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Relaxed Strongly Nonconvex Functions*

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Abstract

In this paper, we introduce some new classes of convex functions, which are called relaxed strongly φ-convex and relaxed strongly φ-invex functions. We study some properties of these new classes of nonconvex functions. Results obtained in this paper can be viewed as refinement and improvement of previously known results.

1 Introduction

In recent years, several extensions and generalizations have been considered for classical convexity. A significant generalization of convex functions is φ-convex functions, which was introduced by Noor [2] recently. It is well-known that the φ-convex functions and φ-convex sets may not be convex functions and convex sets. In particular, this generalization of the convex functions is quite different from other generalizations of the convex functions. In this paper, we consider and introduce a new class of φ-convex functions. This class of nonconvex function is called the relaxed strongly φ-convex (φ-invex) functions. Several new concepts of φ-monotonicity are introduced. We establish the relationship between these classes and derive some new results. As special cases, one can obtain some new classes of relaxed strongly convex functions. Results obtained in this paper can be viewed as refinement and improvement of previous known results.

2 Preliminaries

Let K be a nonempty closed set in a real Hilbert space H. We denote by $\langle ., . \rangle$ and $\|.\|$ the inner product and norm respectively. Let $F : K \to H$ be continuous function. Let $\varphi : K \longrightarrow R$ be a continuous function.

DEFINITION 2.1 [2]. Let $u \in K$. Then the set K is said to be φ-convex at u with respect to $\varphi(.)$, if

$$u + te^{i\varphi}(v - u) \in K, \quad \forall u, v \in K, t \in [0, 1].$$

K is said to be an φ-convex set with respect to and φ, if K is φ-convex at each $u \in K$. Note that the convex set with $\varphi = 0$ is a convex set, but the converse is not true.

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From now onward K is a nonempty closed φ-convex set in H with respect to φ unless otherwise specified.

DEFINITION 2.2. The function F on the φ-convex set K is said to be relaxed strongly φ-convex with respect to φ, if there exists a constant $\mu > 0$ such that

$$F(u + te^{i\varphi}(v-u)) \leq (1-t)F(u) + tF(v) + \mu t(1-t)\|v-u\|^2, \quad \forall u, v \in K, \quad t \in [0,1].$$

The function F is said to be relaxed strongly φ-concave if and only if $-F$ is relaxed strongly φ-convex. Note that every relaxed strongly convex function is a relaxed strongly φ-convex function, but the converse is not true.

DEFINITION 2.3. The function F on the φ-convex set K is called relaxed strongly quasi φ-convex with respect to φ, if there exists a constant $\mu > 0$ such that

$$F(u + te^{i\varphi}(v-u)) \leq max\{F(u), F(v)\} + \mu t(1-t)\|v-u\|^2, \quad \forall u, v \in K, \quad t \in [0,1].$$

DEFINITION 2.4. The function F on the φ-convex set K is said to be relaxed strongly logarithmic φ-convex with respect to φ, if there exists a constant $\mu > 0$ such that

$$F(u + te^{i\varphi}(v-u)) \leq (F(u))^{1-t}(F(v))^t + \mu t(1-t)\|v-u\|^2, \quad u, v \in K, \quad t \in [0,1],$$

where $F(.) > 0$.

DEFINITION 2.5. The differentiable function F on the φ-convex set K is said to be a relaxed strongly φ-invex function with respect to φ, if there exists a constant $\mu > 0$ such that

$$F(v) - F(u) \geq \langle F'_\varphi(u), v-u\rangle - \mu\|v-u\|^2, \quad \forall u, v \in K,$$

where $F'_\varphi(u)$ is the differential of F at u in the direction of $e^{i\varphi}(v-u) \in K$.

From definitions 2.2-2.4, we have

$$\begin{aligned} F(u + te^{i\varphi}(v-u)) &\leq (F(u))^{1-t}(F(v))^t + \mu t(1-t)\|v-u\|^2 \\ &\leq (1-t)F(u) + tF(v) + \mu t(1-t)\|v-u\|^2 \\ &\leq \max\{F(u), F(v)\} + \mu t(1-t)\|v-u\|^2 \\ &< \max\{F(u), F(v)\} + \mu t(1-t)\|v-u\|^2. \end{aligned}$$

For $t = 1$, Definitions 2.2 and 2.4 reduce to the following, which is mainly due to Noor [2]:

Condition A. $F(u + e^{i\varphi}(v-u)) \leq F(v), \quad \forall u, v \in K,$

which plays an important part in studying the properties of the φ-convex functions.

