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Zeros Of Real Symmetric Polynomials*

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Abstract

We consider a generalization of the symmetric polynomials and we give a sufficient condition in order to have that their real zero set contains a vector subspace of a certain dimension, which generalizes a result of R. Aron, R. Gonzalo and A. Zagorodnyuk. Furthermore, we investigate an application to elementary symmetric polynomials.

1 Introduction

The purpose of this paper is to study the (real) zero set of polynomials in n variables invariant under the action of a given subgroup of S_n. Although this is quite a natural question (at least for symmetric polynomials, namely polynomials in n variables invariant under the action of the whole group S_n), it does not seem to have been studied very much. In particular, we treat on this paper the following special case of the problem: *given a certain subgroup $\mathbb{G}$ of S_n and a polynomial F in n variables invariant under the action of $\mathbb{G}$, determine when $F^{-1}(0)$ contains a vector subspace*. The only known result about this problem is given in [1] for a special class of symmetric polynomials: if $F \in \mathbb{R}[X_1, \ldots, X_n]$ is a symmetric homogeneous polynomial having odd degree then $F^{-1}(0)$ contains an $\lfloor \frac{n}{2} \rfloor$–dimensional real vector subspace.

Furthermore we investigate an application of this problem to the elementary symmetric polynomials: we completely describe the real zero set of $e_2(X_1, \ldots, X_n) = \sum_{1 \le j < k \le n} X_j X_k$ for any n, and we present some conjectures and open questions.

Throughout the paper, $\mathbb{K}$ denotes a field of characteristic zero, except where otherwise stated, that is in Theorems 5 and 6, in Corollary 7, and in Conjecture 8, where $\mathbb{K} \subset \mathbb{R}$.

We note that the latter is the most interesting case, because in an algebrically closed field these problems sometimes become trivial using tools from classical algebraic geometry.

Following [4, Chap. 1], we recall the definitions of some classical families of symmetric polynomials.

For any $r \ge 1$, the r–th *elementary symmetric polynomial* e_r is

$$e_r(X_1, \ldots, X_n) = \sum_{1 \le j_1 < j_2 < \ldots < j_r \le n} X_{j_1} X_{j_2} \cdots X_{j_r}$$

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(for $r = 0$, we define $e_0 = 1$). So $e_r(X_1, \ldots, X_n) = 0$ if $r > n$.

For any $r \geq 1$, the r–th *complete symmetric polynomial* $h_r(X_1, \ldots, X_n)$ is the sum of all monomials of degree r in $X_1, \ldots, X_n$ (for $r = 0$, we define $h_0 = 1$).

2 Main Results

Our main result holds for a class of polynomials which is more general than symmetric polynomials, namely the polynomials invariant under the following subgroup of S_n.

DEFINITION 1. Let $\mathbb{G}_k = \langle (1,2), (3,4), \ldots, (2k-1, 2k) \rangle \leq S_n$ and suppose that it acts on $\mathbb{K}[X_1, \ldots, X_n]$, with $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$, switching the variables X_{2j-1}, X_{2j}, for all $j = 1, \ldots, k$.

Note that every symmetric polynomial with n variables is invariant under the action of $\mathbb{G}_k$ for any $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$.

THEOREM 2. Let $F \in \mathbb{K}[X_1, \ldots, X_n]$ be a polynomial invariant under the action of $\mathbb{G}_k$ for some $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$, and such that writing

$$F(X_1, \ldots, X_n) = \sum_{h_1, \ldots, h_{2k} \geq 0} c_{h_1, \ldots, h_{2k}}(X_{2k+1}, \ldots, X_n) X_1^{h_1} \cdots X_{2k}^{h_{2k}}$$

we have that if

$$h_{2j-1} \equiv h_{2j} \pmod 2 \qquad for\ all\ j = 1, \ldots, k$$

then $c_{h_1, \ldots, h_{2k}}(0, \ldots, 0) = 0$.

Then

$$F(\alpha_1, -\alpha_1, \ldots, \alpha_k, -\alpha_k, 0, \ldots, 0) = 0$$

for all $\alpha_j \in \mathbb{K} \quad j = 1, \ldots, k$, which means that $F^{-1}(0)$ contains a vector subspace of dimension k.

