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Preconditioned Diagonally Dominant Property For Linear Systems With H-Matrices*

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Abstract

It is well-known that most iterative methods for linear systems with strictly diagonally dominant coefficient matrix A are convergent. When A is not diagonally dominant, preconditioned techniques can be employed. In this note, a sparse preconditioning matrix with parameters $\alpha_2, \alpha_3, ..., \alpha_n$ is constructed for transforming a general H-matrix into a strictly diagonally dominant matrix. Also, we discuss the relationship between diagonally dominant property and the parameters $\alpha_2, \alpha_3, ..., \alpha_n$.

1 Introduction

For a linear system

$$Ax = b, \tag{1}$$

where A is an $n \times n$ square matrix, and b an n-vector, a basic iterative method for solving equation (1) is

$$Mx^{k+1} = Nx^k + b, \quad k = 0, 1, ..., \tag{2}$$

where $A = M - N$ and M is nonsingular. (2) can also be written as

$$x^{k+1} = Tx^k + c, \quad k = 0, 1, ..., \tag{3}$$

where $T = M^{-1}N$ and $c = M^{-1}b$. Assume A has unit diagonal entries and let $A = I - L - U$, where $-L$ and $-U$ are strictly lower and strictly upper triangular matrices, respectively. Then the iteration matrix of the classical Gauss-Seidel method is given by

$$T = (I - L)^{-1}U.$$

It is well known that many traditional iterative methods for solving linear system (1) work well for diagonally dominant matrix A. Otherwise, preconditioning techniques

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may transform A into a diagonally dominant one. It is clear that after finding the preconditioners P and Q such that PAQ is strictly diagonally dominant, then we can apply iterative methods for solving

$$PAQy = Pb, \tag{4}$$

and

$$x = Qy$$

instead of solving

$$Ax = b.$$

Then, we can define the basic iterative method:

$$M_p y^{k+1} = N_p y^k + Pb, \quad k = 0, 1, ..., \tag{5}$$

where $PAQ = M_p - N_p$ and M_p is nonsingular. (5) can also be written as

$$y^{k+1} = Ty^k + c, \quad k = 0, 1, ...,$$

where $T = M_p^{-1} N_p$ and $c = M_p^{-1} Pb$. Assume

$$PAQ = \hat{D} - \hat{L} - \hat{U},$$

where $-\hat{L}$ and $-\hat{U}$ are strictly lower and strictly upper triangular matrices, respectively. Then the iteration matrix of the classical Gauss-Seidel method is given by $T = (\hat{D} - \hat{L})^{-1} \hat{U}$. Therefore, the main problem is to find 'good' P and Q such that the matrix PAQ is diagonally dominant. For instance, we may require the matrices P, Q, and/or PAQ to be sparse if the original matrix A is sparse. Yuan in [1] and Ying in [2] investigated this problem. Unfortunately, Ying [2] pointed out that the main result of [1] is not true or impractical. In this note, we will construct two sparse preconditioning matrix P and Q involving parameters $\alpha_2, \alpha_3, ..., \alpha_n$ such that PAQ is a strictly diagonally dominant matrix for a general H-matrix. Also, the relationship between diagonally dominant property and the parameters is discussed.

2 Preconditioned Diagonally Dominant Property

Let $A = (a_{ij})$ be an n by n square matrix. The comparison matrix of A is denoted by $\langle A \rangle = (m_{ij})$ defined by

$$m_{ii} = |a_{ii}|, \quad m_{ij} = -|a_{ij}| \text{ for } i \neq j.$$

Let $|A|$ denote the matrix whose elements are the moduli of the elements of the given matrix. A is an H-matrix if and only if its comparison matrix is an M-matrix. A real vector $r = (r_1, ..., r_n)^T$ is called positive and denoted by $r > 0$, if $r_i > 0$ for all i. We consider the following preconditioner in [5]

$$P = \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \\ -\alpha_2 a_{21} & 1 & \ddots & \ddots & \vdots \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ -\alpha_n a_{n1} & 0 & \cdots & \cdots & 1 \end{bmatrix}, \tag{6}$$

where $\alpha = (\alpha_2, \alpha_3, ..., \alpha_n)$ and $\alpha_2, ..., \alpha_n$ are parameters.

If A is an H-matrix, let $r = \langle A\rangle^{-1}e > 0$ and $Q = \text{diag}(r)$, where $e = (1, ..., 1)^T$. When $\alpha_i a_{i1} a_{1i} r_i \neq 1$ for $i = 2, ..., n$, $(\hat{D} - \hat{L})^{-1}$ exists, and hence it is possible to define the Gauss-Seidel iteration matrix for PAQ.

Next, we quote some known results:

LEMMA 1. [4] A is an H-matrix if and only if there exists a vector $r > 0$ such that $\langle A\rangle r > 0$.

