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Fredholm Property For A Parameter Dependent Second Order Operator Differential Equation*

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Abstract

In this paper we consider an abstract elliptic differential problem where the equation is quadratically parameter-dependent and the boundary conditions may contain a spectral parameter also. We establish a Fredholm property for the operator generated by our parameter-dependent abstract differential equation. The results obtained are applied to study some elliptic problems.

1 Introduction

Regular boundary value problems for elliptic partial differential equations with a spectral parameter have been studied by many authors [2, 1, 13, 16]. Such parameter may appear in both the equation and the boundary conditions. So, in the same way, the coerciveness estimate has been established.

However, non-regular problems, not satisfying Sapiro-Lopatinski conditions, are less studied. In the papers and monographs [3, 4, 5, 7, 13, 15, 16] sufficient conditions for coerciveness estimate to hold are given. We quote in particular [16], where a number of such problems is considered.

In this paper we consider a boundary value problem for an abstract differential equation which may depend quadratically on a spectral parameter, the boundary conditions contain a linear operator and a spectral parameter. Moreover, we prove that the operator generated by our problem is a Fredholm between appropriate spaces. Then, we apply the abstract results to some boundary value problems for elliptic partial differential equations in a cylinder.

More precisely, in section 2, we give some background preliminaries. The principal boundary value problem for abstract differential equations is studied in section 3. We prove the Fredholm property for the operator generated by our problem. In section 4, we apply the obtained abstract results to some boundary value problems for elliptic partial differential equations in a cylinder.

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2 Preliminaries

Let H be a Hilbert space, A a linear closed operator in H and D_A its domain. We denote by $B(H)$ the space of bounded operators acting in H, endowed with the usual operator norm, and by $L_p(0,1;H)$ the Banach space of strongly measurable functions $x \longrightarrow u(x) : (0,1) \longrightarrow H$, whose p-th power norms, $\|u\|_{0,p}^p = \int_0^1 \|u(x)\|_H^p \, dx, p \in (1,\infty)$, are summable.

The vector-valued Sobolev space is defined as

$$W_p^n\left(0,1;H\left(A^2\right),H\right) = \left\{u : A^2 u \in L_p(0,1;H);\ u^{(n)} \in L_p(0,1;H)\right\},$$

the norm in this space is given by

$$\|u\|_{W_p^n(0,1;H(A^2),H)} = \left\|A^2 u\right\|_{L_p(0,1;H)} + \left\|u^{(n)}\right\|_{L_p(0,1;H)}.$$

The space $H(A)$ is defined by

$$H(A) = \left\{u \in D_A; \|u\|_{H(A)}^2 = \|u\|_H^2 + \|Au\|_H^2 < \infty\right\}$$

and it is precisely the domain of A equipped with the Hilbertian graph norm.

Let $-A$ be the generator of the analytic semigroup e^{-tA} for $t > 0$, decreasing at infinity, and strongly continuous for $t \geq 0$. We define the interpolation space [12]

$$(H, H(A^m))_{\theta,p} = \left\{u \in H; \|u\|_{m,\theta}^p = \int_0^\infty t^{m(1-\theta)p-1} \left\|A^m e^{-tA} u\right\|_H^p dt < \infty\right\}$$

where $0 < \theta < 1$, $m \in \mathbb{N}$, $1 \leq p < \infty$ and $\|\cdot\|_{m,\theta}$ is the norm in $(H, H(A^m))_{\theta,p}$.

Let us also define the following interpolation spaces.

DEFINITION 1 ([12]). We denote by $B_{p,q}^s(\Omega)$ where Ω is a regular domain of $\mathbb{R}^n$ the space

$$B_{p,q}^s(\Omega) = \left(W_p^{s_0}(\Omega), W_p^{s_1}(\Omega)\right)_{\theta,q}$$

where $0 \leq s_0, s_1$ are integers, $0 < \theta < 1$, $1 < p < \infty$, $1 \leq q \leq \infty$ and $s = (1-\theta)s_0 + \theta s_1$.