PROOF. We put $W_r = \{X_{2r+1}, \ldots, X_n\}$. Write

$$F(X_1, \ldots, X_n) = F(W_0) = \sum_{h_1, h_2 \geq 0} c_{h_1, h_2}(W_1) X_1^{h_1} X_2^{h_2} = f_1(W_0) + F_2(W_0)$$

where

$$f_1(W_0) = \sum_{\substack{h_1, h_2 \geq 0 \\ h_1 \not\equiv h_2 \pmod 2}} c_{h_1, h_2}(W_1) X_1^{h_1} X_2^{h_2} \tag{1}$$

and

$$F_2(W_0) = \sum_{\substack{h_1, h_2 \geq 0 \\ h_1 \equiv h_2 \pmod 2}} c_{h_1, h_2}(W_1) X_1^{h_1} X_2^{h_2}.$$

$F(W_0)$ is a polynomial invariant under the action of $\mathbb{G}_k$, so for any $\alpha_2 \in \mathbb{K}$

$$F(\alpha_2, -\alpha_2, W_1) = F(-\alpha_2, \alpha_2, W_1).$$

But the exponents of X_1, X_2 in F_2 are equivalent $\pmod 2$, so it is also true that

$$F_2(\alpha_2, -\alpha_2, W_1) = F_2(-\alpha_2, \alpha_2, W_1)$$

hence

$$f_1(\alpha_2, -\alpha_2, W_1) = f_1(-\alpha_2, \alpha_2, W_1). \tag{2}$$

From the definition (1), we have that the exponents of X_1 and X_2 in f_1 are not congruent $\pmod 2$, thus

$$f_1(\alpha_2, -\alpha_2, W_1) = -f_1(-\alpha_2, \alpha_2, W_1).$$

This equation and equation (2) immediately imply

$$f_1(\alpha_2, -\alpha_2, W_1) = 0 \qquad \forall \alpha_2 \in \mathbb{K}.$$

Now iterate the same method on F_2.

On step $r \le k \le \left\lfloor \frac{n}{2} \right\rfloor$ we have

$$F(W_0) = \sum_{t=1}^{r-1} f_t(W_0) + f_r(W_0) + F_{r+1}(W_0)$$

where

$$f_r(W_0) = \sum_{\substack{h_1,\dots,h_{2r} \ge 0 \\ h_{2r-1} \not\equiv h_{2r} \pmod 2 \\ h_{2j-1} \equiv h_{2j} \pmod 2 \\ 1 \le j \le r-1}} c_{h_1,\dots,h_{2r}}(W_r) X_1^{h_1} \cdots X_{2r}^{h_{2r}}, \tag{3}$$

$$F_{r+1}(W_0) = \sum_{\substack{h_1,\dots,h_{2r} \ge 0 \\ h_{2j-1} \equiv h_{2j} \pmod 2 \\ 1 \le j \le r}} c_{h_1,\dots,h_{2r}}(W_r) X_1^{h_1} \cdots X_{2r}^{h_{2r}}$$

and for $t = 1, \dots, r-1$

$$f_t(\alpha_2, -\alpha_2, \dots, \alpha_{2t}, -\alpha_{2t}, W_t) = 0 \qquad \forall \alpha_{2j} \in \mathbb{K} \quad j = 1, \dots, t.$$

As before, $F(W_0)$ is a polynomial invariant under the action of $\mathbb{G}_k$ so

$$F(\alpha_2, -\alpha_2, \dots, \alpha_{2r}, -\alpha_{2r}, W_r) = F(-\alpha_2, \alpha_2, \dots, -\alpha_{2r}, \alpha_{2r}, W_r)$$

for any choice of $\alpha_{2j} \in \mathbb{K} \quad j = 1, \dots, r-1$.
But for these values $f_t = 0 \quad t = 1, \dots, r-1$ and studying the exponents in F_{r+1} it is immediate that

$$F_{r+1}(\alpha_2, -\alpha_2, \dots, \alpha_{2r}, -\alpha_{2r}, W_r) = F_{r+1}(-\alpha_2, \alpha_2, \dots, -\alpha_{2r}, \alpha_{2r}, W_r),$$

hence

$$f_r(\alpha_2, -\alpha_2, \dots, \alpha_{2r}, -\alpha_{2r}, W_r) = f_r(-\alpha_2, \alpha_2, \dots, -\alpha_{2r}, \alpha_{2r}, W_r). \tag{4}$$

