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Existence Of Solutions For a Nonlinear System With Applications To Difference Equations*

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Abstract

In this paper, by using the Mountain Pass Theorem, the existence of solutions for a nonlinear system is established. As an application, we obtain the existence of solutions for the boundary value problem involving a second order difference equation.

1 Introduction

Nonlinear systems of equations arise in many applications [7, 10]. Here we consider the following nonlinear system

$$Au - \nabla_u F(u) = 0 \tag{1}$$

where A is a given $m \times m$ positive definite matrix (m is a positive integer), $u = (u_1, u_2, ..., u_m)^T$ is a column vector in the m-dimensional Euclidean Space $\mathbb{R}^m$, $F \in C^1(\mathbb{R}^m, \mathbb{R})$ and $\nabla_u F(u)$ denotes the gradient of $F(u)$ in u.

As a special case of (1), we consider the boundary value problem of the second order difference equation

$$\begin{aligned} &\triangle^2 u_{n-1} + f_n(u_n) = 0, \quad n = 1, 2, ..., m, \\ &u_0 = 0 = u_{m+1}, \end{aligned} \tag{2}$$

which has been studied by many scholars, see for examples, [1, 2, 3, 4, 9, 10]. (2) can be rewritten as a system of the form (1) where

$$F(u) = \sum_{k=1}^{m} \int_1^{u_k} f_k(s)ds \tag{3}$$

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and

$$A = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 \\ 0 & 0 & 0 & \cdots & 2 \end{pmatrix}_{m\times m} \tag{4}$$

is a positive definite matrix.

In recent years, critical point theory has been used successfully to study the periodic solutions and boundary value problems of difference equations[3, 5, 6, 11, 12]. Motivated by these works, in this paper, we make use of the Mountain Pass Theorem to obtain the existence of solutions of the system (1). For this purpose, we define a functional $I : \mathbb{R}^m \to \mathbb{R}$ by

$$I(u) = \frac{1}{2}u^T Au - F(u). \tag{5}$$

It is easy to see that

$$I'(u) = Au - \nabla_u F(u). \tag{6}$$

Therefore, $u \in \mathbb{R}^m$ is a solution of (1) if and only if u is a critical point of $I(u)$. To seek the solutions of the system (1), we only need to find the critical points of $I(u)$.

Let $\|\cdot\|$ denote the Euclidean norm in $\mathbb{R}^m$, that is,

$$\|u\| = \left(\sum_{i=1}^{m} u_i^2\right)^{\frac{1}{2}} \quad \text{for} \quad u = (u_1, u_2, ..., u_m)^T \in \mathbb{R}^m. \tag{7}$$

THEOREM 1. Assume that $F(u) \geq 0$ for $u \in \mathbb{R}^m$ and satisfies the following conditions
(F_1) $\lim_{\|u\|\to 0} \frac{F(u)}{\|u\|^2} = 0$,
(F_2) $\lim_{\|u\|\to +\infty} \frac{F(u)}{\|u\|^2} = +\infty$.
Then system (1) has at least one nontrivial solution.

THEOREM 2. Assume that $F(u) \geq 0$ for $u \in \mathbb{R}^m$ and satisfies the following conditions
(F_3) $\lim_{\|u\|\to 0} \frac{F(u)}{\|u\|^2} = +\infty$,
(F_4) $\lim_{\|u\|\to +\infty} \frac{F(u)}{\|u\|^2} = 0$.
Then system (1) has at least one nontrivial solution.

Consider the boundary value problem (2), we can easily get the following corollaries.

COROLLARY 1. Assume that $f_k \in C(\mathbb{R}, \mathbb{R})$ and $\int_0^z f_k(s)ds \geq 0$ for $k = 1, 2, ..., m$, $z \in \mathbb{R}$. And f_k for $k = 1, 2, ..., m$ satisfy also the following conditions
(H_1) $\lim_{z\to 0} \frac{\int_0^z f_k(s)ds}{z^2} = 0$,
(H_2) $\lim_{z\to \infty} \frac{\int_0^z f_k(s)ds}{z^2} = +\infty$.

