)\pi} \left[K \sin^{4/3} t - (2 + \cos t)|\cos t|^{4/3}\right] dt \\ &= \int_{2k\pi}^{(2k+1)\pi} (K - 2)\sin^{4/3} t dt > 0 \end{aligned}$$

for $K > 2$. Similarly, for $s_2 = (2k+1)\pi$ and $t_2 = (2k+2)\pi$, we can show that the integral inequality $\bar{Q}_2(H) > 0$. So Equation (26) is oscillatory by Corollary 2.3 for $K > 2$.

EXAMPLE 2.7. Consider the following forced nonlinear differential equation

$$\left[e^t|y(t)|^{3-\alpha}|y'(t)|^{\alpha-1}y'(t)\right]' + 5Ke^t y^3(t) = \cos t, \quad t \geq 1, \tag{27}$$

where α is a quotient of positive odd integers such that $0 < \alpha < 3$, the zeros of the forcing term $\cos t$ are $n\pi + \pi/2$, $n \in \mathbb{Z}$ with $H(t) = \cos t$ and $\rho(t) = e^{-t}$, Cakmak and Tiryaki in [3] obtain oscillation for Equation (27) when $K \geq K_0 = (3/(\alpha+1))^{\alpha+1}$. By Theorem 2.2, for any $T \geq 1$, choose n sufficiently large so that $(2k+1)\pi/2 \geq T$ and $s_1 = (2k+1)\pi/2$ and $t_1 = (2k+3)\pi/2$. It is easy to verify that

$$\begin{aligned} A_{s_1}^{t_1}(q;t) &= 5K \int_{(2k+1)\pi/2}^{(2k+3)\pi/2} \cos^{\alpha+1} t dt \\ &\geq \begin{cases} 5K \int_{(2k+1)\pi/2}^{(2k+3)\pi/2} \cos^2 t dt = \frac{5}{2}K\pi & 0 < \alpha \leq 1 \\ 5K \int_{(2k+1)\pi/2}^{(2k+3)\pi/2} \cos^4 t dt = \frac{15}{4}K & 1 < \alpha < 3 \end{cases} ; \end{aligned}$$

and

$$\begin{aligned} \left(\frac{\alpha}{\gamma}\right)^\alpha A_{s_1}^{t_1}\left(p\left|\frac{H'}{H} + \frac{\rho'}{(\alpha+1)\rho}\right|^{\alpha+1}; t\right) &= \left(\frac{\alpha}{3}\right)^\alpha \int_{(2k+1)\pi/2}^{(2k+3)\pi/2} \left|\sin t + \frac{1}{\alpha+1}\right|^{\alpha+1} dt \\ &< \left(\frac{\alpha}{3}\right)^\alpha \int_{(2k+1)\pi/2}^{(2k+3)\pi/2} \left(1 + \frac{1}{\alpha+1}\right)^{\alpha+1} dt \\ &= \left(\frac{\alpha}{3}\right)^\alpha \left(\frac{\alpha+2}{\alpha+1}\right)^{\alpha+1} \pi. \end{aligned}$$

When $0 < \alpha \leq 1$ and $K > K_1 = \frac{2}{5}\left(\frac{\alpha}{3}\right)^\alpha \left(\frac{\alpha+2}{\alpha+1}\right)^{\alpha+1}$, or when $1 < \alpha < 3$ and $K > K_2 = \frac{4}{15}\left(\frac{\alpha}{3}\right)^\alpha \left(\frac{\alpha+2}{\alpha+1}\right)^{\alpha+1} \pi$, we have (13) is true for $i = 1$. Similarly, for $s_2 = (2k+3)\pi/2$ and $t_2 = (2k+5)\pi/2$, we can show that (13) is also true for $i = 2$. So Equation (27) is oscillatory by Theorem 2.2 for $K > K_1$ in case of $0 < \alpha \leq 1$, and $K > K_2$ in case of $1 < \alpha < 3$. Moreover, we note that $K_0 > K_1$, so our results are better than that of Cakmak and Tiryaki [3].

