Applied Mathematics E-Notes, 6(2006), 211-218 © ISSN 1607-2510
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Vandermonde-Type Determinants And Inequalities*

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Received 9 September 2005

Abstract

Two inequalities involving Vandermonde-type determinants are established.

1 Introduction

The determinant of the Vandermonde matrix of order n is well-known:

$$V_n(x) = \begin{vmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{vmatrix} = \prod_{1 \leqslant i < j \leqslant n} (x_j - x_i). \tag{1}$$

Obviously, if $x_1 < x_2 < \cdots < x_n$, then $V_n(x) > 0$.

In [1], Xiao and Zhang gave a general form of the Vandermonde determinant of order n:

$$V_n(x; r) = \begin{vmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-2} & x_1^{n+r-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-2} & x_2^{n+r-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-2} & x_n^{n+r-1} \end{vmatrix}, \tag{2}$$

where $0 < x_i$ for $1 \leqslant i \leqslant n$ and $r \in \mathbb{R}$. And they proved an integral identity relating $V_n(x; r)$ with $V_n(x)$:

$$V_n(x; r) = \prod_{i=1}^{n-1} (r + i) \cdot V_n(x) \int_E \left(\sum_{i=1}^{n} x_i t_i \right)^r dt_1 dt_2 \cdots dt_{n-1}, \tag{3}$$

where $r \neq -1, -2, ..., -(n-1)$, $t_n = 1 - \sum_{i=1}^{n-1} t_i$ and

$$E = \left\{ (t_1, t_2, \cdots, t_{n-1}) : \sum_{i=1}^{n-1} t_i \leqslant 1, t_i \geqslant 0, i = 1, 2, \cdots, n-1 \right\}.$$

The main purpose of this paper is to define a new type of Vandermonde determinant and investigate the corresponding inequalities.

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*Mathematics Subject Classifications: 26D15, 26D10.

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2 Main Results

We first begin with some definitions.

DEFINITION 1. If $x_i > 0$ and $\alpha_i \in \mathbb{R}$ for $1 \leqslant i \leqslant n$, then

$$V_n(x,\alpha) = \begin{vmatrix} x_1^{\alpha_1} & x_2^{\alpha_1} & \cdots & x_n^{\alpha_1} \\ x_1^{\alpha_2} & x_2^{\alpha_2} & \cdots & x_n^{\alpha_2} \\ \cdots & \cdots & \cdots & \cdots \\ x_1^{\alpha_n} & x_2^{\alpha_n} & \cdots & x_n^{\alpha_n} \end{vmatrix} \tag{4}$$

is called a generalized Vandermonde determinant of order n.

DEFINITION 2 ([2]). Let $A = (a_{ij})_{n\times n}$ be a matrix of order n , where $a_{ij} \in \mathbb{C}$ for $1 \leqslant i \leqslant n, 1 \leqslant j \leqslant n$. Then the permanent of order n of A, written $perA$, is defined by

$$perA := \sum_{i_1 i_2 \cdots i_n} a_{i_1 1} a_{i_2 2} \cdots a_{i_n n} = \sum_{j_1 j_2 \cdots j_n} a_{1 j_1} a_{2 j_2} \cdots a_{n j_n}.$$

In this paper we shall also use other symbols: the permanent of A is denoted by $\mid a_{ij} \mid_n^+$ or

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}_n^+ := \sum_{i_1 i_2 \cdots i_n} a_{i_1 1} a_{i_2 2} \cdots a_{i_n n} = \sum_{j_1 j_2 \cdots j_n} a_{1 j_1} a_{2 j_2} \cdots a_{n j_n},$$

where each sum is taken over all the permutations of $\{1, 2, \cdots, n\}$.

THEOREM 1. Let $0 < x_1 < x_2 < \cdots < x_n$, and $\alpha_1 < \alpha_2 < \cdots < \alpha_n$. Then we have

$$V_n(x,\alpha) > 0. \tag{5}$$

THEOREM 2. Let $x_i > 0$ and $\alpha_i \geqslant 0$ for $1 \leqslant i \leqslant n$. Then

$$h_n(x,\alpha) := \frac{1}{n!} \cdot \begin{vmatrix} x_1^{\alpha_1} & x_2^{\alpha_1} & \cdots & x_n^{\alpha_1} \\ x_1^{\alpha_2} & x_2^{\alpha_2} & \cdots & x_n^{\alpha_2} \\ \cdots & \cdots & \cdots & \cdots \\ x_1^{\alpha_n} & x_2^{\alpha_n} & \cdots & x_n^{\alpha_n} \end{vmatrix}_n^+ \leqslant \prod_{i=1}^{n} \left(\frac{1}{n} \sum_{j=1}^{n} x_j^{\alpha_i} \right), \tag{6}$$

where $h_n(x,\alpha) = \dfrac{1}{n!} \sum_{i_1 i_2 \cdots i_n} \prod_{j=1}^{n} x_{i_j}^{\alpha_j}$ is called a Hardy function with respect to x and α.