Then the boundary value problem (2) has at least one nontrivial solution.

COROLLARY 2. Assume that $f_k \in C(\mathbb{R},\mathbb{R})$ and $\int_0^z f_k(s)ds \geq 0$ for $k = 1,2,...,m$, $z \in \mathbb{R}$. And f_k for $k = 1,2,...,m$ satisfy also the following conditions
(H_3) $\lim_{z\to 0} \frac{\int_0^z f_k(s)ds}{z^2} = +\infty$,
(H_4) $\lim_{z\to \infty} \frac{\int_0^z f_k(s)ds}{z^2} = 0$.
Then the boundary value problem (2) has at least one nontrivial solution.

We note that, for the boundary value problem (2), if there exist positive constants a_1, a_2, M and $\mu > 2$ such that

$$\int_0^z f_k(s)ds \geq a_1|z|^{\mu} - a_2 \quad \text{for} \quad |z| \geq M,\ k = 1,2,...,m. \tag{8}$$

Then, f_k satisfies (H_2). And if there exist positive constants a_3, a_4, M_1 and $1 < \mu_1 < 2$ such that

$$0 \leq \int_0^z f_k(s)ds \leq a_3|z|^{\mu_1} + a_4 \quad \text{for} \quad |z| \geq M_1,\ k = 1,2,...,m. \tag{9}$$

Then, f_k satisfies (H_4). Therefore, Corollary 1 and Corollary 2 improve the results of Theorem 1 and Theorem 2 in [10], respectively.

Now, we provide two examples to show that $F(u)$ may have many other complicated forms beside the form (3).

EXAMPLE 1. Let

$$F(u) = \sum_{i=1}^{m} b_i|u_i|^{\alpha_i} + \alpha_0(u_1^4 + u_1^2u_2^2 + u_4^4)$$

where $b_i > 0, \alpha_i > 2$ for $i = 1,2,...,m$ and $\alpha_0 \geq 0$. Then $F(u) \geq 0$ and satisfies the conditions (F_1) and (F_2).

EXAMPLE 2. Let

$$F(u) = \sum_{i=1}^{m} c_i|u_i|^{\beta_i} + \beta_0|u_1|^{\frac{5}{3}}(\sin u_2 + 1)$$

where $c_i > 0, \beta_i \in (1,2)$ for $i = 1,2,...,m$ and $\beta_0 \geq 0$. Then $F(u) \geq 0$ and satisfies the conditions (F_3) and (F_4).

2 Proofs of Main Results

In this section, we will prove Theorems 1 and 2. To this end, we first state the Mountain pass theorem which can be found in [8]. Suppose E is a real Banach space. Let $C^1(E,\mathbb{R})$ denote the set of functionals that are Fréchet differentiable and whose Fréchet derivatives are continuous on E. For $I \in C^1(E,\mathbb{R})$, we say I satisfies the Palais-Smale condition (henceforth denoted by P. S. condition) if any sequence $(u_m) \subset E$ for which $I(u_m)$ is bounded and $I'(u_m) \to 0$ as $m \to \infty$ possesses a convergent subsequence.

LEMMA 1 (Mountain Pass Theorem). Let E be a real Banach space and assume that $I \in C^1(E, \mathbb{R})$ satisfies the P.S. condition. Suppose $I(0) = 0$ and
(I_1) there are constants $\rho, \alpha > 0$ such that $I_{\partial B_\rho} \geq \alpha$, and
(I_2) there is an $e \in E \backslash B_\rho$ such that $I(e) \leq 0$.
Then I possesses a critical value $c \geq \alpha$. Moreover, c can be characterized as

$$c = \inf_{g \in \Gamma} \max_{u \in g([0,1])} I(u),$$

where

$$\Gamma = \{g \in C([0,1], E) |\ g(0) = 0,\ g(1) = e\}.$$

Since the matrix A is positive definite, the eigenvalues $\lambda_1, \lambda_2, ..., \lambda_m$ of A are positive. Let

$$\lambda_{\min} = \min\{\lambda_1, \lambda_2, \cdots, \lambda_m\}, \quad \lambda_{\max} = \max\{\lambda_1, \lambda_2, \cdots, \lambda_m\}.$$