For $\varphi = 0$, φ-convex set K becomes a convex set K and consequently Definitions 2.2-2.5 reduce to the following new concepts for the relaxed strongly convex functions.

DEFINITION 2.6. The function F on the convex set K is said to be relaxed strongly convex, if there exists a constant $\mu > 0$ such that

$$F(u + t(v-u)) \leq (1-t)F(u) + tF(v) + \mu t(1-t)\|v-u\|^2, \quad \forall u, v \in K, \quad t \in [0,1].$$

The function F is said to be relaxed strongly concave if and only if $-F$ is relaxed strongly convex.

DEFINITION 2.7. The function F on the φ-convex set K is called relaxed strongly quasi convex, if there exists a constant $\mu > 0$ such that

$$F(u+t(v-u)) \leq max\{F(u), F(v)\} + \mu t(1-t)\|v-u\|^2, \quad \forall u, v \in K, \quad t \in [0,1].$$

DEFINITION 2.8. The function F on the convex set K is said to be logarithmic convex, if there exists a constant $\mu > 0$ such that

$$F(u+t(v-u)) \leq (F(u))^{1-t}(F(v))^t + \mu t(1-t)\|v-u\|^2, \quad u, v \in K, \quad t \in [0,1],$$

where $F(.) > 0$.

It is well known that the concepts of relaxed strongly convex functions play a significant role in the mathematical programming and optimization theory, see [1,3,4] and the references therein.

REMARK 2.2. Note that for $\mu = 0$, Definitions 2.2-2.5 reduce to the ones in [2].

DEFINITION 2.9. An operator $T : K \longrightarrow H$ is said to be:

(i). relaxed strongly monotone, iff, there exists a constant $\mu > 0$ such that

$$\langle Tu - Tv, u - v\rangle \geq -\mu\|v-u\|^2, \quad \forall u, v \in K.$$

(ii). monotone, iff,

$$\langle Tu - Tv, u - v\rangle \geq 0, \quad \forall u, v \in K.$$

(iii). relaxed strongly pseudomonotone, iff, there exists a constant $\mu > 0$ such that

$$\langle Tu, v-u\rangle \geq 0 \quad \Longrightarrow \quad \langle Tv, v-u\rangle + \mu\|v-u\|^2 \geq 0, \quad \forall u, v \in K.$$

(iv). relaxed weakly pseudomonotone, iff, there exists a constant $\mu > 0$ such that

$$\langle Tu, v-u\rangle + \mu\|v-u\|^2 \geq 0 \quad \Longrightarrow \quad \langle Tv, v-u\rangle \geq 0, \quad \forall u, v \in K.$$

(v). pseudomonotone, iff,

$$\langle Tu, v-u\rangle \geq 0 \quad \Longrightarrow \quad \langle Tv, v-u\rangle \geq 0, \quad \forall u, v \in K.$$

(vi). quasi monotone, iff,

$$\langle Tu, v-u\rangle > 0 \quad \Longrightarrow \quad \langle Tv, v-u\rangle \geq 0, \quad \forall u, v \in K.$$

DEFINITION 2.10. A differentiable function F on an φ-convex set K is said to be relaxed weakly pseudo φ-convex function, iff, there exists a constant $\mu > 0$ such that

$$\langle F'_\varphi(u), v-u\rangle - \mu\|v-u\|^2 \geq 0 \quad \Longrightarrow \quad F(v) - F(u) \geq 0, \quad \forall u, v \in K.$$

DEFINITION 2.11. A differentiable function F on the K is said to be relaxed strongly quasi φ-convex, if there exists a constant $\mu > 0$ such that

$$F(v) \leq F(u) \quad \Longrightarrow \quad \langle F'_{\varphi}(u), v-u\rangle \geq \mu\|v-u\|^2, \quad \forall u, v \in K.$$

DEFINITION 2.12. The function F on the set K is said to be relaxed strongly pseudo φ-convex, if

$$\langle F'_{\varphi}(u), v-u\rangle \geq 0, \quad \Longrightarrow \quad F(v) \geq F(u) - \mu\|v-u\|^2, \quad \forall u, v \in K.$$

DEFINITION 2.13. A differentiable function F on the K is said to be quasi φ-convex, if

$$F(v) \leq F(u) \quad \Longrightarrow \quad \langle F'_{\varphi}(u), v-u\rangle \leq 0, \quad \forall u, v \in K.$$

All the concepts defined above play an important and fundamental part in the mathematical programming and optimization problems.

