Applied Mathematics E-Notes, 6(2006), 244-250 © ISSN 1607-2510
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Generalizations Of A Theorem Of I. Schur*

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Received 11 August 2006

Abstract

In this article, the monotonicities of two functions $\left(1+\frac{1}{x}\right)^{x+\alpha}$ and $\left(1+\frac{\alpha}{x}\right)^{x+\beta}$ and their corresponding sequences $\left(1+\frac{1}{n}\right)^{n+\alpha}$ and $\left(1+\frac{\alpha}{n}\right)^{n+\beta}$ are presented, and equivalent relations between the monotonicities of either these two functions or these two sequences are verified. As by-products, some new inequalities for the natural logarithm are obtained.

1 Introduction

In standard textbooks of calculus or advanced mathematical analysis, in order to show that the limits $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$ and $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$ exist, it is sufficient to verify the sequence $\left(1+\frac{1}{n}\right)^n$ and the function $\left(1+\frac{1}{x}\right)^x$ are bounded and increasing respectively. These can be done traditionally by Newton's binomial expansion, by the arithmetic-geometric-harmonic means inequalities ([5] and [6, pp. 223–226]), by Bernoulli's inequality [7], by Young's inequality [4], by mathematical induction [6], and so on. See also [15]. As is well-known, the number e is contained in the interval $\left(1+\frac{1}{n}\right)^n < e < \left(1+\frac{1}{n}\right)^{n+1}$, where the sequence $\left(1+\frac{1}{n}\right)^{n+1}$ is decreasing ([8, pp. 357–371] and [10, pp. 266–268]).

A theorem of I. Schur [11, pages 30 and 186] states that the sequence $\left(1+\frac{1}{n}\right)^{n+\alpha}$ is decreasing if and only if $\alpha \ge \frac{1}{2}$. In [3] it was verified that the sequence $\left(1+\frac{1}{n}\right)^{n+\alpha}$ is increasing if and only if $\alpha \le \frac{2\ln 3-3\ln 2}{2\ln 2-\ln 3}$. In [6, 7, 8, 10] and [12, Vol. I, Part I, Chapter 4, p. 38] it was proved that the sequence $\left(1+\frac{1}{n}\right)^n\left(1+\frac{\beta}{n}\right)$ is decreasing if and only if $\beta \ge \frac{1}{2}$; the sequence $\left(1+\frac{\gamma}{n}\right)^{n+1}$ decreases for $0<\gamma\le 2$ and increases for $\gamma>2$ and $n \ge \frac{\gamma}{\gamma-2}+1$; the sequence $\left(1+\frac{\theta}{n}\right)^n$ increases for $\theta>0$. It is easy to see that the function $\left(1+\frac{\alpha}{x}\right)^x$ is increasing with respect to $x>\max\{0,-\alpha\}$ for $\alpha\ne 0$. In the proof in [10, 3.6.3 on p. 267], it was presented that for fixed $x>0$ the function

*Mathematics Subject Classifications: 26A48, 26D07, 33B15, 40A05, 40A30.

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$\left(1+\frac{x}{p}\right)^p$ is increasing with $p \in (0,\infty)$ and the function $\left(1+\frac{x}{p}\right)^{p+x/2}$ is decreasing with $p \in (0,\infty)$. Some related generalizations of I. Schur's theorem have been studied in [1, 2, 13, 14, 16]. Some recent developments on this topic can also be found in [9, pp. 86–88 and pp. 291–292].

It is natural to pose the following problem: What about the monotonicities of the functions $\left(1+\frac{1}{x}\right)^{x+\alpha}$ and $\left(1+\frac{\alpha}{x}\right)^{x+\beta}$ and the corresponding sequences $\left(1+\frac{1}{n}\right)^{n+\alpha}$ and $\left(1+\frac{\alpha}{n}\right)^{n+\beta}$ for all $\alpha \neq 0$ and $\beta \in \mathbb{R}$, respectively?

In this article, using analytic method, we will prove the following theorems which answer the problem posed above.