3 The Case Where $\beta > \alpha$

We now handle the case where $\beta > \alpha$.

THEOREM 3.1. Assume (S3) holds. Suppose further that for any $T \geq t_0$, there exist $T \leq s_1 < t_1 \leq s_2 < t_2$ such that $e(t) \leq 0$ for $t \in [s_1, t_1]$ and $e(t) \geq 0$ for $t \in [s_2, t_2]$. If there exist $H \in D(s_i, t_i)$ and a positive function $\rho \in C^1([t_0, \infty), \mathbb{R})$ such that

$$A_{s_i}^{t_i}(Q_e; t) > M \cdot A_{s_i}^{t_i}\left(p\left|\frac{H'}{H} + \frac{\rho'}{(\alpha+1)\rho}\right|^{\alpha+1}; t\right) \tag{28}$$

for $i = 1, 2$, where

$$Q_e(t) = \alpha^{-\alpha/\beta}\beta(\beta-\alpha)^{(\alpha-\beta)/\beta}[\delta q(t)]^{\alpha/\beta}|e(t)|^{(\beta-\alpha)/\beta}. \tag{29}$$

Then Equation (3) is oscillatory.

PROOF. Suppose to the contrary that there is a nontrivial non-oscillatory solution. We assume that $y(t) > 0$ on $[T_0, \infty)$ for some $T_0 \geq t_0$. Set

$$w(t) = \frac{p(t)\Psi(y(t))|y'(t)|^{\alpha-1}y'(t)}{|y(t)|^{\alpha-1}y(t)}, \quad t \geq T_0. \tag{30}$$

Then differentiating (30) and making use of Equation (3), it follows that for all $t \geq T_0$, we have

$$\begin{aligned} w'(t) &= -\left[q(t)\frac{f(y(t))}{|y(t)|^{\alpha-1}y(t)} - \frac{e(t)}{|y(t)|^{\alpha-1}y(t)}\right] - \alpha\frac{|w(t)|^{(\alpha+1)/\alpha}}{[p(t)\Psi(y(t))]^{1/\alpha}} \\ &= -\left[q(t)\frac{f(y(t))}{|y(t)|^{\beta-1}y(t)}|y(t)|^{\beta-\alpha} - \frac{e(t)}{|y(t)|^{\alpha-1}y(t)}\right] - \alpha\frac{|w(t)|^{(\alpha+1)/\alpha}}{[p(t)\Psi(y(t))]^{1/\alpha}} \\ &\leq -\left[\delta q(t)|y(t)|^{\beta-\alpha} - \frac{e(t)}{|y(t)|^{\alpha-1}y(t)}\right] - \frac{\alpha}{M^{1/\alpha}}\frac{|w(t)|^{(\alpha+1)/\alpha}}{p^{1/\alpha}(t)}. \end{aligned} \tag{31}$$

By our assumption, we can choose $t_1 > s_1 \geq T_0$ so that $e(t) \leq 0$ on the interval $I_1 = [s_1, t_1]$. For given $t \in I_1$, set $F(x) = \delta q(t)x^{\beta-\alpha} - \frac{e(t)}{x^\alpha}$, we have $F'(x^*) = 0$, $F''(x^*) > 0$, where $x^* = \left[\frac{-\alpha e(t)}{(\beta-\alpha)\delta q(t)}\right]^{1/\beta}$. So $F(x)$ attains its minimum at x^* and

$$F(x) \geq F(x^*) = Q_e(t). \tag{32}$$

So (31) and (32) imply that $w(t)$ satisfies

$$Q_e(t) \leq -w'(t) - \frac{\alpha}{M^{1/\alpha}}\frac{|w(t)|^{(\alpha+1)/\alpha}}{p^{1/\alpha}(t)}. \tag{33}$$