PROOF of Theorem 1. We shall prove it by mathematical induction. For $n = 1$, inequality (5) is obvious. For $n = 2$, using Lagrange's mean value theorem, we have

$$\begin{aligned} V_2(x,\alpha) &= \begin{vmatrix} x_1^{\alpha_1} & x_2^{\alpha_1} \\ x_1^{\alpha_2} & x_2^{\alpha_2} \end{vmatrix} = x_1^{\alpha_1} x_2^{\alpha_2} - x_1^{\alpha_2} x_2^{\alpha_1} \\ &= x_1^{\alpha_1} x_2^{\alpha_1} \left(x_2^{\alpha_2-\alpha_1} - x_1^{\alpha_2-\alpha_1} \right) = x_1^{\alpha_1} x_2^{\alpha_1} \left(x_2 - x_1 \right) \cdot \left. \frac{dx^{\alpha_2-\alpha_1}}{dx} \right|_{x=\xi} \\ &= x_1^{\alpha_1} x_2^{\alpha_1} \left(x_2 - x_1 \right) \left(\alpha_2 - \alpha_1 \right) \cdot \xi^{\alpha_2-\alpha_1-1} > 0, \end{aligned}$$

where $x_1 < \xi < x_2$. Hence (5) holds. Suppose inequality (5) is true for $n-1$ with $n > 2$, i.e.,

$$V_{n-1}(\xi,\alpha) = \begin{vmatrix} \xi_2^{\alpha_2} & \xi_3^{\alpha_2} & \cdots & \xi_n^{\alpha_2} \\ \xi_2^{\alpha_3} & \xi_3^{\alpha_3} & \cdots & \xi_n^{\alpha_3} \\ \cdots & \cdots & \cdots & \cdots \\ \xi_2^{\alpha_n} & \xi_3^{\alpha_n} & \cdots & \xi_n^{\alpha_n} \end{vmatrix} > 0, \tag{7}$$

where $0 < \xi_2 < \xi_3 < \cdots < \xi_n$ and $\alpha_2 < \alpha_3 < \cdots < \alpha_n$. Let

$$\varphi(x_n) = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_1^{\alpha_2-\alpha_1} & x_2^{\alpha_2-\alpha_1} & \cdots & x_n^{\alpha_2-\alpha_1} \\ \cdots & \cdots & \cdots & \cdots \\ x_1^{\alpha_n-\alpha_1} & x_2^{\alpha_n-\alpha_1} & \cdots & x_n^{\alpha_n-\alpha_1} \end{vmatrix}. \tag{8}$$

By Lagrange's mean value theorem, we get

$$\begin{aligned} \varphi(x_n) &= \varphi(x_n) - \varphi(x_{n-1}) = (x_n - x_{n-1})\varphi^{'}(\xi_n) \\ &= (x_n - x_{n-1}) \begin{vmatrix} 1 & 1 & \cdots & 0 \\ x_1^{\alpha_2-\alpha_1} & x_2^{\alpha_2-\alpha_1} & \cdots & \left.\frac{dt^{\alpha_2-\alpha_1}}{dt}\right|_{t=\xi_n} \\ \cdots & \cdots & \cdots & \cdots \\ x_1^{\alpha_n-\alpha_1} & x_2^{\alpha_n-\alpha_1} & \cdots & \left.\frac{dt^{\alpha_n-\alpha_1}}{dt}\right|_{t=\xi_n} \end{vmatrix} \\ &= (x_n - x_{n-1}) \begin{vmatrix} 1 & 1 & \cdots & 1 & 0 \\ x_1^{\alpha_2-\alpha_1} & x_2^{\alpha_2-\alpha_1} & \cdots & x_{n-1}^{\alpha_2-\alpha_1} & (\alpha_2-\alpha_1)\,\xi_n^{\alpha_2-\alpha_1-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ x_1^{\alpha_n-\alpha_1} & x_2^{\alpha_n-\alpha_1} & \cdots & x_{n-1}^{\alpha_n-\alpha_1} & (\alpha_n-\alpha_1)\,\xi_n^{\alpha_n-\alpha_1-1} \end{vmatrix}, \end{aligned}$$