Then

$$\lambda_{\min} \|u\|^2 \leq u^T A u \leq \lambda_{\max} \|u\|^2.$$

PROOF OF THEOREM 1. We will use Mountain Pass Theorem to prove the results. In fact, let $E = \mathbb{R}^m$. Clearly, the functional $I(u)$ defined in (5) satisfies that $I \in C^1(E, \mathbb{R})$ and $I(0) = 0$. We now show that I satisfies P. S. condition. Let $\{I(u^{(k)})\}$ be a bounded sequence. Then there exists a positive constant C such that

$$|I(u^{(k)})| \leq C \quad \text{for} \quad k = 1, 2, \cdots, m.$$

According to (F_2), there exist constants $\alpha > \frac{1}{2}\lambda_{\max}$ and $R > 0$ such that

$$F(u) \geq \alpha \|u\|^2 \quad \text{for} \quad \|u\| \geq R.$$

So,

$$-C \leq I(u) = \frac{1}{2} u^T A u - F(u) \leq (\frac{1}{2}\lambda_{\max} - \alpha)\|u\|^2, \quad \|u\| \geq R.$$

Thus, $\{u^{(k)}\}$ is bounded and has a convergent subsequence. I satisfies P. S. condition.

Next, we show that I satisfies (I_1) in Lemma 1. By (F_1), there exist positive constants $\epsilon_1 < \frac{1}{2}\lambda_{\min}$ and ρ_1 such that

$$F(u) \leq \epsilon_1 \|u\|^2 \quad \text{for} \quad \|u\| \leq \rho_1.$$

Thus,

$$I(u) = \frac{1}{2} u^T A u - F(u) \geq (\frac{1}{2}\lambda_{\min} - \epsilon_1)\|u\|^2 = (\frac{1}{2}\lambda_{\min} - \epsilon_1)\rho_1^2 > 0$$

for $\|u\| = \rho_1$. So, (I_1) holds.

Last, we show that (I_2) holds. According to (F_2) again, there exist constants $\alpha_1 > \frac{1}{2}\lambda_{\max}$ and $R_1 > \rho_1$ such that

$$F(u) \geq \alpha_1 \|u\|^2 \quad \text{for} \quad \|u\| \geq R_1.$$

Thus,

$$I(u) = \frac{1}{2}u^T Au - F(u) \leq (\frac{1}{2}\lambda_{\max} - \alpha_1)\|u\|^2 = (\frac{1}{2}\lambda_{\max} - \alpha_1)R_1^2 < 0$$

for $\|u\| = R_1$ and (I_2) follows.

According to the Mountain Pass Theorem, $I(u)$ possesses a critical value $c > 0$ which implies that $I(u)$ has a nonzero critical points. The proof of Theorem 1 is complete.

PROOF OF THEOREM 2. We will show that the functional $-I(u)$ satisfies all conditions of the Mountain Pass Theorem. In fact, let $E = \mathbb{R}^m$, then $-I \in C^1(E, \mathbb{R})$ and $-I(0) = 0$. We will show that $-I$ satisfies P. S. condition first. Let $\{-I(u^{(k)})\}$ be a bounded sequence. Then there exists a positive constant C^* such that

$$|I(u^{(k)})| \leq C^* \quad \text{for} \quad k = 1, 2, \cdots, m.$$

According to (F_4), there exist constants $0 < \epsilon_2 < \frac{1}{2}\lambda_{\min}$ and $R_2 > 0$ such that

$$F(u) \leq \epsilon_2 \|u\|^2 \quad \text{for} \quad \|u\| \geq R_2.$$

So,

$$-C^* \leq -I(u) = -\frac{1}{2}u^T Au + F(u) \leq (-\frac{1}{2}\lambda_{\min} + \epsilon_2)\|u\|^2, \quad \|u\| \geq R_2.$$

Thus, $\{u^{(k)}\}$ is bounded and has a convergent subsequence. And $-I$ satisfies P. S. condition.