From the definition (3), we have that in f_r the exponents of X_{2r-1} and X_{2r} are not congruent $\pmod 2$, whereas the exponents of X_{2j-1} and X_{2j} for all $1 \leq j \leq r-1$, have always the same parity, therefore

$$f_r\left(\alpha_2, -\alpha_2, \ldots, \alpha_{2r}, -\alpha_{2r}, W_r\right) = -f_r\left(-\alpha_2, \alpha_2, \ldots, -\alpha_{2r}, \alpha_{2r}, W_r\right).$$

This equation and equation (4) immediately imply

$$f_r\left(\alpha_2, -\alpha_2, \ldots, \alpha_{2r}, -\alpha_{2r}, W_r\right) = 0 \qquad \forall \alpha_{2j} \in \mathbb{K} \quad j = 1, \ldots, r.$$

So we have proved that for any $k \leq \left\lfloor \frac{n}{2} \right\rfloor$ it is possible to write

$$F\left(X_1, \ldots, X_n\right) = \sum_{t=1}^{k} f_t\left(X_1, \ldots, X_n\right) + F_{k+1}\left(X_1, \ldots, X_n\right)$$

where

$$F_{k+1}\left(X_1, \ldots, X_n\right) = \sum_{\substack{h_1, \ldots, h_{2k} \geq 0 \\ h_{2j-1} \equiv h_{2j} \pmod 2 \\ 1 \leq j \leq k}} c_{h_1, \ldots, h_{2k}}\left(W_k\right) X_1^{h_1} \cdots X_{2k}^{h_{2k}}$$

and for $t = 1, \ldots, k$

$$f_t\left(\alpha_2, -\alpha_2, \ldots, \alpha_{2t}, -\alpha_{2t}, W_t\right) = 0 \qquad \forall \alpha_{2j} \in \mathbb{K} \quad j = 1, \ldots, t.$$

If k is as in the statement of the theorem and we put $W_k = 0$ (in other words $X_{2k+1} = \ldots = X_n = 0$), we have that every $c_{h_1, \ldots, h_{2k}}\left(W_k\right)$ in F_{k+1} is zero because each has constant term equal to zero.

Therefore

$$F\left(\alpha_2, -\alpha_2, \ldots, \alpha_{2k}, -\alpha_{2k}, 0, \ldots, 0\right) = 0 \qquad \forall \alpha_{2j} \in \mathbb{K} \quad j = 1, \ldots, k.$$

Theorem 2 considers a generic polynomial $F \in \mathbb{K}\left[X_1, \ldots, X_n\right]$ invariant under the action of $\mathbb{G}_k$ for some $1 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor$, which avoids monomials with even degree of the form

$$cX_1^{h_1} \cdots X_{2k}^{h_{2k}} \quad \text{with } c \in \mathbb{K} \text{ and } \quad h_{2j-1} \equiv h_{2j} \pmod 2 \quad j = 1, \ldots, k. \tag{5}$$

Note that it is impossible to replace this hypothesis with the weaker assumption to avoid monomials of the form

$$c \prod_{j=1}^{k} \left(X_{2j-1} X_{2j}\right)^{h_j}, \tag{6}$$

therefore to omit the condition $\pmod 2$, transforming the congruence in an equality. In fact, let us consider the following example: let $\mathbb{K} \subset \mathbb{R}$ and $p_r\left(X_1, \ldots, X_n\right) = \sum_{j=1}^{n} X_j^r$ for a generic even $r > 0$. Then p_r is a symmetric polynomial, hence invariant under $\mathbb{G}_k$ for any $1 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor$, it contains monomials as in (5) and no monomial as in (6), and $p_r^{-1}(0) = (0, \dots, 0)$.

The following Corollary is immediate.

COROLLARY 3. Let $F \in \mathbb{K}[X_1, \dots, X_n]$ be a polynomial invariant under the action of $\mathbb{G}_k$ for some $1 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor$, and such that F does not contain monomials of even degree. Then there is a k–dimensional vector subspace contained in $F^{-1}(0)$.

Corollary 3 immediately implies the following result, which was first proved in [1].