where $x_{n-1} < \xi_n < x_n$. By similar considerations, it is easy to see that

$$\begin{aligned} \varphi(x_n) &= \prod_{j=k-1}^{n}(x_j - x_{j-1}) \begin{vmatrix} 1 & 0 & \cdots & 0 \\ x_1^{\alpha_2-\alpha_1} & (\alpha_2-\alpha_1)\,\xi_2^{\alpha_2-\alpha_1-1} & \cdots & (\alpha_2-\alpha)\,\xi_n^{\alpha_2-\alpha_1-1} \\ \cdots & \cdots & \cdots & \cdots \\ x_1^{\alpha_n-\alpha_1} & (\alpha_n-\alpha)\,\xi_2^{\alpha_n-\alpha_1-1} & \cdots & (\alpha_n-\alpha)\,\xi_n^{\alpha_n-\alpha_1-1} \end{vmatrix} \\ &= \prod_{j=2}^{n}(x_j - x_{j-1}) \prod_{i=2}^{n}(\alpha_i - \alpha_1) \begin{vmatrix} \xi_2^{\alpha_2-\alpha_1-1} & \xi_3^{\alpha_2-\alpha_1-1} & \cdots & \xi_n^{\alpha_2-\alpha_1-1} \\ \xi_2^{\alpha_3-\alpha_1-1} & \xi_3^{\alpha_3-\alpha_1-1} & \cdots & \xi_n^{\alpha_3-\alpha_1-1} \\ \cdots & \cdots & \cdots & \cdots \\ \xi_2^{\alpha_n-\alpha_1-1} & \xi_3^{\alpha_n-\alpha_1-1} & \cdots & \xi_n^{\alpha_n-\alpha_1-1} \end{vmatrix} \\ &= \prod_{j=2}^{n}\left[(\alpha_j-\alpha_1)\,(x_j - x_{j-1})\,\xi_j^{-\alpha_1-1}\right] V_{n-1}(\xi,\alpha), \end{aligned}$$

where $0 < x_1 < \xi_2 < x_2 < \cdots < x_{n-1} < \xi_n < x_n$ and $\alpha_2 < \alpha_3 < \cdots < \alpha_n$. For general

n, from (7), (8) and the result above, we obtain

$$V_n(x,\alpha) = \varphi(x_n)\prod_{j=1}^{n} x_j^{\alpha_1}$$
$$= V_{n-1}(\xi,\alpha)\prod_{j=1}^{n} x_j^{\alpha_1}\prod_{j=2}^{n}\left[(\alpha_j-\alpha_1)(x_j-x_{j-1})\,\xi_j^{-\alpha_1-1}\right] > 0, \tag{9}$$

where $0 < x_1 < \xi_2 < x_2 < \cdots < x_{n-1} < \xi_n < x_n$ and $\alpha_1 < \alpha_2 < \alpha_3 < \cdots < \alpha_n$. The proof of Theorem 1 is completed.

From Theorem 1, it is easy to obtain an interesting result:

COROLLARY 1. If $0 < x_1 < x_2 < \cdots < x_n, \alpha_1 < \alpha_2 < \cdots < \alpha_n$ and $t > 0$, then the following polynomial of one variable which pass n points (x_i, u_i) for $1 \leqslant i \leqslant n$ is unique:

$$u(t) = \sum_{i=1}^{n} b_i t^{\alpha_i}. \tag{10}$$

COROLLARY 2. Let $0 < x_1 < x_2 < \cdots < x_n < x_{n+1}$ and $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_n$. Then

$$v(x,\alpha) = \begin{vmatrix} x_2^{\alpha_1}-x_1^{\alpha_1} & x_3^{\alpha_1}-x_2^{\alpha_1} & \cdots & x_{n+1}^{\alpha_1}-x_n^{\alpha_1} \\ x_2^{\alpha_2}-x_1^{\alpha_2} & x_3^{\alpha_2}-x_2^{\alpha_2} & \cdots & x_{n+1}^{\alpha_2}-x_n^{\alpha_2} \\ \cdots & \cdots & \cdots & \cdots \\ x_2^{\alpha_n}-x_1^{\alpha_n} & x_3^{\alpha_n}-x_2^{\alpha_n} & \cdots & x_{n+1}^{\alpha_n}-x_n^{\alpha_n} \end{vmatrix} > 0.$$