LEMMA 2.1. Let T be a relaxed monotone operator with a constant $\mu > 0$. Then T is a relaxed strongly pseudomonotone operator.

PROOF. Let T be a relaxed monotone operator with a constant $\mu > 0$. Then

$$\begin{aligned} \langle Tv, v-u\rangle &= \langle Tv - Tu, v-u\rangle + \langle Tu, v-u\rangle \\ &\geq -\mu\|v-u\|^2, \end{aligned}$$

which shows that the operator T is relaxed strongly pseudomonotone.

From Lemma 2.1, it follows that the relaxed strongly monotonicity implies relaxed strongly pseudo monotonicity, but the converse is not true. In a similar way, one can show that the relaxed strongly monotonicity implies the relaxed weakly pseudo monotonicity, but the converse is not true.

3 Main Results

In this section, we consider some basic properties of relaxed strongly φ-convex (invex) functions on the φ-convex set K.

THEOREM 3.1. Let F be a differentiable function on the φ-convex set K in H. Then the function F is a relaxed strongly φ-convex function if and only if F is a relaxed strongly φ-invex function.

PROOF. Let F be a relaxed strongly φ-convex function on the convex set K. Then there exists a constant $\mu > 0$ such that

$$F(u + te^{i\varphi}(v-u)) \leq (1-t)F(u) + tF(v) + \mu t(1-t)\|v-u\|^2, \quad \forall u, v \in K,$$

which can be written as

$$F(v) - F(u) \geq \frac{F(u + te^{i\varphi}(v-u)) - F(u)}{t} - \mu(1-t)\|v-u\|^2.$$

Letting $t \longrightarrow 0$ in the above inequality, we have

$$F(v) - F(u) \geq \langle F'_{\varphi}(u), v-u \rangle - \mu \|v-u\|^2,$$

which implies that F is a relaxed strongly φ-invex functions.

Conversely, let F be a relaxed strongly φ-invex function on the φ-convex set K. Then $\forall u, v \in K, t \in [0,1], \quad v_t = u + te^{i\varphi}(v-u) \in K$, we have

$$\begin{aligned} F(v) \quad - \quad & F(u + te^{i\varphi}(v-u)) \geq \langle F'_{\varphi}(u + te^{i\varphi}(v-u)), v - v_t \rangle - \mu \|v - v_t\|^2 \\ = \quad & (1-t)\langle F'_{\varphi}(u + te^{i\varphi}(v-u)), v-u \rangle - \mu(1-t)^2 \|v-u\|^2. \end{aligned} \tag{1}$$

In a similar way, we have

$$\begin{aligned} F(u) \quad - \quad & F(u + te^{i\varphi}(v-u)) \geq \langle F'_{\varphi}(u + te^{i\varphi}(v-u)), u - v_t \rangle \\ = \quad & -t\langle F'_{\varphi}(u + te^{i\varphi}(v-u)), v_t - u \rangle - \mu t^2 \|v-u\|^2. \end{aligned} \tag{2}$$

Multiplying (1) by t and (2) by $(1-t)$ and adding the resultant, we have

$$F(u + te^{i\varphi}(v-u)) \leq (1-t)F(u) + tF(v) + \mu t(1-t)\|v-u\|^2,$$

showing that F is a relaxed strongly φ-convex function.

THEOREM 3.2. Let F be differentiable on the φ-convex set K. Let Condition A hold. Then F is a relaxed strongly φ-invex function if and only if its differential F'_{φ} is relaxed strongly φ-monotone.

PROOF. Let F be a relaxed strongly φ-invex function on the φ-convex set K. Then

$$F(v) - F(u) \geq \langle F'_{\varphi}(u), v-u \rangle - \mu \|v-u\|^2, \quad \forall u, v \in K. \tag{3}$$

Changing the role of u and v in (3), we have

$$F(u) - F(v) \geq \langle F'_{\varphi}(v), u-v \rangle - \mu \|u-v\|^2, \quad \forall u, v \in K. \tag{4}$$

Adding (3) and (4), we have

$$\langle F'_{\varphi}(u) - F'_{\varphi}(v), u-v \rangle \geq -2\mu \|v-u\|^2, \tag{5}$$

which shows that F'_{φ} is relaxed strongly φ-monotone.