LEMMA 2. [4] If $A = I - B$, where $B \geq 0$, is an M-matrix if and only if $\rho(B) < 1$, where $\rho(\cdot)$ denotes the spectral radius of a matrix.

LEMMA 3. [4] If $\rho(B) < 1$, then $(I - B)^{-1} = \sum_{k=0}^{\infty} B^k$.

LEMMA 4. If A is an H-matrix with unit diagonal elements, then $\langle A\rangle^{-1} = (m_{ij})$ satisfies

$$\sum_{j=1}^{n} m_{ij} \geq 1,\ i = 1, ..., n.$$

PROOF. Since $\langle A\rangle = I - B$ where $B \geq 0$, is an M-matrix, by Lemma 2, we have $\rho(B) < 1$, and

$$\langle A\rangle^{-1} = (I - B)^{-1} = \sum_{k=0}^{\infty} B^k \geq I,\ i = 1, ..., n.$$

Therefore, $\sum_{j=1}^{n} m_{ij} \geq 1$ for $i = 1, ..., n$.

LEMMA 5. [3] If A is an H-matrix, then $|A^{-1}| \leq \langle A\rangle^{-1}$.

LEMMA 6. If A is an H-matrix with unit diagonal elements, then $\rho((I - L)^{-1}U) < 1$.

PROOF. Since $A = I - L - U$ is an H-matrix, we see that $I - L$ is also an H-matrix and $\langle A\rangle = I - |L| - |U|$ is an M-matrix. Furthermore, we have $\rho((I - |L|)^{-1}|U|) < 1$. From Lemma 5, $|(I - L)^{-1}| \leq (I - |L|)^{-1}$. Therefore, we have

$$|(I - L)^{-1}U| \leq |(I - L)^{-1}||U| \leq (I - |L|)^{-1}|U|,$$

so that

$$\rho((I-L)^{-1}U) \le \rho(|(I-L)^{-1}U|) \le \rho((I-|L|)^{-1}|U|) < 1.$$

Now we give the main results as follows:

THEOREM 1. If A is an H-matrix with unit diagonal elements, then

$$\frac{1+2r_1\,|a_{i1}|}{|a_{i1}|\,(2r_1-1)} > 1,\ i=2,...,n,$$

where $r=(r_1,...,r_n)^T=\langle A\rangle^{-1}e$.

Indeed, let $r_1=\sum\limits_{j=1}^{n} m_{1j} \ge 1$. Then Lemma 4 implies that: $2r_1-1>0$, and

$$\frac{1+2r_1\,|a_{i1}|}{|a_{i1}|\,(2r_1-1)} > \frac{1+2r_1\,|a_{i1}|}{|a_{i1}|\,(2r_1)} > 1.$$

THEOREM 2. For an H-matrix A with unit diagonal elements, if

$$0<\alpha_i\le 1,\ \text{or } 1<\alpha_i<\frac{|a_{i1}|\,(2r_1-1)}{1+2r_1\,|a_{i1}|},$$

then PA is an H-matrix and PAQ is a strictly diagonally dominant matrix, where $\alpha_2,\alpha_3,...,\alpha_n$ are constants.

PROOF. Let

$$PA=\begin{cases} a_{ij} & i=1 \\ a_{ij}-\alpha_i a_{i1}a_{1j} & i\ne 1 \end{cases}.$$

For $i=1$, we have $(\langle A\rangle r)_1=1$ and $(\langle PA\rangle r)_1=1>0$. For $i=2,...,n$, we have

$$\begin{aligned}(\langle PA\rangle r)_i &= -\left|(1-\alpha_i)a_{i1}\right|r_1+\left|a_{ii}-\alpha_i a_{i1}a_{1i}\right|r_i-\sum_{j\ne i,1}^{n}|a_{ij}-\alpha_i a_{i1}a_{1j}|r_j \\ &\ge -\left|(1-\alpha_i)a_{i1}\right|r_1+|a_{ii}|\,r_i-|\alpha_i a_{i1}a_{1i}|\,r_i-\sum_{j\ne 1,i}^{n}|a_{ij}|r_j-\sum_{j\ne 1,i}^{n}|\alpha_i a_{i1}a_{1j}|\,r_j.\end{aligned}$$

If $0<\alpha_i\le 1$, then the last expression is

$$\begin{aligned}&-|a_{i1}|\,r_1+\alpha_i\,|a_{i1}|\,r_1+a_{ii}r_i-\alpha_i\,|a_{i1}a_{1i}|\,r_i-\sum_{j\ne 1,i}^{n}|a_{ij}|r_j-\sum_{j\ne 1,i}^{n}|\alpha_i a_{i1}a_{1j}|\,r_j \\ =\ & (\langle A\rangle r)_i+\alpha_i\,|a_{i1}|\,(r_1-|a_{1i}|\,r_i-\sum_{j\ne 1,i}^{n}|a_{1j}|r_j) \\ =\ & 1+\alpha_i\,|a_{i1}|>0.\end{aligned}$$