THEOREM 1. For $x>0$, the function $f_\alpha(x)=\left(1+\frac{1}{x}\right)^{x+\alpha}$ increases if and only if $\alpha \le 0$ and decreases if and only if $\alpha \ge \frac{1}{2}$. For $x<-1$, the function $f_\alpha(x)$ decreases if and only if $\alpha \ge 1$ and increases if and only if $\alpha \le \frac{1}{2}$. The necessary and sufficient conditions such that the sequence $a_n=\left(1+\frac{1}{n}\right)^{n+\alpha}$ decreases or increases are $\alpha \ge \frac{1}{2}$ or $\alpha \le \frac{2\ln 3-3\ln 2}{2\ln 2-\ln 3}$ respectively.

THEOREM 2. Let $b_n=\left(1+\frac{\alpha}{n}\right)^{n+\beta}$ for $\alpha>-1$ and $\alpha \neq 0$ and $F_{\alpha,\beta}(x)=\left(1+\frac{\alpha}{x}\right)^{x+\beta}$ for $\alpha \neq 0$ and either $x>\max\{0,-\alpha\}$ or $x<\min\{0,-\alpha\}$.

1. For $x>\max\{0,-\alpha\}$, the function $F_{\alpha,\beta}(x)$ increases if and only if either $\alpha>0\ge\beta$, or $\alpha<0\le\beta$, or $\alpha\le 2\beta<0$; the function $F_{\alpha,\beta}(x)$ decreases if and only if either $2\beta\ge\alpha>0$ or $\beta\le\alpha<0$.

2. For $x<\min\{0,-\alpha\}$, the function $F_{\alpha,\beta}(x)$ increases if and only if either $\alpha>0\ge\beta$, or $0<2\beta\le\alpha$, or $\alpha<0\le\beta$; the function $F_{\alpha,\beta}(x)$ decreases if and only if either $0>\alpha\ge 2\beta$ or $0<\alpha<\beta$.

3. The sequence b_n increases if and only if either $\alpha>0$ and $\beta \le \frac{\ln(1+\alpha)-2\ln(1+\alpha/2)}{\ln(1+\alpha/2)-\ln(1+\alpha)}$ or $-1<\alpha<0$ and $\alpha\le 2\beta$. The sequence b_n decreases if and only if either $-1<\alpha<\beta\le\frac{\ln(1+\alpha)-2\ln(1+\alpha/2)}{\ln(1+\alpha/2)-\ln(1+\alpha)}<0$ or $0<\alpha\le 2\beta$.

THEOREM 3. Theorem 1 and Theorem 2 are equivalent to each other.

REMARK 1. As by-products of the proofs of Theorem 1 and Theorem 2, some inequalities for the natural logarithm $\ln t$ are obtained as follows.

$$\ln t \ge \frac{t+3}{t+1}\ln\frac{1+t}{2}, \qquad t>0; \tag{1}$$

$$\ln t \ge \frac{1+t}{t}\ln\frac{1+t}{2}, \qquad t\in(0,1); \tag{2}$$

$$\frac{2t}{2+t} \le \ln(1+t) \le \frac{t(2+t)}{2(1+t)}, \qquad t>0; \tag{3}$$

$$\ln(1+t) > \frac{t(1-t)}{1+t}, \qquad t\in(-1,1). \tag{4}$$

When $-1<t<0$, inequality (3) is reversed.

These inequalities from (1) to (4) play important roles in theory of gamma functions. The left hand side of inequality (3), which is the same as the left hand side of [10, 3.6.19] essentially, improves a related problem of the 11th William Lowell Putnam Mathematical Competition. The right hand side of inequality (3) is weaker than the right hand side of [10, 3.6.18 and 3.6.19]. For further information, please refer to [8, pp. 367–368] or [10].

REMARK 2. Note that some errors of mathematical expression in [1, 2, 13, 16] are corrected by Theorem 2.

2 Proofs

2.1 Proof of Theorem 1

Direct calculation gives

$$[\ln f_\alpha(x)]' = \ln\left(1+\frac{1}{x}\right) - \frac{x+\alpha}{x(x+1)} \text{and} [\ln f_\alpha(x)]'' = \frac{(2\alpha-1)x+\alpha}{x^2(x+1)^2}.$$

For $x>0$, it is easy to see that $[\ln f_\alpha(x)]''>0$ and $[\ln f_\alpha(x)]'$ increases if and only if $\alpha \geq \frac{1}{2}$. Since $\lim_{x\to\infty}[\ln f_\alpha(x)]' = 0$ for any $\alpha \in \mathbb{R}$, thus $[\ln f_\alpha(x)]' < 0$ for $\alpha \geq \frac{1}{2}$ (This implies the right hand side of inequality (3), which means $f'_\alpha(x) < 0$ and $f_\alpha(x)$ decreases. This implies also that the sequence a_n is decreasing for $\alpha \geq \frac{1}{2}$.