Indeed, from inequality (5), we get

$$v(x,\alpha) = \frac{1}{\prod_{j=1}^{n}\alpha_j}\begin{vmatrix} \int_{x_1}^{x_2} x^{\alpha_1-1}dx & \int_{x_2}^{x_3} x^{\alpha_1-1}dx & \cdots & \int_{x_n}^{x_{n+1}} x^{\alpha_1-1}dx \\ \int_{x_1}^{x_2} x^{\alpha_2-1}dx & \int_{x_2}^{x_3} x^{\alpha_2-1}dx & \cdots & \int_{x_n}^{x_{n+1}} x^{\alpha_2-1}dx \\ \cdots & \cdots & \cdots & \cdots \\ \int_{x_1}^{x_2} x^{\alpha_n-1}dx & \int_{x_2}^{x_3} x^{\alpha_n-1}dx & \cdots & \int_{x_n}^{x_{n+1}} x^{\alpha_n-1}dx \end{vmatrix}$$
$$= \frac{1}{\prod_{j=1}^{n}\alpha_j}\int_{x_1}^{x_2}\int_{x_2}^{x_3}\cdots\int_{x_n}^{x_{n+1}}\begin{vmatrix} t_1^{\alpha_1-1} & t_2^{\alpha_1-1} & \cdots & t_n^{\alpha_1-1} \\ t_1^{\alpha_2-1} & t_2^{\alpha_2-1} & \cdots & t_n^{\alpha_2-1} \\ \cdots & \cdots & \cdots & \cdots \\ t_1^{\alpha_n-1} & t_2^{\alpha_n-1} & \cdots & t_n^{\alpha_n-1} \end{vmatrix} dt_1 dt_2\cdots dt_n > 0.$$

PROOF of Theorem 2. We will obtain its general form:

$$h_{m,n}(x,\alpha) := \frac{1}{n!}\cdot\begin{vmatrix} x_1^{\alpha_1} & x_2^{\alpha_1} & \cdots & x_n^{\alpha_1} \\ \cdots & \cdots & \cdots & \cdots \\ x_1^{\alpha_m} & x_2^{\alpha_m} & \cdots & x_n^{\alpha_m} \\ 1 & 1 & \cdots & 1 \\ \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & \cdots & 1 \end{vmatrix}_n^+ \leqslant \prod_{i=1}^{m}\left(\frac{1}{n}\sum_{j=1}^{n} x_j^{\alpha_i}\right), \tag{11}$$

where all numbers of $n-m$ rows equal 1. For m, it will be verified by mathematical induction also. Without loss of generality, we may assume that $0 < x_1 \leqslant x_2 \leqslant \cdots \leqslant x_n$. Then, from $\alpha_i > 0$ for $1 \leqslant i \leqslant n$,

$$0 < x_1^{\alpha_i} \leqslant x_2^{\alpha_i} \leqslant \cdots \leqslant x_n^{\alpha_i}. \tag{12}$$

Obviously, for $m = 1$, the equality of (11) holds. For $m > 1$, we delete the element at i-th row and j-th column from the $| \, a_{ij} \, |_n^+$, then we obtain a permanent of order $(n-1)$. And it is called cofactor of a_{ij} denoted by M_{ij}. For $m = 2$, by $x_1^{\alpha_2} \leqslant x_2^{\alpha_2} \leqslant \cdots \leqslant x_2^{\alpha_2}$, it is easy to get the following:

$$\frac{1}{(n-1)!} M_{1j} = \frac{1}{n-1} \sum_{1 \leqslant k \leqslant n, k \neq j} x_k^{\alpha_2} = \frac{1}{n-1} \left[\left(\sum_{k=1}^{n} x_k^{\alpha_2} \right) - x_j^{\alpha_2} \right] \tag{13}$$

and

$$\frac{1}{(n-1)!} M_{11} \geqslant \frac{1}{(n-1)!} M_{12} \geqslant \cdots \geqslant \frac{1}{(n-1)!} M_{1n}. \tag{14}$$