If $\alpha_i > 1$, then the last expression is

$$\begin{aligned}
& (1-\alpha_i)\,|a_{i1}|\,r_1 + a_{ii}r_i - \alpha_i\,|a_{i1}a_{1i}|\,r_i - \sum_{j\neq i,1}^{n} |a_{ij} - \alpha_i a_{i1} a_{1j}| r_j \\
= \;& (|a_{i1}|\,r_1 + r_i - \sum_{j\neq 1,i}^{n} |a_{ij}| r_j) - \alpha_i\,|a_{i1}|\,(r_1 + |a_{1i}|\,r_i + \sum_{j\neq 1,i}^{n} |a_{1j}| r_j) \\
= \;& (1 + 2\,|a_{i1}|\,r_1) - \alpha_i\,|a_{i1}|\,(2r_1 - 1) > 0.
\end{aligned}$$

We have thus shown that $(\langle PA\rangle\, r)_i > 0$ for $i = 1, ..., n$. Hence PA is also an H-matrix. Furthermore, PAQ is a strictly diagonally dominant matrix. The proof is complete.

THEOREM 3. For an H-matrix A with unit diagonal elements, if

$$0 < \alpha_i \leq 1 \text{ or } 1 < \alpha_i < \frac{|a_{ik}|\,(2r_1 - 1)}{1 + 2r_1\,|a_{ik}|},$$

then $P_k A$ is an H-matrix and $P_k AQ$ is a strictly diagonally dominant matrix, where

$$P_k = \begin{bmatrix}
1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & 1 & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & -\alpha_{k+1}a_{k+1,k} & 1 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \vdots & 0 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
0 & \cdots & -\alpha_n a_{nk} & 0 & \cdots & \cdots & 1
\end{bmatrix}, \tag{7}$$

and $\alpha_{k+1}, \alpha_{k+2}, ..., \alpha_n$ are constants.

For $i = 1, ..., k$, it is clear that $((P_k A)r)_i = 1 > 0$. For $i = k+1, ...n$, the proof is similar to that of Theorem 2.

By Theorem 2 and Theorem 3, we immediately have the following.

THEOREM 4. If A is an H-matrix with unit diagonal elements and satisfies conditions of Theorem 2 or Theorem 3, then PAQ $(P_k AQ)$ is a strictly diagonally dominant matrix and $\rho((\hat{D} - \hat{L})^{-1}\hat{U}) < 1$.

We can show that $(\langle PA\rangle r)_i$ is not decreasing with respect to any $\alpha_i \in [0, 1]$ for an M-matrix A.

THEOREM 5. For an M-matrix A with unit diagonal elements, let

$$[0, ..., 0]^T \leq \alpha = [\alpha_2, ..., \alpha_n]^T \leq \hat{\alpha} = [\hat{\alpha_2}, ..., \hat{\alpha_n}]^T \leq [1, ..., 1]^T.$$

Then

$$((PA)r)_i \leq ((\hat{P}A)r)_i, \; i = 1, ..., n,$$

where, P and r are the same as those in the above theorems, $\hat{P}$ can be obtained by substituting α_i for $\hat{\alpha_i}$ in matrix P.

PROOF. According to the proof of Theorem 2, we have

$$((PA)r)_i = 1 - \alpha_i a_{i1},$$

and

$$((\hat{P}A)r)_i = 1 - \hat{\alpha_i} a_{i1}, \quad i = 2, ..., n.$$

Since $a_{i1} \le 0$ and $0 \le \alpha_i \le \hat{\alpha_i} \le 1$, we see that

$$((PA)r)_i \le ((\hat{P}A)r)_i.$$

For $i = 1$, we have

$$((\hat{P}A)r)_1 = ((PA)r)_1 = 1.$$

Therefore

$$((PA)r)_i \le ((\hat{P}A)r)_i.$$

Similarly we can also show that $(\langle PA \rangle r)_i$ is nonincreasing in any $\alpha_i > 1$ for an M-matrix A.

THEOREM 6. For an M-matrix A with unit diagonal elements, let

$$[1, ..., 1]^T < \alpha = [\alpha_2, ..., \alpha_n]^T \le \hat{\alpha} = [\hat{\alpha_2}, ..., \hat{\alpha_n}]^T,$$

and

$$\hat{\alpha_i} \le \min\left(\frac{1}{a_{i1}a_{1i}}, \frac{1 + 2r_1 |a_{i1}|}{|a_{i1}| (2r_1 - 1)}\right).$$

Then

$$(\langle PA \rangle r)_i \ge (\langle \hat{P}A \rangle r)_i, \ i = 1, ..., n,$$

where $P, \hat{P}$ and r are the same as those in Theorem 5.