3 Fredholm Property

Consider, in $L_p(0,1;H)$, the following boundary value problem

$$\begin{cases} L(\lambda)u = -\lambda^2 u + u'' - Au = f,\ x \in (0,1) \\ L_1(\lambda)u = \delta u'(0) + \lambda u(0) = f_1 \\ L_2(\lambda)u = u'(1) + Bu(0) = f_2 \end{cases} \tag{1}$$

A is a positive operator with a compact resolvent, B is a continuous operator, $f \in L_p(0,1;H)$, $\delta \in \mathbb{C}$, and $f_1, f_2 \in (H, H(A))_{\frac{1}{2}-\frac{1}{2p},p}$.

Let us introduce the following operator

$$\mathcal{L}(\lambda) : u \to (L(\lambda), L_1(\lambda), L_2(\lambda))$$

from $W_p^2(0,1;H(A),H)$ to $L_p(0,1;H) \oplus (H,H(A))_{\frac{1}{2}-\frac{1}{2p},p} \oplus (H,H(A))_{\frac{1}{2}-\frac{1}{2p},p}$

We shall show that this operator is a Fredholm between the above mentioned spaces.

THEOREM 1. Suppose that

- A is a closed, positive and densely defined linear operator on H,
- B is a linear continuous from $H(A^{\frac{1}{2}})$ into H and from $H(A)$ into $H(A^{\frac{1}{2}})$,
- the injection $H(A) \subset H$ is compact,
- $\exists \epsilon \in (0, \frac{\pi}{2})$ such that $\delta \neq 0$ and $|\arg \frac{\delta - 1}{\delta}| < \epsilon$.

Then for λ such that $|\arg \lambda| \leq \frac{\pi}{2} - \epsilon$ and $|\lambda|$ great enough, the operator $\mathcal{L}(\lambda)$ is a Fredholm from $W_p^2(0,1;H(A),H)$ into $L_p(0,1;H) \oplus (H,H(A))_{\frac{1}{2}-\frac{1}{2p},p} \oplus (H,H(A))_{\frac{1}{2}-\frac{1}{2p},p}$.

To establish the main result let us consider, in $L_p(0,1;H)$, the following auxiliary boundary value problem

$$\begin{cases} L_0(\lambda)u = u'' - (T + \lambda I)^2 u = f, \ x \in (0,1) \\ L_{10}(\lambda)u = \delta u'(0) + \lambda u(0) = f_1 \\ L_{20}(\lambda)u = u'(1) + Bu(0) = f_2. \end{cases} \tag{2}$$

We have the following coercive estimate.

THEOREM 2 ([3]). Suppose that

- T is a closed, densely defined linear operator on H and $\|R(\mu, T)\| \leq c(1 + |\mu|)^{-1}$ for $|\arg \mu| \geq \frac{\pi}{2}$ and $\Re \mu \mapsto \infty$,
- B is a linear continuous from $H(T)$ into H and from $H(T^2)$ into $H(T)$,
- $\exists \epsilon \in (0, \frac{\pi}{2})$ such that $\delta \neq 0$ and $|\arg \frac{\delta - 1}{\delta}| \leq \epsilon$.

Then for λ such that $|\arg \lambda| \leq \frac{\pi}{2} - \epsilon$ and $|\lambda|$ sufficiently large, the operator $\mathcal{L}_0(\lambda) : u \mapsto (L_0(\lambda)u, L_{10}(\lambda)u, L_{20}(\lambda)u)$ is an isomorphism from $W_p^2(0,1;H(T^2),H)$ onto $L_p(0,1;H) \oplus (H,H(T))_{1-\frac{1}{p},p} \oplus (H,H(T))_{1-\frac{1}{p},p}$. Furthermore, for these λ, the following coerciveness estimate holds true.

$$\begin{aligned} & \|u''\|_{0,p} + \|T^2 u\|_{0,p} + |\lambda|^2 \|u\|_{0,p} \\ \leq \ & C \left(\|f\|_{0,p} + \|f_1\|_{(H,H(T))_{1-\frac{1}{p},p}} + \|f_2\|_{(H,H(T))_{1-\frac{1}{p},p}} + |\lambda|^{1-\frac{1}{p}} \|f_1\|_H + |\lambda|^{1-\frac{1}{p}} \|f_2\|_H \right) \end{aligned}$$

