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An Upper Bound Of A Function With Two Independent Variables*

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Abstract

An upper bound for a function with two independent variables is obtained.

1 Introduction

The following terminology was explicitly introduced in [7], formally published in [6], immediately studied or cited by [2, 3, 9, 10, 11]: A function f is said to be logarithmically completely monotonic on an interval I if f has derivatives of all orders on I and its logarithm $\ln f$ satisfies $(-1)^k[\ln f(x)]^{(k)} \geq 0$ for all $k \in \mathbb{N}$ on I. Recently, it is pointed out that this notion has appeared in [1] without definition.

For our own convenience, let $\mathcal{L}[I]$ stand for the set of all logarithmically completely monotonic functions on I. Among other things, it is proved in [2, 6, 7, 13] that a logarithmically completely monotonic function is always completely monotonic, that is, $\mathcal{L}[I] \subset \mathcal{C}[I]$, but not conversely, where $\mathcal{C}[I]$ denotes the set of all completely monotonic functions on I. Further, it is shown in [2] that $\mathcal{S} \setminus \{0\} \subset \mathcal{L}[(0,\infty)] \subset \mathcal{C}[(0,\infty)]$, where $\mathcal{S}$ denotes the set of all Stieltjes transforms. In [2, Theorem 1.1] and [3, 9] it is pointed out that the logarithmically completely monotonic functions on $(0,\infty)$ can be characterized as the infinitely divisible completely monotonic functions investigated by Horn in [4, Theorem 4.4]. In [8], among other things, the following basic property of the logarithmically completely monotonic functions is obtained: If $h'(x) \in \mathcal{C}[I]$ and $f(x) \in \mathcal{L}[h(I)]$, then $f(h(x)) \in \mathcal{L}[I]$. For more related information, please refer to [5] and the references therein.

Let Γ denote the classical Euler's gamma function. Let

$$\tau(s,t) = \frac{1}{s}\left[t - (t+s+1)\left(\frac{t}{t+1}\right)^{s+1}\right] \tag{1}$$

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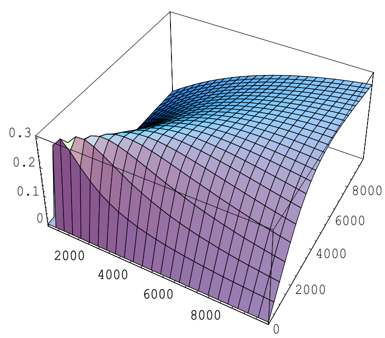
for $(s,t) \in (0,\infty) \times (0,\infty)$, and let $\tau_0 = \tau(s_0,t_0)$ be the maximum of $\tau(s,t)$ on the set $\mathbb{N} \times (0,\infty)$. In [7, 8], it was proved that $\tau(s,t) > 0$ by using the well known Bernoulli's inequality and, for any given real number α satisfying $\alpha \le \frac{1}{1+\tau_0}$, the function $\frac{(x+1)^\alpha}{[\Gamma(x+1)]^{1/x}} \in \mathcal{L}[(-1,\infty)]$.

It is clear that $\lim_{t\to 0+} \tau(s,t) = 0$ for any $s \in (0,\infty)$. Now it is natural to ask for the maximum of $\tau(s,t)$ on $(0,\infty)\times(0,\infty)$. To the best of our knowledge, it is not easy and trivial to give an upper bound for $\tau(s,t)$ in $(0,\infty)\times(0,\infty)$. However, by using a novel approach, an endeavor was made in [12] and an upper bound of $\tau(s,t)$ was obtained: $\tau(s,t) < 1$.

The numerical calculation of $\tau(s,t)$ can be carried out by the well known software MATHEMATICA easily. However, it is believed that an accurate upper bound or the maximum of $\tau(s,t)$ cannot be found by numerical method, since the domain of (s,t) is an infinite region. A plot below and a numerical computation by the MATHEMATICA version 5.2 reveals that the maximum of the function $\tau(s,t)$ should be less than $\frac{3}{10}$.

In[3]:= $\texttt{Plot3D}\left[\frac{1}{s}\left(t-(t+s+1)\left(\frac{t}{t+1}\right)^{s+1}\right), \{s, 0.000000001, 9999\}, \{t, 0, 9999\}\right]$



Out[3]= - SurfaceGraphics -

In this short note, as a subsequence of [12], we shall give a more accurate upper bound for the function $\tau(s,t)$ on $(0,\infty)\times(0,\infty)$. Our main result is

THEOREM 1. For $(s,t) \in (0,\infty)\times(0,\infty)$, we have $0 < \tau(s,t) < \frac{3}{10}$.