Conversely, let F'_{φ} be relaxed strongly φ-monotone. From (5), we have

$$\langle F'_{\varphi}(v), v-u \rangle \leq \langle F'_{\varphi}(u), v-u \rangle - 2\mu \|v-u\|^2. \tag{6}$$

Since K is a φ-convex set, $\forall u, v \in K, t \in [0,1] \quad v_t = u + te^{i\varphi}(v-u) \in K$. Taking $v = v_t$ in (6), we have

$$\langle F'_{\varphi}(v_t), v-u \rangle \geq \langle F'_{\varphi}(u), v-u \rangle - 2\mu t \|v-u\|^2. \tag{7}$$

Let $g(t) = F(u + te^{i\varphi}(v-u))$. Then from (7), we have

$$\begin{aligned} g'(t) \quad = \quad & \langle F'_{\varphi}(u + te^{i\varphi}(v-u)), v-u \rangle \\ \geq \quad & \langle F'_{\varphi}(u), v-u \rangle - 2\mu t \|v-u\|^2. \end{aligned} \tag{8}$$

Integrating (8) between 0 and 1, we have

$$g(1) - g(0) \geq \langle F'_\varphi(u), v - u\rangle - \mu\|v - u\|^2,$$

that is,

$$F(u + e^{i\varphi}(v - u)) - F(u) \geq \langle F'_\varphi(u), v - u\rangle - \mu\|v - u\|^2.$$

By using Condition A, we have

$$F(v) - F(u) \geq \langle F'_\varphi(u), v - u\rangle + \mu\|v - u\|^2,$$

which shows that F is a relaxed strongly φ-invex function on the φ-convex set K.

From Theorem 3.1 and Theorem 3.2, we have:
relaxed strongly φ-convex functions F $\Longrightarrow$ relaxed strongly φ-invex functions F $\Longrightarrow$ strongly φ-monotonicity of the differential F'_φ and conversely if condition A holds.

For $\mu = 0$, Theorems 3.1 and 3.2 reduce to the following results, which appear to be new ones for φ-convex functions.

THEOREM 3.3. Let F be a differentiable function on the φ-invex set K in H. Then the function F is a φ-convex function if and only if F is a φ-invex function.

THEOREM 3.4. Let F be differentiable function and let Condition A hold. Then the function F is φ-convex (invex) function if and only if its differential F'_φ is φ-monotone.

For $\varphi = 0$, the φ-convex set K becomes the convex set K. Consequently, Theorem 3.1 and Theorem 3.2 reduces to the following results for relaxed strongly convex functions and appears to be new ones. These results show that the concept of the relaxed strongly monotonicity is related to the relaxed strongly convex functions. For the applications of the relaxed strongly monotonicity in the hemivariational inequalities, see [1] and the references therein.

THEOREM 3.5. Let F be a differentiable function on the convex set K in H. Then the following statements are equivalent.
(a). The functions F is relaxed strongly φ-convex function with a constant $\mu > 0$.
(b). The function F satisfies:

$$F(v) - F(u) \geq \langle F'(u), v - u\rangle - \mu\|v - u\|^2, \quad \forall u, v \in K.$$

(c). The differential $F'(u)$ of the function F is relaxed strongly monotone with a constant $\mu > 0$, that is,

$$\langle F'(u) - F'(v), u - v\rangle \geq -\mu\|u - v\|^2, \quad \forall u, v \in K.$$

We now give a necessary condition for relaxed strongly pseudo φ-convex function.

THEOREM 3.6. Let F'_φ be relaxed strongly φ-pseudomonotone and Condition A hold. Then F is strongly pseudo φ-convex function.

PROOF. Let F'_φ be relaxed strongly φ-pseudomonotone. Then, $\forall u, v \in K$,

$$\langle F'_\varphi(u), v-u\rangle \geq 0,$$

implies that

$$\langle F'_\varphi(v), v-u\rangle - \alpha\|v-u|^2 \geq 0. \tag{9}$$

Since K is an φ-convex set, $\forall u, v \in K, t \in [0,1], v_t = u + te^{i\varphi}(v-u) \in K$. Taking $v = v_t$ in (9), we have

$$\langle F'_\varphi(u + te^{i\varphi}(v-u)), v-u\rangle \geq -t\alpha\|v-u\|^2. \tag{10}$$

Let $g(t) = F(u + te^{i\varphi}(v-u)), \quad \forall u, v \in K, t \in [0,1]$. Then, using (10), we have

$$g'(t) = \langle F'_\varphi(u + te^{i\varphi}(v-u)), v-u\rangle \geq -t\alpha\|v-u\|^2.$$

Integrating the above relation between 0 and 1, we have

$$g(1) - g(0) \geq \frac{\alpha}{2}\|v-u\|^2,$$

that is,

$$F(u + e^{i\varphi}(v-u)) - F(u) \geq -\frac{\alpha}{2}\|v-u\|^2,$$

which implies, using Condition A,

$$F(v) - F(u) \geq -\frac{\alpha}{2}\|v-u\|^2,$$

showing that F is relaxed strongly pseudo φ-convex function.