PROOF of Theorem 1. Let us write the operator $\mathcal{L}(\lambda) = \mathcal{L}_0(\lambda) + \mathcal{L}_1(\lambda)$ where

$$\mathcal{L}_0(\lambda) : u \mapsto \left(u'' - (A^{\frac{1}{2}} + \lambda I)^2 u, \delta u'(0) + \lambda u(0), u'(1) + Bu(0) \right)$$

from $W_p^2(0,1;H(A),H)$ to

$$L_p(0,1;H) \oplus (H,H(A))_{\frac{1}{2}-\frac{1}{2p},p} \oplus (H,H(A))_{\frac{1}{2}-\frac{1}{2p},p},$$

and

$$\mathcal{L}_1(\lambda) : u \mapsto \left(2\lambda A^{\frac{1}{2}}u, 0, 0\right)$$

from $W_p^2(0,1;H(A),H)$ to

$$L_p(0,1;H) \oplus (H,H(A))_{\frac{1}{2}-\frac{1}{2p},p} \oplus (H,H(A))_{\frac{1}{2}-\frac{1}{2p},p}.$$

To prove that $\mathcal{L}_0(\lambda)$ is an isomorphism, we remark that $\mathcal{L}_0(\lambda)$ is the operator generated by the auxiliary problem (2) when $T = A^{\frac{1}{2}}$, hence it suffices to check that $A^{\frac{1}{2}}$ satisfies the conditions fulfilled by the operator T, which follows immediately from the properties of the operator A. Then $\mathcal{L}_0(\lambda)$ is a Fredholm operator.

$\mathcal{L}_1(\lambda)$ is compact due to the condition 3 of Theorem 1, then $\mathcal{L}_0(\lambda) + \mathcal{L}_1(\lambda)$ is a Fredholm as a perturbation of a Fredholm by a compact operator.

4 Applications

Consider, in $L_p([0,1]\times[0,1])$, the following boundary value problem

$$\begin{cases} -\Delta u + \lambda^2 u = f,\ (x,y) \in [0,1]\times[0,1] \\ \delta \frac{\partial}{\partial x}u(0,y) + \lambda u(0,y) = f_1(y);\ y \in [0,1] \\ \frac{\partial}{\partial x}u(1,y) + b(y)\frac{\partial}{\partial y}u(0,y) = f_2(y);\ y \in [0,1] \\ u(x,0) = u(x,1) = 0,\ x \in [0,1] \end{cases} \tag{3}$$

$f \in L_p(0,1;L_2(0,1))$; $f_1, f_2 \in B_{2,p}^{2-\frac{2}{p}}(0,1) \cap H_0^1(0,1)$, for $p > 2$.

Let us introduce the following operator

$$\mathcal{P}_1(\lambda) : u \to (f, f_1, f_2)$$

from $W_p^2(0,1;W_2^2(0,1)\cap H_0^1(0,1), L_2(0,1))$ into $L_p(0,1;L_2(0,1)) \oplus B_{2,p}^{2-\frac{2}{p}}(0,1)\cap H_0^1(0,1) \oplus B_{2,p}^{2-\frac{2}{p}}(0,1) \cap H_0^1(0,1)$

THEOREM 3. Assume that for $p > 2$ we have $f \in L_p(0,1;L_2(0,1)), f_1, f_2 \in B_{2,p}^{2-\frac{2}{p}}(0,1) \cap H_0^1(0,1)$. Moreover, suppose that there exists $\epsilon \in (0,\frac{\pi}{2})$ such that $\delta \neq 0$ and $|\arg \frac{\delta-1}{\delta}| \leq \epsilon$. Then for λ such that $|\arg\lambda| \leq \frac{\pi}{2} - \epsilon$ and $|\lambda|$ sufficiently large, the operator $\mathcal{P}_1(\lambda)$ is a Fredholm operator.