COROLLARY. Let $F \in \mathbb{R}[X_1, \dots, X_n]$ be a symmetric homogeneous polynomial of odd degree. Then there is an $\left\lfloor \frac{n}{2} \right\rfloor$–dimensional vector subspace contained in $F^{-1}(0)$.

The following question is quite natural:

OPEN QUESTION 4. Find other types of subgroups $G < S_n$ such that similar results hold for polynomials invariant under their action.

We note that it seems extremely hard to have an analogue of Corollary 3 for a polynomial $F \in \mathbb{R}[X_1, \dots, X_n]$ having even degree, even if F is symmetric, that is it is invariant under the action of the whole group S_n.

In fact, the following result for complete symmetric polynomials with even degree is known, see [2] for the proof.

THEOREM 5. Let $h_{2r}(X_1, \dots, X_n) \in \mathbb{R}[X_1, \dots, X_n]$ be the real complete symmetric polynomial in n variables with even degree equal to $2r$.
If $\sum_{j=1}^{n} X_j^2 = 1$ then

$$h_{2r}(X_1, \dots, X_n) \geq \frac{1}{2^r r!}.$$

Hence from the homogeneity of h_{2r} we have that $h_{2r}(X_1, \dots, X_n) > 0$ if $(X_1, \dots, X_n) \neq (0, \dots, 0)$.

Nevertheless, we believe that it is possible to find an analogue of Corollary 3 focused on the elementary symmetric polynomials $e_{2r}(X_1, \dots, X_n)$ with even degree $2r$, where on the contrary, the dimension of the maximal real vector subspace contained in $e_{2r}^{-1}(0)$ depends only on the degree $2r$ and not on the number of variables n.

In this sense we prove the following result.

THEOREM 6. For any $n \geq 2$, the real zero set of $e_2(X_1, \dots, X_n)$ is the circular cone

$$\{X \in \mathbb{R}^n : \cos^2(X, u) = \frac{1}{n}\},$$

where $u = (1, \dots, 1) \in \mathbb{R}^n$.

PROOF. Let n be the number of variables in e_2, $\langle\ ,\ \rangle$ be the standard scalar product in $\mathbb{R}^n$, specifically $\langle v, w \rangle = \sum_{j=1}^{n} v_j w_j$ for any $v = (v_1, \dots, v_n), w = (w_1, \dots, w_n) \in \mathbb{R}^n$, $\|v\|$ be the euclidean norm, that is to say $\|v\| = \sqrt{\langle v, v \rangle}$, and

$$\cos(v, w) = \frac{\langle v, w \rangle}{\|v\| \cdot \|w\|}$$

be the cosine of the convex angle between the two vectors $v, w \in \mathbb{R}^n$.

We remark that $2e_2(X_1, \dots, X_n) = \left(\sum_{j=1}^n X_j\right)^2 - \sum_{j=1}^n X_j^2$, therefore $X = (X_1, \dots, X_n) \in e_2^{-1}(0)$ if and only if

$$\left(\sum_{j=1}^n X_j\right)^2 = \sum_{j=1}^n X_j^2.$$

But this means that

$$\cos^2(X, u) = \frac{\langle X, u\rangle^2}{\|X\|^2 \cdot \|u\|^2} = \frac{1}{n},$$

where $u = (1, \dots, 1)$, and the desired result follows.

In particular Theorem 6 implies

COROLLARY 7. The real zero set of e_2 is a set of straight lines passing through the origin such that no three lines are on the same plane, so it contains no real vector subspace of dimension 2.

We feel that the following more general conjecture holds.

CONJECTURE 8. If r is even then $e_r^{-1}(0)$ contains no real vector subspace of dimension r.

More precisely, symbolic computations and heuristic reasons lead us to suppose that if $r \equiv 0 \pmod 2$ then any vector subspace in $e_r^{-1}(0)$ containing an element with at least r nonzero coordinates is a straight line, and obviously any vector in $\mathbb{R}^n$ with at most $r-1$ nonzero coordinates is a zero of $e_r(X_1, \dots, X_n)$.

We remark that the two extremal cases of Conjecture 8 are the Corollary 7 and the case $e_{n-1}(X_1, \dots, X_n)$, $\quad n \equiv 1 \pmod 2$ which becomes a task quite hard to tackle; see [3] for a result about $e_{n-1}(X_1, \dots, X_n)$ with the same flavour.

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