From Chebysef's inequality, we observe that

$$\begin{aligned} h_{2,n}(x,\alpha) &= \frac{1}{n} \cdot \sum_{j=1}^{n} x_j^{\alpha_1} \cdot \frac{1}{(n-1)!} M_{1j} \\ &\leqslant \left[\frac{1}{n} \cdot \sum_{j=1}^{n} x_j^{\alpha_1} \right] \cdot \left[\frac{1}{n} \cdot \sum_{j=1}^{n} \frac{1}{(n-1)!} M_{1j} \right] = \prod_{i=1}^{2} \left(\frac{1}{n} \sum_{j=1}^{n} x_j^{\alpha_i} \right). \end{aligned} \tag{15}$$

It follows that inequality (11) is true for $m = 2$. Assume that (11) holds for $m-1$. Then

$$h_{m-1,n}(x,\alpha) \leqslant \prod_{i=1}^{m-1} \left(\frac{1}{n} \sum_{j=1}^{n} x_j^{\alpha_i} \right) \tag{16}$$

For general m, we first prove that (14) holds. It is easy to see that

$$M_{11} = \begin{vmatrix} x_2^{\alpha_2} & x_3^{\alpha_2} & \cdots & x_n^{\alpha_2} \\ \cdots & \cdots & \cdots & \cdots \\ x_2^{\alpha_m} & x_3^{\alpha_m} & \cdots & x_n^{\alpha_m} \\ 1 & 1 & \cdots & 1 \\ \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & \cdots & 1 \end{vmatrix}_{n-1}^{+} = \sum_{i=2}^{n} x_2^{\alpha_i} M_{i2}^{*}$$

$$M_{12} = \begin{vmatrix} x_1^{\alpha_2} & x_3^{\alpha_2} & \cdots & x_n^{\alpha_2} \\ \cdots & \cdots & \cdots & \cdots \\ x_1^{\alpha_m} & x_3^{\alpha_m} & \cdots & x_n^{\alpha_m} \\ 1 & 1 & \cdots & 1 \\ \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & \cdots & 1 \end{vmatrix}_{n-1}^{+} = \sum_{i=2}^{n} x_1^{\alpha_i} M_{i1}^{*},$$

where $x_j^{\alpha_i} = 1$ for $m+1 \leqslant i \leqslant n, 1 \leqslant j \leqslant n$ and $M_{i1}^* = M_{i2}^* > 0$ for $2 \leqslant i \leqslant n$. Therefore $M_{11} - M_{12} = \sum_{i=2}^{n} (x_2^{\alpha_i} - x_1^{\alpha_i}) M_{i1}^* \geqslant 0$. That is

$$\frac{1}{(n-1)!} M_{11} \geqslant \frac{1}{(n-1)!} M_{12}.$$

By similar arguments, we see that (14) is true.

Now by the properties of a permanent, we get

$$\frac{1}{n} \cdot \sum_{j=1}^{n} \frac{1}{(n-1)!} M_{1j} = \frac{1}{n!} \cdot \begin{vmatrix} x_1^{\alpha_2} & x_2^{\alpha_2} & \cdots & x_n^{\alpha_2} \\ \cdots & \cdots & \cdots & \cdots \\ x_1^{\alpha_m} & x_2^{\alpha_m} & \cdots & x_m^{\alpha_m} \\ 1 & 1 & \cdots & 1 \\ \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & \cdots & 1 \end{vmatrix}_n^+ . \tag{17}$$

Combining (12) and (14) with the proved result above, by Chebysef's inequality and inductive hypothesis, we have

$$\begin{aligned} h_{m,n}(x,\alpha) &= \frac{1}{n} \cdot \sum_{j=1}^{n} x_j^{\alpha_1} \cdot \frac{1}{(n-1)!} M_{1j} \\ &\leqslant \left[\frac{1}{n} \cdot \sum_{j=1}^{n} x_j^{\alpha_1} \right] \cdot \left[\frac{1}{n} \cdot \sum_{j=1}^{n} \frac{1}{(n-1)!} M_{1j} \right] \leqslant \prod_{i=1}^{m} \left(\frac{1}{n} \sum_{j=1}^{n} x_j^{\alpha_i} \right). \end{aligned}$$

Hence, inequality (11) is true for m. Setting $m = n$, we obtain the conclusion of Theorem 2. The proof is completed.