For $x>0$, it is clear that $[\ln f_\alpha(x)]'' < 0$ and $[\ln f_\alpha(x)]'$ decreases if and only if $\alpha \leq 0$. Then $[\ln f_\alpha(x)]' > 0$, $f'_\alpha(x) > 0$ and $f_\alpha(x)$ increases for $\alpha \leq 0$. This implies that the sequence $\left(1+\frac{1}{n}\right)^{n+\alpha}$ is increasing for $\alpha \leq 0$.

For $x>0$, when $0<\alpha<\frac{1}{2}$, the function $[\ln f_\alpha(x)]''$ has a unique zero point $x_0 = \frac{\alpha}{1-2\alpha} > 0$ which is a supremum point of $[\ln f_\alpha(x)]'$, this supremum equals $[\ln f_\alpha(x_0)]' = \ln\left(\frac{1}{\alpha}-1\right) + 2(2\alpha-1) > 0$ (This implies the left hand side of inequality (3). Since $\lim_{x\to 0^+}[\ln f_\alpha(x)]' = -\infty$ for $\alpha > 0$ and $\lim_{x\to\infty}[\ln f_\alpha(x)]' = 0$ for any $\alpha \in \mathbb{R}$, it follows that the functions $[\ln f_\alpha(x)]'$ and $f'_\alpha(x)$ have only one zero point $x_1 > 0$, which is a unique infimum point of $f_\alpha(x)$ on $(0,\infty)$. Consequently, the sufficient and necessary condition of the sequence a_n being increasing is $f_\alpha(1) \leq f_\alpha(2)$ which is equivalent to $\alpha \leq \frac{2\ln 3 - 3\ln 2}{2\ln 2 - \ln 3}$.

For $x<-1$, the function $[\ln f_\alpha(x)]''>0$ and $[\ln f_\alpha(x)]'$ is increasing if and only if $\alpha \leq \frac{1}{2}$. From $\lim_{x\to-\infty}[\ln f_\alpha(x)]' = 0$ it is deduced that $[\ln f_\alpha(x)]' > 0$ and $f'_\alpha(x) > 0$ in $(-\infty,-1)$. Consequently, the function $f_\alpha(x)$ is increasing in $(-\infty,-1)$ if $\alpha \leq \frac{1}{2}$.

For $x<-1$, the function $[\ln f_\alpha(x)]''<0$ and $[\ln f_\alpha(x)]'$ is decreasing if and only if $\alpha \geq 1$. From $\lim_{x\to-\infty}[\ln f_\alpha(x)]' = 0$ it follows that $[\ln f_\alpha(x)]' < 0$ and $f'_\alpha(x) < 0$ in $(-\infty,-1)$. Accordingly, the function $f_\alpha(x)$ decreases in $(-\infty,-1)$ if $\alpha \geq 1$.

For $x<-1$ and $\frac{1}{2}<\alpha<1$, the function $[\ln f_\alpha(x)]''$ has a unique zero point $x_0 = \frac{\alpha}{1-2\alpha} < -1$ which is a minimum point of $[\ln f_\alpha(x)]'$. Since $\lim_{x\to(-1)^-}[\ln f_\alpha(x)]' = \infty$ and $\lim_{x\to-\infty}[\ln f_\alpha(x)]' = 0$, then the functions $[\ln f_\alpha(x)]'$ and $f'_\alpha(x)$ have only one zero point $x_1 > 0$, which is a unique infimum point of $f_\alpha(x)$ on $(0,\infty)$. This completes the proof of Theorem 1.

