Applied Mathematics E-Notes, 6(2006), 225-234 © ISSN 1607-2510
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A Challenging Test For Convergence Accelerators: Summation Of A Series With A Special Sign Pattern*

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Received 5 November 2005

Abstract

Slowly convergent series that have special sign patterns have been used in testing the efficiency of convergence acceleration methods. In this paper, we study the series $S_{m,n} = \sum_{k=0}^{\infty} \frac{(-1)^{\lfloor k/m \rfloor}}{(k+1)^{2n+1}}$ when $m \geq 1$, $n \geq 0$, which has m positive terms followed by m negative terms periodically. Using special functions, we first derive its sum in simple terms involving only the Riemann Zeta function and trigonometric functions. With the exact sum available, we next use this series to test the efficiency of various nonlinear convergence acceleration methods in summing it numerically. We conclude that the Shanks transformation and the Levin–Sidi $d^{(m)}$-transformation are two acceleration methods that produce highly accurate approximations to the sum of $S_{m,n}$, the latter being the more effective.

1 Introduction

The infinite series

$$S_1 = \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \log 2 \tag{1}$$

has been frequently used in testing the efficiency of the various nonlinear convergence acceleration methods, such as the Shanks transformation [7], the $\mathcal{L}$-transformation of Levin [4], and the θ-algorithm of Brezinski [1]. It is well-known that all three transformations (and several others as well) are very effective on this series, in the sense that they accelerate the convergence of the sequence of the partial sums of the series and produce excellent approximations to its limit log 2. See, e.g., Smith and Ford [10].

A more general version of this series with a special sign pattern, namely, the series

$$S_2 = \sum_{k=0}^{\infty} \frac{(-1)^{\lfloor k/2 \rfloor}}{k+1} = 1 + \frac{1}{2} - \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + \frac{1}{6} - \cdots = \frac{\pi}{4} + \frac{1}{2}\log 2, \tag{2}$$

*Mathematics Subject Classifications: 33E20, 40A99, 41A58, 41A99.

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was considered by Lubkin [6] in the study of his W-transformation. Lubkin concluded that the W-transformation, either in its simple form or in iterated form, was not effective in accelerating the convergence of the partial sums of this series. This series poses a real challenge for most of the known convergence acceleration methods. It turns out that the Shanks transformation produces convergence acceleration while the Levin transformation and the θ-algorithm are not effective at all. See Wimp [11, p. 171]. (Incidentally, the special cases of the Levin transformation and of the θ-algorithm, namely, the $\mathcal{L}_2$-transformation and the θ_2-algorithm, respectively, are identical to the W-transformation.) Another nonlinear method that accelerates the convergence of the series S_2 is the $d^{(2)}$-transformation of Levin and Sidi [5].

In view of the fact that the series S_2, with its special sign pattern, poses a real challenge for convergence acceleration methods, we ask whether we can find additional series *with known sums* that are more general and challenging than S_2, as far as convergence acceleration methods are concerned. The aim of this paper is, first of all, to provide such infinite series that are nontrivial and that can be used as test cases by researchers in the field of convergence acceleration.

The following generalization of Lubkin's series was considered in Sidi [9, Section 25.9]:

$$S_m = \sum_{k=0}^{\infty} \frac{(-1)^{\lfloor k/m \rfloor}}{k+1}, \quad m = 1, 2, \ldots\ . \tag{3}$$

It was stated there that

$$S_m = \frac{1}{m} \log 2 + \frac{\pi}{2m} \sum_{k=1}^{m-1} \cot \frac{k\pi}{2m} = \frac{1}{m} \log 2 + \frac{\pi}{2m} \sum_{k=1}^{m-1} \tan \frac{k\pi}{2m}. \tag{4}$$

This series too has a special sign pattern similar to that of S_2; its first m terms are positive and are followed by m negative terms, which are followed by m positive terms, and so on. Therefore, it is a very appropriate test series for the study of convergence acceleration methods. The result in (4) was given without proof in [9]; we give its complete proof in Section 2 of this note. In view of the fact that special functions are involved in the derivation, that the sum of S_m can be expressed in the simple form given here seems quite surprising.

We note that, so far, the only nonlinear convergence acceleration methods that are effective on this series are the Shanks transformation and the $d^{(m)}$-transformation of Levin and Sidi [5].