COROLLARY 3. Let $f : [0,1] \to R_{++}^1$, $\ln f : [0,1] \to R$ such that f and $\ln f$ are integrable. If $\alpha \in R_+^m$, and $\lim_{n\to+\infty} h_{m,n}(f;\alpha)$ exists, where

$$h_{m,n}(f;\alpha) := \frac{1}{n!} \cdot \begin{vmatrix} f^{\alpha_1}\left(\frac{1}{n}\right) & f^{\alpha_1}\left(\frac{2}{n}\right) & \cdots & f^{\alpha_1}\left(\frac{n}{n}\right) \\ \cdots & \cdots & \cdots & \cdots \\ f^{\alpha_m}\left(\frac{1}{n}\right) & f^{\alpha_m}\left(\frac{2}{n}\right) & \cdots & f^{\alpha_m}\left(\frac{n}{n}\right) \\ 1 & 1 & \cdots & 1 \\ \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & \cdots & 1 \end{vmatrix}_n^+ ,$$

then

$$\left[\exp\left(\int_0^1 \ln f(x) dx \right) \right]^{\sum_{j=1}^{m} \alpha_j} \leqslant \lim_{n\to\infty} h_{m,n}(f;\alpha) \leqslant \prod_{j=1}^{m} \int_0^1 [f(x)]^{\alpha_j} dx.$$

PROOF. Let $A(a) = \frac{1}{n} \sum_{i=1}^{n} a_i, G(a) = \sqrt[n]{\prod_{i=1}^{n} a_i}$. From inequality (11) and the arithmetic-geometric mean inequality, we have

$$[G(f_n)]^{\sum_{j=1}^{m} \alpha_j} \leqslant h_n(a,\alpha) \leqslant \prod_{j=1}^{m} A(f_n^{\alpha_j}).$$

where $f_n = \left(f\left(\frac{1}{n}\right), f\left(\frac{2}{n}\right), \cdots, f\left(\frac{n}{n}\right)\right)$, and $f_n^{\alpha_j} = \left(f^{\alpha_j}\left(\frac{1}{n}\right), f^{\alpha_j}\left(\frac{2}{n}\right), \cdots, f^{\alpha_j}\left(\frac{n}{n}\right)\right)$ for $1 \leqslant j \leqslant n$. It is easy to see that the integral analogues of the above inequalities are just our desired (17). The proof of Corollary 3 is completed.

COROLLARY 4. Let $0 \leqslant \alpha_j \leqslant 1$ for $1 \leqslant j \leqslant n$ and

$$G \subset \Omega_n := \left\{ x \left| \sum_{i=1}^{n} x_i \leqslant n, x \in R_+^n \right.\right\}.$$

Then

$$\int_G h_n(x,\alpha) dx_1 dx_2 \cdots dx_n \leqslant \frac{n^n}{n!}.$$

PROOF. From inequality (6) and the well known power mean inequality ([9], [10]), for $0 \leqslant \alpha_j \leqslant 1$ with $1 \leqslant j \leqslant n$, we have

$$h_n(x,\alpha) \leqslant \prod_{j=1}^{n} \left(\frac{1}{n} \sum_{i=1}^{n} x_i^{\alpha_j} \right) \leqslant \prod_{j=1}^{n} \left(\frac{1}{n} \sum_{i=1}^{n} x_i \right)^{\alpha_j} \leqslant 1.$$

From the above and $x \in G \subset \Omega_n$, it follows that

$$\int_G h_n(x,\alpha) dx_1 dx_2 \cdots dx_n \leqslant \int_G dx_1 dx_2 \cdots dx_n \leqslant \int_{\Omega_n} dx_1 dx_2 \cdots dx_n = \frac{n^n}{n!}.$$

Thus, Corollary 4 is true.

For more information of the Hardy function, please refer to ([3]-[8]). It would also be of interest to show that

$$\frac{h_n(a,\alpha)}{h_n(b,\alpha)} \leqslant \prod_{i=1}^{n} \left(\frac{\sum_{j=1}^{n} a_j^{\alpha_i}}{\sum_{j=1}^{n} b_j^{\alpha_i}} \right), \tag{18}$$

where $b_1 \geqslant b_2 \geqslant \cdots \geqslant b_n > 0$, $a_1/b_1 \geqslant a_2/b_2 \geqslant \cdots \geqslant a_n/b_n > 0$, which is true for $n = 1$ and 2.

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