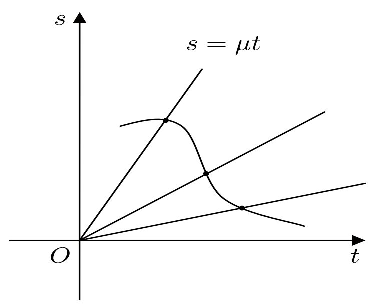
REMARK 1. The proof of Theorem 1 is dependent on an improved upper bound of the function $\Psi(x) = \frac{1}{x}\left(1 - \frac{1+x}{e^x}\right)$ defined for $x \in (0,\infty)$ by (9) below. It is noted that the upper bound for the function $\Psi(x)$ can be further improved by numerical method, theoretically or practically. Since $\Psi'(x) = \frac{1+x+x^2-e^x}{x^2e^x}$, it is easy to obtain by the MATHEMATICA version 5.2 numerically the unique root $x_0 = 1.79328213290076\cdots$ of equation $e^x = 1+x+x^2$ and $\Psi(x) \le 0.2984256075256390\cdots$ for $x \in (0,\infty)$. So, we would like to pose an open problem: Can one find a best possible upper bound or show that the maximum is less than $\frac{3}{10}$ for the function $\tau(s,t)$ on the domain $(0,\infty)\times(0,\infty)$ by non-numerical method? Here we wish to obtain a "nice" upper bound for the considered function.

2 Proof

Let $s=\mu t$ for $\mu\in(0,\infty)$ and $t\in(0,\infty)$. Then we have

$$\tau(\mu t,t)=\frac{1}{\mu}\left[1-\frac{(\mu+1)t+1}{1+t}\left(\frac{t}{1+t}\right)^{\mu t}\right]\triangleq\frac{1}{\mu}[1-q_\mu(t)], \tag{2}$$

$$\begin{aligned}q'_\mu(t)&=\mu\left(\frac{t}{1+t}\right)^{\mu t}\frac{2+t+\mu t+(1+t)(1+t+\mu t)\ln\frac{t}{1+t}}{(1+t)^2}\\&\triangleq\frac{\mu p_\mu(t)}{(1+t)^2}\left(\frac{t}{1+t}\right)^{\mu t}.\end{aligned} \tag{3}$$



Let

$$\phi_\mu(x)=\ln(1+x)-\frac{x}{1+x}-\frac{x^2}{(1+x)(x+\mu+1)} \tag{4}$$

for $x\ge 0$. Then we have

$$\phi'_\mu(x)=\frac{x(x^2+\mu x+\mu^2-1)}{(x+\mu+1)^2(1+x)^2}\triangleq\frac{xg_\mu(x)}{(x+\mu+1)^2(1+x)^2}. \tag{5}$$

It is clear that if $\mu\ge 1$ then $g_\mu(x)>0$ in $(0,\infty)$. As a result, we have $\phi'_\mu(x)>0$, and then $\phi_\mu(x)$ is strictly increasing in $(0,\infty)$. Since $\phi_\mu(0)=0$, it follows that $\phi_\mu(x)>0$ in $(0,\infty)$ for any given $\mu>0$.

When $0<\mu<1$, the function $g_\mu(x)$ has a unique positive zero point $x_0=\left(\sqrt{4-3\mu^2}-\mu\right)/2$, and then $g_\mu(x)$ is negative in $(0,x_0)$ and positive in (x_0,∞). This means that the function $\phi'_\mu(x)$ is negative in $(0,x_0)$ and positive in (x_0,∞), that is, $\phi_\mu(x)$ is strictly decreasing in $(0,x_0)$ and strictly increasing in (x_0,∞). Since $\phi_\mu(0)=0$, we have $\phi_\mu(x)<0$ in $(0,x_0)$. Since $\lim_{x\to\infty}\phi_\mu(x)=\infty$, then there exists a unique point $x_1\in(x_0,\infty)$, which is dependent on μ, such that $\phi_\mu(x)<0$ in $(0,x_1)$ and $\phi_\mu(x)>0$ in (x_1,∞).