PROOF. Set $H = L_2(0,1),\ H(A) = W_2^2(0,1) \cap H_0^1(0,1),\ Au = -\frac{d^2u}{dy^2}$ and $B = b(y)\frac{d}{dy}$. We have, from [12],

$$(H,H(A))_{\theta,p} = (L_p(0,1), W_2^2(0,1)\cap H_0^1(0,1))_{\theta,p} = \begin{cases} B_{2,p}^{2\theta}(0,1) \cap H_0^1(0,1) \text{ if } \theta < \frac{1}{4} \\ W_2^{\frac{1}{2}}(0,1) \text{ if } \theta = \frac{1}{4} \\ B_{2,p}^{2\theta}(0,1) \text{ if } \theta > \frac{1}{4}. \end{cases}$$

Hence, in our case, we obtain

$$(H, H(A))_{\frac{1}{2}-\frac{1}{2p},p} = \begin{cases} B_{2,p}^{1-\frac{1}{p}}(0,1) \cap H_0^1(0,1) \text{ if } p > 2 \\ W_2^{\frac{1}{2}}(0,1) \text{ if } p = 2 \\ B_{2,p}^{1-\frac{1}{p}}(0,1) \text{ if } p < 2. \end{cases}$$

The problem (3) can be re-written in the form

$$\begin{cases} -u''(x) + Au(x) + \lambda^2 u(x) = f(x), \\ \delta u'(0) + \lambda u(0) = f_1 \\ u'(1) + Bu(0) = f_2. \end{cases} \tag{4}$$

It is clear that the operator A satisfies the condition 1 of Theorem 1, and the operator B satisfies the condition 2 of the same theorem. Then we obtain the Fredholm property for our problem, from the abstract result.

Consider, in $L_p([0,1] \times G)$, the following boundary value problem, G is a regular bounded domain of $\mathbb{R}^n$,

$$\begin{cases} -\Delta u + \lambda^2 u = f, \ (x,y) \in [0,1] \times G \\ \delta \frac{\partial}{\partial x} u(0,y) + \lambda u(0,y) = f_1(y); \ y \in G \\ \frac{\partial}{\partial x} u(1,y) + b(y) \frac{\partial}{\partial y} u(0,y) = f_2(y); \ y \in G \\ u(x,y) = 0, \ x \in [0,1], y \in \partial G \end{cases} \tag{5}$$

$f \in L_p(0,1; L_2(G))$; $f_1, f_2 \in B_{2,p}^{2-\frac{2}{p}}(G) \cap H_0^1(G)$, for $p > 2$.

Let us introduce the following operator

$$\mathcal{P}_2(\lambda) : u \to (f, f_1, f_2)$$

from $W_p^2(0,1; W_2^2(G) \cap H_0^1(G), L_2(G))$ into $L_p(0,1; L_2(G)) \oplus B_{2,p}^{2-\frac{2}{p}}(G) \cap H_0^1(G) \oplus B_{2,p}^{2-\frac{2}{p}}(G) \cap H_0^1(G)$

THEOREM 4. Assume that for $p > 2$ we have $f \in L_p(0,1; L_2(G)), f_1, f_2 \in B_{2,p}^{2-\frac{2}{p}}(G) \cap H_0^1(G)$. Moreover, suppose that there exists $\epsilon \in (0, \frac{\pi}{2})$ such that $\delta \neq 0$ and $|\arg \frac{\delta - 1}{\delta}| \leq \epsilon$. Then for λ such that $|\arg \lambda| \leq \frac{\pi}{2} - \epsilon$ and $|\lambda|$ great enough, the operator $\mathcal{P}_2(\lambda)$ is a Fredholm operator.

PROOF. Set $H = L_2(G)$, $H(A) = W_2^2(G) \cap H_0^1(G)$, $Au = -\frac{d^2 u}{dy^2}$ and $B = b(y) \frac{d}{dy}$. We have

$$(H, H(A))_{\frac{1}{2}-\frac{1}{2p},p} = \begin{cases} B_{2,p}^{1-\frac{1}{p}}(G) \cap H_0^1(G) \text{ if } p > 2 \\ W_2^{\frac{1}{2}}(G) \text{ if } p = 2 \\ B_{2,p}^{1-\frac{1}{p}}(G) \text{ if } p < 2. \end{cases}$$

The problem (5) can be re-written in the form

$$\begin{cases} -u''(x) + Au(x) + \lambda^2 u(x) = f(x), \\ \delta u'(0) + \lambda u(0) = f_1 \\ u'(1) + Bu(0) = f_2. \end{cases}$$

It is clear that the operator A satisfies the condition 1 of Theorem 1, and the operator B satisfies the condition 2 of the same theorem. Then we obtain the Fredholm property for our problem, from the abstract result.

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