PROOF. By assuming the diagonal elements of PA ($\hat{P}A$) are positive and the first column elements of PA ($\hat{P}A$) are nonnegative, other off-diagonal elements of PA ($\hat{P}A$) are nonpositive. Namely,

$$PA(\hat{P}A) = b_{ij}(\hat{b_{ij}}) \begin{cases} > 0 & i \ne j \\ \ge 0 & i \ne 1, j = 1 \\ \le 0 & j \ne 1, j \ne i \end{cases}.$$

Hence,

$$((PA)r)_i = (1 - 2a_{i1}r_1) + \alpha_i a_{i1}(2r_1 - 1),$$

and

$$((\hat{P}A)r)_i = (1 - 2a_{i1}r_1) + \hat{\alpha}_i a_{i1}(2r_1 - 1),\ i = 2, ..., n.$$

Since $(2r_1 - 1) > 0$ and $a_{i1} < 0$, in view of our assumptions,

$$((PA)r)_i \geq ((\hat{P}A)r)_i, \quad i = 2, ..., n.$$

For $i = 1$, we have

$$((PA)r)_1 = ((\hat{P}A)r)_1 = 1.$$

Therefore,

$$((PA)r)_i \geq ((\hat{P}A)r)_i, \quad i = 1, ..., n.$$

REMARK: From Theorem 5 and Theorem 6, we notice that the diagonally dominance of the matrix PAQ is better than the other α for an M-matrix A, whenever $\alpha = (1, 1, \ldots, 1)^T$. The convergence rate of the Gauss-Seidel iterative method of PAQ is faster than the other α for the linear system (4) for an M-matrix A.

3 Numerical Example

First, we apply the proposed method to construct the diagonally dominant matrix for the following test matrix. Without loss of generality, we take $\alpha_i = 1$ for $i = 2, \ldots, n$ in (6). Let

$$A = \begin{pmatrix} 1.0000 & 0.1000 & -0.2000 & 0.1000 \\ -0.9000 & 1.0000 & 0.7000 & -0.8000 \\ 0.1000 & -0.1000 & 1.0000 & 0.3000 \\ 0.3000 & -0.5000 & 0.2000 & 1.0000 \end{pmatrix}.$$

We may prove A is an H-matrix. By using preconditioner P and Q, we have the following matrix:

$$PAQ = \begin{pmatrix} 16.4275 & 6.6183 & -4.4611 & 4.3481 \\ 0 & 72.1397 & 11.5988 & -30.8715 \\ 0 & -7.2802 & 22.7515 & 12.6095 \\ 0 & -35.0771 & 5.7994 & 42.1765 \end{pmatrix},$$

clearly, PAQ is a strictly diagonally dominant matrix.

Next, we test the Gauss-Seidel iterative method for the linear system (4). For comparison, we also consider the Gauss-Seidel iterative method for the linear system

(1). For the following H-matrix A,

$$A = \begin{pmatrix} 1 & c_1 & c_2 & 2/n & c_1 & c_2 & c_3 & \cdots & \cdots & c_1 & c_2 & c_3 \\ c_3 & 1 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & c_2 \\ c_2 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & c_1 \\ c_1 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ c_3 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 2/n \\ c_2 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & c_2 \\ c_1 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & c_1 \\ 2/n & c_1 & c_2 & c_3 & \cdots & \cdots & \cdots & \cdots & c_1 & c_2 & c_3 & 1 \end{pmatrix},$$

where $c_1 = \frac{-1}{n+1}$, $c_2 = \frac{1}{n}$, $c_3 = \frac{-1}{n+1}$. We set b so that the solution of (1) is $x^T = (1, 2, ..., n)$. Let the convergence criterion be $\left\|x^{k+1} - x^k\right\| / \left\|x^{k+1}\right\| \leq 10^{-6}$. For $n = 4, 10, 15, 20, 15$, we show the spectral radius of the Gauss-Seidel iterative matrices, and the number of Gauss-Seidel iterations for coefficient matrices A and PAQ. We use $GS(A)$ to denote the Gauss-Seidel iterative method for matrix A, and use $GS(PAQ)$ to denote the Gauss-Seidel iterative method for the matrix PAQ. Below, we summarize our finding in a table

	GS(A)		GS(PAQ)	
	spectral radius	iterations	spectral radius	iterations
n=4	0.5225	29	0.2871	15
n=10	0.3230	18	0.3012	13
n=15	0.2870	17	0.2612	11
n=20	0.2685	16	0.2529	11
n=50	0.2914	17	0.2853	10

Table 1: the spectral radius and the number of iterations

From Table 1, we find that the Gauss-Seidel method for the coefficient matrix PAQ is better than that for A.

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