Next, we show that $-I$ satisfies (I_1) in Lemma 1. By (F_3), there exist positive constants $M_2 > \frac{1}{2}\lambda_{\max}$ and ρ_2 such that

$$F(u) \geq M_2 \|u\|^2 \quad \text{for} \quad \|u\| \leq \rho_2.$$

Thus,

$$I(u) = -\frac{1}{2}u^T Au + F(u) \geq (-\frac{1}{2}\lambda_{\max} + M_2)\|u\|^2 = (-\frac{1}{2}\lambda_{\max} + M_2)\rho_2^2 > 0$$

for $\|u\| = \rho_2$. So, (I_1) holds.

Last, we show that (I_2) holds. According to (F_4) again, there exist constants $0 < \epsilon_3 < \frac{1}{2}\lambda_{\min}$ and $R_3 > \rho_2$ such that

$$F(u) \leq \epsilon_3 \|u\|^2 \quad \text{for} \quad \|u\| \geq R_3.$$

Thus,

$$-I(u) = -\frac{1}{2}u^T Au + F(u) \leq (-\frac{1}{2}\lambda_{\min} + \epsilon_3)\|u\|^2 = (-\frac{1}{2}\lambda_{\min} + \epsilon_3)R_3^2 < 0$$

for $\|u\| = R_3$ and (I_2) follows.

According to the Mountain Pass Theorem, $-I(u)$ possesses a critical value $c > 0$ which implies that $I(u)$ has a nonzero critical points. The proof of Theorem 2 is complete.

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References

[1] R. P. Agarwal and D. O'Regan, A fixed-point approach for nonlinear discrete boundary value problems, Computers Math. Applic., 36 (1998), 115–121.

[2] R. P. Agarwal and D. O'Regan, Singular discrete boundary value problems, Appl. Math. Lett., 12(1999), 127–131.

[3] R. P. Agarwal, K. Perera and D. O'Regan, Multiple positive solutions of singular and nonsingular discrete problems via variational methods, Nonlinear Anal., 58(2004), 69–73.

[4] F. M. Atici, Existence of positive solutions of nonlinear discrete Sturm-Liouville problems, Math. Comput. Modelling, 32 (2000), 599–607.

[5] Z. M. Guo and J. S. Yu, Existence of periodic solutions and subharmonic solutions on second order superlinear difference equations, Sci. China Ser. A, 46(2003), 506–515.

[6] Z. M. Guo and J. S. Yu, The existence of periodic and subharmonic solutions to subquadratic second-order difference equations, J. London Math. Soc., 68(2003), 419-430.

[7] W. G. Kelly and A. P. Peterson, Difference Equations, Academic Press, London, 1991.

[8] P. H. Rabinowitz, Minimax Methods in Critical Point Theory with Applications to Differential Equations, CBMS Issues Math. Ed., 65, 1986.

[9] G. Zhang and S. S. Cheng, Existence of solutions for a nonlinear system with a parameter, J. Math. Anal. Appl., 314(2006), 311–319.

[10] G. Zhang and Z. L. Yang, Existence of 2^n nontrivial solutions for discrete two-point boundary value problems, Nonlinear Anal., 59(2004), 1181–1187.

[11] Z. Zhou, J. S. Yu and Z. M. Guo, Periodic solutions of higher dimensional discrete system, Proceedings of the Royal Society of Edinburgh Ser. A, 134(2004), 1013–1022.

[12] Z. Zhou, J. S. Yu and Z. M. Guo, The existence of periodic and subharmonic solutions to subquadratic discrete Hamiltonian Systems, ANZIAM J., 47(2005), 89–102.