In Section 3, we consider a further generalization, namely, the series

$$S_{m,n} = \sum_{k=0}^{\infty} \frac{(-1)^{\lfloor k/m \rfloor}}{(k+1)^{2n+1}}, \quad m = 1, 2, \ldots\ ,\ n = 0, 1, \ldots\ . \tag{5}$$

[Thus, $S_{m,0}$ of (5) is simply S_m of (3).] We show that the sum of $S_{m,n}$ too can be expressed in simple terms involving the Riemann Zeta function and trigonometric functions: Using the techniques of Section 2, we show that

$$S_{m,n} = \frac{1-2^{-2n}}{m^{2n+1}}\zeta(2n+1) + \frac{1}{(2n)!}\left(\frac{\pi}{2m}\right)^{2n+1}\sum_{k=1}^{m-1}\left(\frac{d^{2n}}{d\xi^{2n}}\cot\xi\right)\Big|_{\xi=\frac{k\pi}{2m}}, \tag{6}$$

where $\zeta(s)$ is the Riemann Zeta function defined by

$$\zeta(s) = \sum_{k=1}^{\infty} k^{-s}, \quad \Re s > 1. \tag{7}$$

Our numerical experiments seem to indicate that, with $m > 1$, the convergence of $S_{m,n}$ too can be accelerated by the Shanks transformation and the $d^{(m)}$-transformation of Levin and Sidi; other nonlinear methods known at present are not effective.

It would be interesting to know whether the sum of the "companion" series

$$T_{m,n} := \sum_{k=0}^{\infty} \frac{(-1)^{\lfloor k/m \rfloor}}{(k+1)^{2n}}, \quad m,n = 1,2,\dots,$$

can also be expressed in such simple terms. So far, this does not seem to be the case. The convergence of $T_{m,n}$, just as that of $S_{m,n}$, can be accelerated by the Shanks transformation and the Levin–Sidi $d^{(m)}$-transformation; again, other nonlinear methods known so far are not effective.

In Section 4, we present brief descriptions of the $\mathcal{L}$-transformation, the θ-algorithm, the Shanks transformation, and the Levin–Sidi $d^{(m)}$-transformation. Finally, in Section 5, we compare numerically these nonlinear convergence acceleration methods as they are applied to S_m, with $m = 2$ and $m = 3$. Our numerical results indicate that of the two effective methods, the Shanks transformation and the Levin–Sidi $d^{(m)}$-transformation, the latter is the more effective in that it requires less terms of the series to produce a required level of accuracy.

2 Derivation of Eq. (4)

The result given in (4) can be verified very simply for the case $m = 2$ by rewriting S_2 in the form

$$S_2 = \left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right) + \left(\frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \cdots\right), \tag{8}$$

and invoking (1) and $\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} = \arctan 1 = \frac{\pi}{4}$. This splitting technique can be extended to arrive at the result given in (4) for $m \geq 3$. However, we need to use some amount of special function theory for this purpose.

We start by rewriting (3) in the form

$$S_m = \sum_{k=0}^{\infty}(-1)^k \sum_{i=1}^{m} \frac{1}{km+i} = \sum_{i=1}^{m}\sum_{k=0}^{\infty}\frac{(-1)^k}{km+i} = \frac{1}{m}\sum_{i=1}^{m}\beta\left(\frac{i}{m}\right) \tag{9}$$

where the function $\beta(x)$ is defined via (see Gradshteyn and Ryzhik [3, p. 947, formula 8.372.1])

$$\beta(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{x+k}. \tag{10}$$

Note that (9) is a rearrangement of the (conditionally convergent) series S_m that does *not* change the sum of S_m. Now,

$$\beta(1) = \log 2, \quad \beta(x) = \frac{1}{2}\left[\psi\left(\frac{x+1}{2}\right) - \psi\left(\frac{x}{2}\right)\right], \tag{11}$$

where $\psi(z) = \frac{d}{dz}\log\Gamma(z)$ (see [3, p. 947, formula 8.370]). Thus, (9) becomes

$$S_m = \frac{1}{m}\log 2 + \frac{1}{2m}\sum_{i=1}^{m-1}\left[\psi\left(\frac{m+i}{2m}\right) - \psi\left(\frac{i}{2m}\right)\right]. \tag{12}$$