The remaining argument is the same as the proof of Theorem 2.2, so we obtain a desired contradiction with (28) when $y(t) > 0$ eventually. On the other hand, if $y(t)$ is a negative solution for $t \geq T_0 > t_0$, we define the Riccati transformation (30) to yield (31). In this case, we choose $t_2 > s_2 \geq T_0$ so that $e(t) \geq 0$ on the interval $I_2 = [s_2, t_2]$. For given $t \in I_2$, set $F(x) = \delta q(t)x^{\beta-\alpha} - \frac{e(t)}{x^\alpha}$, we have $F(x) \geq F(x^*) = Q_e(t)$. The remaining proof is similar to that of Theorem 2.2. The proof is complete.

COROLLARY 3.2. If $\rho(t) \equiv 1$ in Theorem 3.1, and the hypothesis (28) is replaced by

$$\tilde{Q}_i(H) := \int_{s_i}^{t_i} \left[Q_e(t)H^{\alpha+1}(t) - p(t)|H'(t)|^{\alpha+1}\right] dt > 0 \tag{34}$$

for $i = 1, 2$. Then Equation (3) is oscillatory.

We remark that Corollary 3.2 is closely related to the $(\alpha+1)$-degree functional. Furthermore, in Theorem 3.1, there is no restriction on the positive constant α, plus, Theorem 2.4 can be treated as its limiting case when $\beta \to \alpha + 0$ with the convention that $0^0 = 1$.

EXAMPLE 3.5. Consider the following forced quasi-linear differential equation

$$(\gamma t^{\lambda/3} y'(t))' + t^{\lambda}|y(t)|^2 y(t) = -\sin^3 t, \quad t \geq 1, \tag{35}$$

where $\gamma, \lambda > 0$ are constants. We see that $\Psi(u) \equiv 1$, which implies $M = 1$, and $\alpha = 1$, $\beta = 3$ in Theorem 3.1. Since $\alpha < \beta$, Theorems 1.1, 1.2 and 2.2 cannot be applied. However, we can obtain oscillation for Equation (35) with $H(t) = \sin t$ and $\rho(t) = t^{-\lambda/3}$. For any $T \geq 1$, choose n sufficiently large so that $n\pi = 2k\pi \geq T$ and $s_1 = 2k\pi$ and $t_1 = (2k+1)\pi$. It is easy to verify that $Q_e(t) = \frac{3}{2}\sqrt[3]{2} t^{\lambda/3} \sin^2 t$,

$$A_{s_1}^{t_1}(Q_e(t); t) = \frac{3}{2}\sqrt[3]{2}\int_{2k\pi}^{(2k+1)\pi} \sin^4 t dt = \frac{3}{2}\sqrt[3]{2}\int_0^{\pi} \sin^4 t dt = \frac{9}{8}\sqrt[3]{2},$$

and

$$\begin{aligned} A_{s_1}^{t_1}\left(p\left|\frac{H'}{H} + \frac{\rho'}{(\alpha+1)\rho}\right|^{\alpha+1}; t\right) =& \gamma\int_{2k\pi}^{(2k+1)\pi} \left|\cos t - \frac{\lambda \sin t}{6t}\right|^2 dt \\ <& \gamma\int_{2k\pi}^{(2k+1)\pi} (1+\frac{\lambda}{6})^2 dt = \gamma(1+\frac{\lambda}{6})^2\pi. \end{aligned}$$

So we have (28) is true for $i = 1$ provided $0 < \gamma < \frac{9\sqrt[3]{2}}{8(1+\lambda/6)^2}$. Similarly, for $s_2 = (2k+1)\pi$ and $t_2 = (2k+2)\pi$, we can show that (28) is true for $i = 2$. So Equation (35) is oscillatory for $0 < \gamma < \frac{9\sqrt[3]{2}}{8(1+\lambda/6)^2}$ by Theorem 3.1.

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