Let $x=\frac{1}{t}$. Then $\phi_\mu(x)>0$ is equivalent to

$$p_\mu(t)=2+(\mu+1)t+(1+t)[(\mu+1)t+1]\ln\frac{t}{1+t}<0, \tag{6}$$

and $\phi_\mu(x)<0$ is equivalent to

$$p_\mu(t)=2+(\mu+1)t+(1+t)[(\mu+1)t+1]\ln\frac{t}{1+t}>0. \tag{7}$$

Therefore, we have the following conclusions:

1. If $\mu \geq 1$, we have $p_\mu(t) < 0$, then $q'_\mu(t) < 0$ in $(0, \infty)$, and $q_\mu(t)$ is strictly decreasing in $(0, \infty)$, thus $q_\mu(t) > \lim_{t\to\infty} q_\mu(t) = \frac{1+\mu}{e^\mu}$.

2. If $0 < \mu < 1$, we have $p_\mu(t) > 0$ in $(0, x_1)$ and $p_\mu(t) < 0$ in (x_1, ∞). These are equivalent to $q'_\mu(t) > 0$ in $(0, x_1)$ and $q'_\mu(t) < 0$ in (x_1, ∞). Hence $q_\mu(t)$ is strictly increasing in $(0, x_1)$ and $q_\mu(t)$ is strictly decreasing in (x_1, ∞). Therefore, we have

$$q_\mu(t) > \min\left\{\lim_{t\to 0} q_\mu(t), \lim_{t\to\infty} q_\mu(t)\right\} = \min\left\{1, \frac{1+\mu}{e^\mu}\right\} = \frac{1+\mu}{e^\mu}. \tag{8}$$

These tell us that $q_\mu(t) > \frac{1+\mu}{e^\mu}$ for any $t \in (0, \infty)$ and $\mu \in (0, \infty)$. Then

$$\tau(\mu t, t) < \frac{1}{\mu}\left(1 - \frac{1+\mu}{e^\mu}\right) \triangleq \Psi(\mu) \tag{9}$$

for $t \in (0, \infty)$ and $\mu \in (0, \infty)$.

In order to prove Theorem 1, it is sufficient to show $\Psi(\mu) < \frac{3}{10}$, which is equivalent to $g(\mu) = 3\mu e^\mu - 10e^\mu + 10\mu + 10 > 0$.

Easy calculation gives $g'(\mu) = 3\mu e^\mu - 7e^\mu + 10$ and $g''(\mu) = e^\mu(3\mu - 4)$. Hence, the function $g'(\mu)$ is decreasing in $(0, 4/3)$ and increasing in $(4/3, \infty)$. This means that the function $g'(\mu)$ attains its minimum at the point $\mu = 4/3$ and $g'(4/3) = -1.38100368\cdots < 0$. Since $g'(0) = 3$ and $\lim_{\mu\to\infty} g'(\mu) = \infty$, the function $g'(\mu)$ has two zero points $\mu_1 \in (0, 4/3)$ and $\mu_2 \in (4/3, \infty)$. It is clear that $g'(\mu) > 0$ and $g(\mu)$ is increasing for $\mu \notin (\mu_1, \mu_2)$. Since $g(0) = 0$, it is easily concluded that μ_1 is a point of local maximum and μ_2 is a point of local minimum for the function $g(\mu)$. An elementary reasoning now yields that it is sufficient to prove that $g(\mu_2) \geq 0$ if we wish to prove what we want. For this purpose, we calculate

$$g(\mu_2) = 3\mu_2 e^{\mu_2} - 7e^{\mu_2} + 10 - 3e^{\mu_2} + 10\mu_2 = -3e^{\mu_2} + 10\mu_2, \tag{10}$$

since $g'(\mu_2) = 3\mu_2 e^{\mu_2} - 7e^{\mu_2} + 10 = 0$.

We now consider the function $h(\mu) = 10\mu - 3e^\mu$ such that $h(\mu_2) = g(\mu_2)$. It is clear that $h'(\mu) = 10 - 3e^\mu$ and $h''(\mu) = -3e^\mu < 0$. It is not difficult to see that the function $h(\mu)$ has a maximum at the point $\nu = \ln\frac{10}{3}$ and it has two zero points ν_1 and ν_2 such that $h(\mu) > 0$ for $\mu \in (\nu_1, \nu_2)$. Now it is sufficient to prove that $\mu_2 \in (\nu_1, \nu_2)$. A simple verification will show that $\nu_1 < 1 < \mu_2 < \frac{44}{25} < \nu_2$. This completes the proof.

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