Rewriting the summation on the right-hand side of (12) in the form

$$\begin{aligned}\sum_{i=1}^{m-1}\left[\psi\left(\frac{m+i}{2m}\right) - \psi\left(\frac{i}{2m}\right)\right] &= \sum_{s=1}^{m-1}\psi\left(\frac{2m-s}{2m}\right) - \sum_{i=1}^{m-1}\psi\left(\frac{i}{2m}\right) \\ &= \sum_{k=1}^{m-1}\left[\psi\left(1-\frac{k}{2m}\right) - \psi\left(\frac{k}{2m}\right)\right],\end{aligned} \tag{13}$$

and invoking (see [3, p. 945, formula 8.365.8])

$$\psi(1-z) = \psi(z) + \pi\cot\pi z, \tag{14}$$

we arrive at

$$\sum_{i=1}^{m-1}\left[\psi\left(\frac{m+i}{2m}\right) - \psi\left(\frac{i}{2m}\right)\right] = \pi\sum_{k=1}^{m-1}\cot\frac{k\pi}{2m}. \tag{15}$$

We also have that

$$\sum_{k=1}^{m-1}\cot\frac{k\pi}{2m} = \sum_{s=1}^{m-1}\cot\frac{(m-s)\pi}{2m} = \sum_{s=1}^{m-1}\cot\left(\frac{\pi}{2}-\frac{s\pi}{2m}\right) = \sum_{s=1}^{m-1}\tan\frac{s\pi}{2m}. \tag{16}$$

Combining (13)–(16) in (12), the result in (4) follows.

3 Derivation of Eq. (6)

We now turn to the derivation of the result in (6) pertaining to $S_{m,n}$ with $n = 1, 2, \dots$. Proceeding as in the preceding section, we rewrite $S_{m,n}$ in the form

$$\begin{aligned}
S_{m,n} &= \sum_{k=0}^{\infty}(-1)^k \sum_{i=1}^{m} \frac{1}{(km+i)^{2n+1}} \\
&= \sum_{i=1}^{m}\sum_{k=0}^{\infty} \frac{(-1)^k}{(km+i)^{2n+1}} \\
&= \sum_{i=1}^{m-1}\sum_{k=0}^{\infty} \frac{(-1)^k}{(km+i)^{2n+1}} + \sum_{k=0}^{\infty} \frac{(-1)^k}{(km+m)^{2n+1}} \\
&= \frac{1}{m^{2n+1}} \sum_{i=1}^{m-1}\sum_{k=0}^{\infty} \frac{(-1)^k}{(k+i/m)^{2n+1}} + \frac{1}{m^{2n+1}} \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)^{2n+1}}.
\end{aligned} \tag{17}$$

Denoting

$$\eta(s) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)^s}, \tag{18}$$

and making use of the fact that

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(x+k)^{p+1}} = \frac{(-1)^p}{p!}\beta^{(p)}(x), \quad p = 1, 2, \dots, \tag{19}$$

(see [3, p. 947, formula 8.374]), (17) becomes

$$S_{m,n} = \frac{1}{m^{2n+1}}\eta(2n+1) + \frac{1}{m^{2n+1}(2n)!} \sum_{i=1}^{m-1} \beta^{(2n)}\left(\frac{i}{m}\right). \tag{20}$$

Invoking (11), we obtain

$$\begin{aligned}
\sum_{i=1}^{m-1} \beta^{(2n)}\left(\frac{i}{m}\right) &= \frac{1}{2}\sum_{i=1}^{m-1} \frac{d^{2n}}{dx^{2n}}\left[\psi\left(\frac{x+1}{2}\right) - \psi\left(\frac{x}{2}\right)\right]\Bigg|_{x=\frac{i}{m}} \\
&= \frac{1}{2^{2n+1}} \sum_{i=1}^{m-1} \left[\psi^{(2n)}\left(\frac{m+i}{2m}\right) - \psi^{(2n)}\left(\frac{i}{2m}\right)\right] \\
&= \frac{1}{2^{2n+1}} \sum_{k=1}^{m-1} \left[\psi^{(2n)}\left(1-\frac{k}{2m}\right) - \psi^{(2n)}\left(\frac{k}{2m}\right)\right] \\
&= \frac{1}{2^{2n+1}} \sum_{k=1}^{m-1} \left(\frac{d^{2n}}{dz^{2n}}\left[\psi(1-z) - \psi(z)\right]\right)\Bigg|_{z=\frac{k}{2m}}.
\end{aligned} \tag{21}$$