2.2 Proofs of Inequalities

Let $G(t)=(2+t)\ln(1+t)-(4+t)\ln\left(1+\frac{t}{2}\right)$ for $t>-1$. Then

$$G'(t)=\ln(1+t)-\ln\left(1+\frac{t}{2}\right)+\frac{1}{1+t}-\frac{2}{2+t}\text{and}G''(t)=\frac{t(3+2t)}{(1+t)^2(2+t)^2}.$$

It is clear that $G''(t)$ has a unique zero $t=0$ for $t>-1$ and $G'(t)$ takes the infimum $G'(0)=0$, thus $G'(t)>0$ (This implies the inequality (4)) and $G(t)$ is increasing. From $G(0)=0$, it follows that $G(t)>0$ for $t>0$ and $G(t)<0$ for $t<0$. From this, we conclude the inequality (1) and

$$\frac{\ln(1+t)-2\ln\left(1+\frac{t}{2}\right)}{\ln\left(1+\frac{t}{2}\right)-\ln(1+t)}<\frac{t}{2} \tag{5}$$

for $t>-1$ and $t\neq 0$.

The inequality (2) follows from standard arguments.

2.3 Proof of Monotonicity of $\left(1+\frac{\alpha}{x}\right)^{x+\beta}$

Direct calculation yields

$$\ln F_{\alpha,\beta}(x)=(x+\beta)\ln\left(1+\frac{\alpha}{x}\right), \tag{6}$$

$$[\ln F_{\alpha,\beta}(x)]'=\ln\left(1+\frac{\alpha}{x}\right)-\frac{\alpha(x+\beta)}{x(x+\alpha)}, \tag{7}$$

$$[\ln F_{\alpha,\beta}(x)]''=\frac{\alpha[(2\beta-\alpha)x+\alpha\beta]}{x^2(x+\alpha)^2}. \tag{8}$$

2.3.1 The Case of $x>\max\{0,-\alpha\}$

It is not difficult to see that the function $F_{\alpha,0}(x)$ is increasing for $x>\max\{0,-\alpha\}$. Direct calculation also yields

$$\lim_{x\to\infty}[\ln F_{\alpha,\beta}(x)]''=0, \qquad \lim_{x\to\infty}[\ln F_{\alpha,\beta}(x)]'=0, \tag{9}$$

$$\lim_{x\to 0^+}[\ln F_{\alpha,\beta}(x)]'=-\operatorname{sgn}\beta\cdot\infty \qquad \text{if } \alpha>0, \tag{10}$$

$$\lim_{x\to(-\alpha)^+}[\ln F_{\alpha,\beta}(x)]'=-\infty \qquad \text{if } \beta=\alpha<0, \tag{11}$$

$$\lim_{x\to(-\alpha)^+}[\ln F_{\alpha,\beta}(x)]'=\operatorname{sgn}(\beta-\alpha)\cdot\infty \qquad \text{if } \alpha<0,\ \beta\neq 0 \text{ and } \beta\neq\alpha. \tag{12}$$

From (8), it follows that $[\ln F_{\alpha,\beta}(x)]''>0$ and $[\ln F_{\alpha,\beta}(x)]'$ is increasing for $\alpha=2\beta>0$. Further, from (9), it follows also that $[\ln F_{\alpha,\beta}(x)]'<0$ (From this, we can obtain the right hand side of inequality (3)) and $\ln F_{\alpha,\beta}(x)$ decreases. Therefore $F_{\alpha,\beta}(x)$ decreases for $\alpha=2\beta>0$. By the same argument, it can be deduced that if $\alpha=2\beta<0$ the function $F_{\alpha,\beta}(x)$ increases.

From (8), if $\alpha \neq 2\beta$, the function $[\ln F_{\alpha,\beta}(x)]''$ may have one zero point $x_0 = \frac{\alpha\beta}{\alpha-2\beta}$ at most.

If $x_0 \leq \max\{0, -\alpha\}$, then the function $[\ln F_{\alpha,\beta}(x)]''$ has no zero point. This means that if $\alpha > 0 > \beta$, or $0 < \alpha < 2\beta$, or $\alpha < 0 < \beta$, or $\alpha < 2\beta < 0$, or $\beta \leq \alpha < 0$, the function $[\ln F_{\alpha,\beta}(x)]''$ keep the same sign and $[\ln F_{\alpha,\beta}(x)]'$ is monotonic. Furthermore, utilizing (10), (11) and (12), it is concluded that when either $\alpha > 0 > \beta$, or $\alpha < 0 < \beta$ or $\alpha < 2\beta < 0$, the function $[\ln F_{\alpha,\beta}(x)]' > 0$, and then $\ln F_{\alpha,\beta}(x)$ and $F_{\alpha,\beta}(x)$ increases; when either $2\beta > \alpha > 0$ or $\beta \leq \alpha < 0$, the function $[\ln F_{\alpha,\beta}(x)]' < 0$, then $\ln F_{\alpha,\beta}(x)$ and $F_{\alpha,\beta}(x)$ decreases.