As a special case of Theorem 3.6, we have the following:

THEOREM 3.7. Let the differential $F'_\varphi(u)$ of a function $F(u)$ on the φ-convex set K be φ-pseudomonotone. If Condition A holds, then F is pseudo φ-convex function.

THEOREM 3.8. Let the differential $F'_\varphi(u)$ of a differentiable φ-convex function $F(u)$ be Lipschitz continuous on the φ-convex set K with a constant $\beta > 0$. If Condition A holds, then

$$F(v) - F(u) \leq \langle F'_\varphi(u), v-u\rangle + \frac{\beta}{2}\|v-u\|^2, \quad \forall u, v \in K.$$

PROOF. $\forall u, v \in K, t \in [0,1], \quad u + te^{i\varphi}(v-u) \in K$, since K is an φ-convex set. Now we consider the function

$$\varphi(t) = F(u + te^{i\varphi}(v-u)) - F(u) - t\langle F'_\varphi(u), v-u\rangle.$$

from which it follows that $\varphi(0) = 0$ and

$$\varphi'(t) = \langle F'_\varphi(u + te^{i\varphi}(v-u)), v-u\rangle - \langle F'_\varphi(u), v-u\rangle. \tag{11}$$

Integrating (10) between 0 and 1, we have

$$\begin{aligned}
\varphi(1) &= F(u+e^{i\varphi}(v-u)) - F(u) - \langle F'_\varphi(u), v-u\rangle \\
&\leq \int_0^1 |\varphi'(t)|dt \\
&= \int_0^1 \left|\langle F'_\varphi(u+te^{i\varphi}(v-u)), v-u\rangle - \langle F'_\varphi(u), v-u\rangle\right| dt \\
&\leq \beta \int_0^1 t\|v-u\|^2 dt \\
&= \frac{\beta}{2}\|v-u\|^2,
\end{aligned}$$

which implies that

$$F(u+e^{i\varphi}(v-u)) - F(u) \leq \langle F'_\varphi(u), v-u\rangle + \frac{\beta}{2}\|v-u\|^2. \tag{12}$$

from which, using Condition A, we obtain

$$F(v) - F(u) \leq \langle F'_\varphi(u), v-u\rangle + \frac{\beta}{2}\|v-u\|^2.$$

REMARK 3.1. For $\varphi = 0$, the φ-convex set K becomes a convex set and consequently Theorem 3.8 reduces to the well known result [4] in convexity.

DEFINITION 3.1. The function F is said to be sharply relaxed strongly pseudo φ-convex, if there exists a constant $\mu > 0$ such that

$$\begin{aligned}
&\langle F'_\varphi(u), v-u\rangle \geq 0 \\
\Longrightarrow \quad & \\
&F(v) \geq F(v+te^{i\varphi}(v-u)) - \mu t(1-t)\|v-u\|^2, \quad \forall u, v \in K, \quad t \in [0,1].
\end{aligned}$$

THEOREM 3.9. Let F be a sharply relaxed strongly pseudo φ-convex function on K with a constant $\mu > 0$. Then

$$\langle F'_\varphi(v), v-u\rangle \geq \mu\|v-u\|^2, \quad \forall u, v \in K.$$

PROOF. Let F be a sharply relaxed strongly pseudo φ-convex function on K. Then

$$F(v) \geq F(v+te^{i\varphi}(v-u)) + \mu t(1-t)\|v-u\|^2, \forall u, v \in K, \quad t \in [0,1].$$

from which we have

$$\frac{F(v+te^{i\varphi}(v-u)) - F(v)}{t} + \mu(1-t)\|v-u\|^2 \leq 0.$$

Taking limit in the above inequality, as $t \longrightarrow 0$, we have

$$\langle F'_\varphi(v), v-u\rangle \geq \mu\|v-u\|^2,$$

the required result.

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