Note that the third equality in (21) is obtained in the same way (13) is obtained. The fourth equality in (21) is made possible by the fact that $2n$ is an *even* integer. (In case of $T_{m,n}$, this is not possible, because $\beta^{(2n)}$ is replaced by $\beta^{(2n-1)}$ and $2n-1$ is an *odd* integer. As a result, a simple expression for the sum of $T_{m,n}$ does not seem to be possible.) Invoking now (14), we get

$$\sum_{i=1}^{m-1} \beta^{(2n)}\left(\frac{i}{m}\right) = \frac{\pi}{2^{2n+1}} \sum_{k=1}^{m-1} \left(\frac{d^{2n}}{dz^{2n}} \cot \pi z\right)\bigg|_{z=\frac{k}{2m}} = \left(\frac{\pi}{2}\right)^{2n+1} \sum_{k=1}^{m-1} \left(\frac{d^{2n}}{d\xi^{2n}} \cot \xi\right)\bigg|_{\xi=\frac{k\pi}{2m}}. \tag{22}$$

Combining (18) and (22) in (20), and invoking

$$\eta(s) = (1-2^{1-s})\zeta(s), \tag{23}$$

the result in (6) follows.

4 Application of Convergence Acceleration Methods

The convergence of the (slowly converging) series S_m, $S_{m,n}$, and $T_{m,n}$ can be accelerated by applying to them nonlinear convergence acceleration methods. The methods that are known for their especially good acceleration properties and that we have tested here are the Levin $\mathcal{L}$-transformation, the Brezinski θ-algorithm, the Shanks transformation, and the Levin–Sidi $d^{(m)}$-transformation. For the sake of completeness, we recall the definitions of these methods here. For more information and recent results on these methods, see [9].

Below, for the $\mathcal{L}$- and $d^{(m)}$-transformations, we let $\sum_{k=1}^{\infty} a_k$ be the series whose convergence is to be accelerated, and $A_n = \sum_{k=1}^{n} a_k$, $n = 1, 2, \ldots$. In keeping with convention, for the Shanks transformation and the θ-algorithm, we let $\sum_{k=0}^{\infty} a_k$ be the series whose convergence is to be accelerated, and $A_n = \sum_{k=0}^{n} a_k$, $n = 0, 1, \ldots$. We also define the forward difference operator Δ such that $\Delta c_j = c_{j+1} - c_j$ for all j.

$\mathcal{L}$-transformation. Letting $\omega_r = r a_r$, we define $\mathcal{L}_n^{(j)}$ via the linear equations

$$A_r = \mathcal{L}_n^{(j)} + \omega_r \sum_{i=0}^{n-1} \frac{\bar{\beta}_i}{r^i}, \quad J \le r \le J+n; \quad J = j+1.$$

Here $\mathcal{L}_n^{(j)}$ is the approximation to the sum of the series and the $\bar{\beta}_i$ are auxiliary unknowns. (With the present ω_r, this is also known as the u-transformation.) Then, $\mathcal{L}_n^{(j)}$ is given by the closed-form expression

$$\mathcal{L}_n^{(j)} = \frac{\Delta^n \left(J^{n-1} A_J/\omega_J\right)}{\Delta^n \left(J^{n-1}/\omega_J\right)} = \frac{\sum_{i=0}^{n}(-1)^i \binom{n}{i}(J+i)^{n-1} A_{J+i}/\omega_{J+i}}{\sum_{i=0}^{n}(-1)^i \binom{n}{i}(J+i)^{n-1}/\omega_{J+i}}; \quad J = j+1.$$

Note that $\mathcal{L}_n^{(j)}$ is determined by the terms A_i, $j+1 \le i \le j+n+1$. Also, it is known that $\mathcal{L}_2^{(j)}$ is the approximation produced by the Lubkin W-transformation. The "diagonal" sequences $\{\mathcal{L}_n^{(j)}\}_{n=1}^{\infty}$ (j fixed) have the best convergence properties.

θ-algorithm. This method is defined via the following recursive scheme.

$$\theta_{-1}^{(j)} = 0, \quad \theta_0^{(j)} = A_j, \quad j \geq 0;$$
$$\theta_{2n+1}^{(j)} = \theta_{2n-1}^{(j+1)} + D_{2n}^{(j)}; \quad D_k^{(j)} = 1/\Delta\theta_k^{(j)} \quad \text{for all } j, k \geq 0,$$
$$\theta_{2n+2}^{(j)} = \theta_{2n}^{(j+1)} - \frac{\Delta\theta_{2n}^{(j+1)}}{\Delta D_{2n+1}^{(j)}} D_{2n+1}^{(j)}, \quad j, n \geq 0.$$

Note that the operator Δ operates only on the upper index, namely, on j. Here, the relevant quantities (i.e., the approximations to the sum of the series) are the $\theta_{2n}^{(j)}$. Note that $\theta_{2n}^{(j)}$ is determined by A_i, $j \leq i \leq j + 3n$. Also, it is known that $\theta_2^{(j)}$ is the approximation produced by the Lubkin W-transformation. The "diagonal" sequences $\{\theta_{2n}^{(j)}\}_{n=0}^{\infty}$ (j fixed) have the best convergence properties.