If $x_0 > \max\{0, -\alpha\}$, the function $[\ln F_{\alpha,\beta}(x)]''$ has a unique zero x_0. If $\alpha(2\beta-\alpha) > 0$, the function $[\ln F_{\alpha,\beta}(x)]'$ has a unique minimum attained at x_0; if $\alpha(2\beta - \alpha) < 0$, the function $[\ln F_{\alpha,\beta}(x)]'$ has a unique maximum attained at x_0. This implies that for $\alpha < \beta < \frac{\alpha}{2} < 0$ the function $[\ln F_{\alpha,\beta}(x)]'$ has a unique zero point which is a maximum point of $\ln F_{\alpha,\beta}(x)$ and $F_{\alpha,\beta}(x)$ and that for $0 < 2\beta < \alpha$ the function $[\ln F_{\alpha,\beta}(x)]'$ has a unique zero point which is a minimum point of $\ln F_{\alpha,\beta}(x)$ and $F_{\alpha,\beta}(x)$.

2.3.2 The Case of $x < \min\{0, -\alpha\}$

It is not difficult to see that the function $F_{\alpha,0}(x)$ increases for $x < \min\{0, -\alpha\}$. Straightforward computation also leads to

$$\lim_{x\to-\infty}[\ln F_{\alpha,\beta}(x)]'' = 0, \qquad \lim_{x\to-\infty}[\ln F_{\alpha,\beta}(x)]' = 0, \tag{13}$$

$$\lim_{x\to(-\alpha)^-}[\ln F_{\alpha,\beta}(x)]' = -\operatorname{sgn}(\beta-\alpha)\cdot\infty \qquad \text{if } \alpha > 0, \tag{14}$$

$$\lim_{x\to 0^-}[\ln F_{\alpha,\beta}(x)]' = -\infty \qquad \text{if } \beta = \alpha < 0, \tag{15}$$

$$\lim_{x\to 0^-}[\ln F_{\alpha,\beta}(x)]' = \operatorname{sgn}\beta\cdot\infty \qquad \text{if } \alpha < 0 \text{ and } \beta \neq \alpha. \tag{16}$$

By (8), if $\alpha = 2\beta > 0$, then the function $[\ln F_{\alpha,\beta}(x)]'' > 0$ and $[\ln F_{\alpha,\beta}(x)]'$ is increasing. Considering (13) gives $[\ln F_{\alpha,\beta}(x)]' > 0$, and then $\ln F_{\alpha,\beta}(x)$ and $F_{\alpha,\beta}(x)$ are increasing for $\alpha = 2\beta > 0$. Similarly, if $\alpha = 2\beta < 0$, the function $F_{\alpha,\beta}(x)$ is decreasing.

Observing (8), when $\alpha \neq 2\beta$, the function $[\ln F_{\alpha,\beta}(x)]''$ may have at most one zero point $x_0 = \frac{\alpha\beta}{\alpha-2\beta}$.

If $x_0 \geq \min\{0, -\alpha\}$, then $[\ln F_{\alpha,\beta}(x)]''$ has no zero point. This implies that if either $0 > \alpha > 2\beta$, or $\alpha > 0 > \beta$, or $0 < 2\beta < \alpha$, or $0 < \alpha < \beta$, or $\alpha < 0 < \beta$ then the function $[\ln F_{\alpha,\beta}(x)]''$ does not change its sign and $[\ln F_{\alpha,\beta}(x)]'$ is monotonic. Employing (14), (15) and (16) concludes that when either $\alpha > 0 > \beta$, or $0 < 2\beta < \alpha$, or $\alpha < 0 < \beta$ the function $[\ln F_{\alpha,\beta}(x)]' > 0$, and then $\ln F_{\alpha,\beta}(x)$ and $F_{\alpha,\beta}(x)$ increases and that when either $0 > \alpha > 2\beta$ or $0 < \alpha < \beta$ the function $[\ln F_{\alpha,\beta}(x)]' < 0$, and then $\ln F_{\alpha,\beta}(x)$ and $F_{\alpha,\beta}(x)$ decreases.