Shanks transformation. This method is defined via the linear equations

$$A_r = e_n(A_j) + \sum_{k=1}^{n} \bar{\alpha}_k \Delta A_{r+k-1}, \quad j \leq r \leq j + n.$$

Here $e_n(A_j)$ is the approximation to the sum of the series and the $\bar{\alpha}_k$ are auxiliary unknowns. The $e_n(A_j)$ can be obtained recursively with the help of the ε-algorithm of Wynn [12] as follows:

$$\varepsilon_{-1}^{(j)} = 0, \quad \varepsilon_0^{(j)} = A_j, \quad j \geq 0; \quad \varepsilon_{k+1}^{(j)} = \varepsilon_{k-1}^{(j+1)} + \frac{1}{\varepsilon_k^{(j+1)} - \varepsilon_k^{(j)}}, \quad j, k \geq 0.$$

Then, $e_n(A_j) = \varepsilon_{2n}^{(j)}$ for all j and n. Another algorithm that is as efficient as the ε-algorithm is the recent FS/qd-algorithm of the author given in [9, Chapter 21]. Note that $e_n(A_j) = \varepsilon_{2n}^{(j)}$ is determined by A_i, $j \leq i \leq j + 2n$. The "diagonal" sequences $\{\varepsilon_{2n}^{(j)}\}_{n=0}^{\infty}$ (j fixed) have the best convergence properties.

$d^{(m)}$-transformation. Pick integers R_l such that $1 \leq R_0 < R_1 < R_2 < \cdots$, let $n = (n_1, n_2, \ldots, n_m)$, and define $d_n^{(m,j)}$ through

$$A_{R_l} = d_n^{(m,j)} + \sum_{k=1}^{m} R_l^k (\Delta^{k-1} a_{R_l}) \sum_{i=0}^{n_k - 1} \frac{\bar{\beta}_{ki}}{R_l^i}, \quad j \leq l \leq j + N; \quad N = \sum_{k=1}^{m} n_k.$$

Here, $d_n^{(m,j)}$ is the approximation to A, while the $\bar{\beta}_{ki}$ are auxiliary unknowns. Note that $d_n^{(m,j)}$ is determined by A_i, $R_j \leq i \leq m + R_{j+N}$. The "diagonal" sequences $\{d_{(\nu,\nu,\ldots,\nu)}^{(m,j)}\}_{\nu=1}^{\infty}$ (j fixed) have the best convergence properties. The simplest and obvious choice for the R_l is $R_l = l + 1$, $l = 0, 1, \ldots$. It is important to note that the R_l can be fixed to achieve best possible convergence acceleration and numerical stability in a systematic way. See [9, Sections 6.5 and 12.7].

When $m = 1$, the $d_n^{(m,j)}$ can be computed recursively via the W-algorithm of Sidi [8] (not to be confused with the Lubkin W-transformation). The W-algorithm (with the notation $A_n^{(j)} \equiv d_n^{(1,j)}$) reads as follows:

(i) Set

$$M_0^{(j)} = \frac{A_{R_j}}{R_j\, a_{R_j}}, \quad N_0^{(j)} = \frac{1}{R_j\, a_{R_j}}, \quad j \geq 0.$$

(ii) Compute $M_n^{(j)}$ and $N_n^{(j)}$ recursively from

$$M_n^{(j)} = \frac{M_{n-1}^{(j+1)} - M_{n-1}^{(j)}}{R_{j+n}^{-1} - R_j^{-1}}, \quad N_n^{(j)} = \frac{N_{n-1}^{(j+1)} - N_{n-1}^{(j)}}{R_{j+n}^{-1} - R_j^{-1}}, \quad j \geq 0,\ n \geq 1.$$

(iii) Set

$$A_n^{(j)} = \frac{M_n^{(j)}}{N_n^{(j)}}, \quad j, n \geq 0.$$

When $m > 1$, the $d_{(\nu,\nu,\dots,\nu)}^{(m,j)}$ can be computed via the W$^{(m)}$-algorithm of Ford and Sidi [2]; we refer the reader to [2] or to [9, Chapter 7] for details. A FORTRAN 77 code that implements the $d^{(m)}$-transformation with the help of the W$^{(m)}$-algorithm is given in [9, Appendix I]; it can also be obtained from the author via e-mail.

When $R_l = l+1,\ l = 0, 1, \dots$, and $m = 1$, the $d^{(m)}$-transformation reduces to the $\mathcal{L}$-transformation. This shows that the $\mathcal{L}$-transformation can be implemented recursively via the W-algorithm.

5 Numerical Example

We have applied the transformations mentioned above to the series S_m, $S_{m,n}$, and $T_{m,n}$. We have used the ε-algorithm to implement the Shanks transformation, and the W$^{(m)}$-algorithm to implement the $d^{(m)}$-transformation. (Note that the θ-algorithm is already defined via a recursive procedure.) Our numerical results show that the $\mathcal{L}$-transformation and the θ-algorithm are effective accelerators only when $m = 1$; they do not produce any acceleration when $m > 1$. The Shanks transformation and the $d^{(m)}$-transformation are very effective for all m. The $d^{(m)}$-transformation seems to be more effective in that it uses a smaller number of sequence elements to produce a required level of accuracy.

Table 1 contains the results obtained in quadruple precision (approximately 35 decimal digits) for the series S_m with $m = 2$ and $m = 3$. In our computations with the $d^{(m)}$-transformation, we have chosen $R_l = l + 1,\ l = 0, 1, \dots$. Note that, in Table 1, we compare $d_{(4k,4k)}^{(2,0)}$ with $\varepsilon_{8k}^{(0)}$, because they both use about the same number of terms of S_2 (approximately $8k$ terms). Similarly, we compare $d_{(4k,4k,4k)}^{(3,0)}$ with $\varepsilon_{12k}^{(0)}$, because they both use about the same number of terms of S_3 (approximately $12k$ terms).

It is interesting to note that, for each k, the approximations $d_{(4k,4k)}^{(2,0)}$ and $d_{(4k,4k,4k)}^{(3,0)}$ have comparable accuracies and so do the approximations $\varepsilon_{8k}^{(0)}$ and $\varepsilon_{12k}^{(0)}$.

k	$\lvert d^{(2,0)}_{(4k,4k)} - S_2\rvert$	$\lvert \varepsilon^{(0)}_{8k} - S_2\rvert$	$\lvert d^{(3,0)}_{(4k,4k,4k)} - S_3\rvert$	$\lvert \varepsilon^{(0)}_{12k} - S_3\rvert$
1	5.414(−5)	2.504(−4)	9.420(−5)	7.459(−4)
2	2.330(−10)	2.512(−7)	5.561(−10)	7.379(−7)
3	1.619(−15)	2.332(−10)	2.735(−16)	6.827(−10)
4	4.590(−20)	2.108(−13)	5.448(−20)	6.164(−13)
5	4.210(−25)	1.880(−16)	7.413(−25)	5.498(−16)
6	4.414(−31)	1.665(−19)	3.681(−30)	4.870(−19)
7	7.704(−34)	1.467(−22)	7.704(−34)	4.294(−22)
8	*	1.288(−25)	*	3.775(−25)
9	*	1.129(−28)	*	3.310(−28)
10	*	9.929(−32)	*	2.286(−31)

Table 1: Results obtained by applying the $d^{(m)}$-transformation and the Shanks transformation to the series S_m with $m = 2$ and $m = 3$.

Concluding Remarks

In this work, we have considered the slowly converging series S_m given in (3) and $S_{m,n}$ given in (5). These series, because of their special sign patterns, are challenging test cases for convergence acceleration methods; in fact, most convergence acceleration methods fail to produce anything meaningful when applied to these series. We have derived the exact sums of S_m and $S_{m,n}$ in simple terms and have also considered their summation numerically via convergence acceleration methods. Based on numerical evidence, we have concluded that, of the nonlinear convergence acceleration methods known at present, the Shanks transformation and the Levin–Sidi $d^{(m)}$-transformation are the only effective summation methods and that the $d^{(m)}$-transformation is the more effective of the two. Other nonlinear methods we have tried have not improved the convergence of S_m and $S_{m,n}$.

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