If $x_0 < \min\{0, -\alpha\}$, the function $[\ln F_{\alpha,\beta}(x)]''$ has a unique zero x_0. If $\alpha(2\beta-\alpha) > 0$, the function $[\ln F_{\alpha,\beta}(x)]'$ has a unique minimum attained at x_0; if $\alpha(2\beta - \alpha) < 0$, the function $[\ln F_{\alpha,\beta}(x)]'$ has a unique maximum attained at x_0. This implies that for $2\beta > \alpha > \beta > 0$ the function $[\ln F_{\alpha,\beta}(x)]'$ has a unique zero point which is a minimum point of $\ln F_{\alpha,\beta}(x)$ and $F_{\alpha,\beta}(x)$ and that for $0 > 2\beta > \alpha$ the function $[\ln F_{\alpha,\beta}(x)]'$ has a unique zero point which is a maximum point of $\ln F_{\alpha,\beta}(x)$ and $F_{\alpha,\beta}(x)$.

2.4 Proof of Monotonicity of $\left(1+\frac{\alpha}{n}\right)^{n+\beta}$

It has been proved above that the function $F_{\alpha,\beta}(x)$ has a unique maximum if $\alpha < \beta < \frac{\alpha}{2} < 0$ and that the function $F_{\alpha,\beta}(x)$ has a unique minimum if $0 < 2\beta < \alpha$. Consequently, if $F_{\alpha,\beta}(1) \le F_{\alpha,\beta}(2)$ for $\alpha > 2\beta > 0$ the sequence b_n increases; if $F_{\alpha,\beta}(1) \ge F_{\alpha,\beta}(2)$ for $2\beta < \alpha < \beta < 0$ the sequence b_n decreases; otherwise, b_n is not monotonic. Namely, when $\alpha > 2\beta > 0$ and $\beta \le \frac{\ln(1+\alpha)-2\ln(1+\alpha/2)}{\ln(1+\alpha/2)-\ln(1+\alpha)}$, the sequence $b_n = \left(1+\frac{\alpha}{n}\right)^{n+\beta}$ increases; when $2\beta < \alpha < \beta < 0$ and $\beta \le \frac{\ln(1+\alpha)-2\ln(1+\alpha/2)}{\ln(1+\alpha/2)-\ln(1+\alpha)}$, the sequence b_n decreases. As a result, using inequality (1) or (5), the sufficient and necessary conditions of the sequence $\left(1+\frac{\alpha}{n}\right)^{n+\beta}$ being monotonic are concluded. The proof of Theorem 2 is complete.

2.5 Proof of Theorem 3

It is clear that Theorem 1 is a special case of Theorem 2. In order to prove Theorem 3, it is sufficient to conclude Theorem 2 directly from Theorem 1. For this purpose, taking $\frac{\alpha}{x} = \frac{1}{y}$ for $\alpha > 0$ yields $F_{\alpha,\beta}(x) = [f_{\beta/\alpha}(y)]^{\alpha}$. This shows that the functions $F_{\alpha,\beta}(x)$ and $f_{\beta/\alpha}(y)$ have the same monotonicity as $\alpha > 0$. On the other hand, if $\alpha < 0$, setting $\frac{-\alpha}{x+\alpha} = \frac{1}{y}$ leads to $F_{\alpha,\beta}(x) = [f_{1-\beta/\alpha}(y)]^{\alpha}$. This tells us that the functions $F_{\alpha,\beta}(x)$ and $f_{1-\beta/\alpha}(y)$ have the opposite monotonicity as $\alpha < 0$. From Theorem 1, the monotonicities of $F_{\alpha,\beta}(x)$ can be deduced immediately.

By similar arguments, the equivalence between the necessary and sufficient conditions of the monotonicities of the sequences a_n and b_n can be obtained easily. The proof of Theorem 3 is complete.

Acknowledgements. The authors would like to express heartily their thanks to the anonymous referees for their valuable comments and corrections to this paper.

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