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Blowup Of Solutions For Evolution Equations With Nonlinear Damping*

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Abstract

The initial boundary value problem for a class of evolution equations with nonlinear damping in a bounded domain is considered. By modifying the method in [8], we prove that any solution, with nonpositive initial energy as well as small positive initial energy, blows up in finite time under some conditions. The estimates of the lifespan of solutions are also given. We improve an earlier result in [12].

1 Introduction

In this paper we are concerned with the blow up of solutions of the initial boundary value problem

$$u_{tt} - \Delta u_t - \sum_{i=1}^{N} \frac{\partial}{\partial x_i} \left[\sigma_i\left(u_{x_i}\right) + \beta_i(u_{x_i t})\right] + h(u_t) = f(u), \tag{1}$$

$$u(x,0) = u_0(x), u_t(x,0) = u_1(x), x \in \Omega, \tag{2}$$

$$u(x,t) = 0, x \in \partial\Omega, t \geq 0, \tag{3}$$

where $\Delta = \sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2}$ and Ω is a bounded domain in R^N, $N \geq 1$, with a smooth boundary $\partial\Omega$ so that the divergence theorem can be applied. Here, $\sigma_i(s) = |s|^{m-2}\, s$ and $\beta_i(s) = |s|^{\gamma-2}\, s$, $i = 1, ..., N$, $m, \gamma > 2$, are continuous functions. $h(s) = |s|^{l-2}\, s$, $l > 2$, is a nonlinear damping term and $f(s) = |s|^{p-2}\, s$, $p > 2$, is a nonlinear source term.

Equations of type (1) are used to describe longitudinal motion in viscoelasticity mechanics, and can also be seen as field equations governing the longitudinal motion of a viscoelastic configuration obeying the nonlinear Voight model [1,2,4,6]. In the case

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where $\beta_i \equiv 0$, $i = 1, ..., N$, $h \equiv 0$ and $f \equiv 0$ and $N = 1$, there has been a rather impressive literature concerning the existence and nonexistence of global solutions and properties of solutions [1,2,5]. When the influence of the nonlinear damping and source terms are considered, there are also many results [6,11,14]. In addition, Clements [3] treated the problem (1)-(3) with $\beta_i \equiv 0, i = 1, ..., N, h \equiv 0$ and $f = f(x,t)$ and obtained the global existence of weak solutions by using monotone operator theory. Later, Ang and Dinh [4] investigated the problem (1)-(3) with $\beta_i \equiv 0$ for $i = 1, ..., N$, $f \equiv 0$ and $h = |u_t|^{\alpha} \operatorname{sgn} u_t$ for $0 < \alpha < 1$. They established the global existence of solutions under some conditions. Recently, in [9,10,13], the authors studied the problem (1)-(3) and obtained global existence results under the growth assumptions on the nonlinear terms and the initial data. These global existence results have been improved by Liu and Zhao [7] by using a new method. As for the nonexistence of global solutions, Yang [12] obtained the blow up properties for the problem (1)-(3) with the following restriction on the initial energy $E(0) < \min\left\{ -\left(\frac{pk_1+mk_2}{p-m}\right)^{1/\delta}, -1 \right\}$, where $E(0)$ is given later in (4), $p > m$ and k_1, k_2 and δ are some positive constants.

In this paper we show that the local solution of the problem (1)-(3) with nonpositive initial energy as well as small positive initial energy blows up in finite time. We modify the method in [8] and obtain nonexistence of global solutions under more relaxed condition on $E(0)$ so that we extend the result of [12].

Let

$$\begin{aligned} U &= L^{\infty}([0,T]; W_0^{1,m}(\Omega)) \cap W^{1,\infty}([0,T]; L^2(\Omega)) \\ &\quad \cap W^{1,l}([0,T]; L^l(\Omega)) \cap W^{1,\gamma}([0,T]; W_0^{1,\gamma}(\Omega)), \end{aligned}$$

where $T > 0$ is a real number. Throughout this paper, $\|\cdot\|_p$ is the norm of $L^p(\Omega)$, $p \geq 1$. We need the following local existence result in [9,10,13].

THEOREM 1. Suppose that $2 < m < p < p^*$ and that $u_0 \in W_0^{1,m}(\Omega)$, $u_1 \in L^2(\Omega)$. Then there exists a unique solution $u \in U$ of (1)-(3) satisfying $u \in U$, where $p^* = \frac{Nm}{N-m}$ if $N > m$, and $p^* = \infty$ if $N \leq m$.

2 Results

We define the energy function associated with a solution u of (1)-(3) by

$$E(t) = \frac{1}{2} \|u_t\|_2^2 + J(t), \ t \geq 0, \tag{4}$$

where

$$J(t) = \frac{1}{m} \|\nabla u(t)\|_m^m - \frac{1}{p} \|u(t)\|_p^p. \tag{5}$$

Note that we have

$$E(t) \geq \frac{1}{m} \|\nabla u(t)\|_m^m - \frac{1}{p} \|u\|_p^p, \ t \geq 0. \tag{6}$$

By Sobolev embedding theorem, we get $E(t) \geq G(\|\nabla u(t)\|_m)$ for $t \geq 0$, where $G(\lambda) = \frac{1}{m}\lambda^m - \frac{B_1^p}{p}\lambda^p$, here B_1 is the Sobolev's constant from $W_0^{1,m}(\Omega)$ to $L^p(\Omega)$. Note that $G(\lambda)$ has the maximum at

$$\lambda_1 = \left(\frac{1}{B_1^p}\right)^{\frac{1}{p-m}}$$

and the maximum value is

$$E_1 = G(\lambda_1) = \left(\frac{1}{m} - \frac{1}{p}\right) B_1^{\frac{-mp}{p-m}}.$$

Furthermore, by means of the divergence theorem, it is easily seen that the following holds.

LEMMA 1. $E(t)$ is a nonincreasing function on $[0, T]$ and

$$E'(t) = -\|u_t\|_l^l - \|\nabla u_t\|_2^2 - \|\nabla u_t\|_\gamma^\gamma. \tag{7}$$

Adapting the idea of Vitillaro [8], we have the following Lemma.

LEMMA 2. Assume that $E(0) < E_1$. (i) If $\|\nabla u_0\|_m < \lambda_1$, then $\|\nabla u(t)\|_m < \lambda_1$ for $t \geq 0$. (ii) If $\|\nabla u_0\|_m > \lambda_1$, then there exists $\lambda_2 > \lambda_1$ such that $\|\nabla u(t)\|_m \geq \lambda_2$ for $t \geq 0$.

THEOREM 2. Suppose that the assumptions of Theorem 1 hold and that $p > \max\{m, l\}$ and $m > \gamma$. If $E(0) < 0$, or, $0 \leq E(0) < E_1$ and $\|\nabla u_0\|_m > \lambda_1$, then the local solution of the problem (1)-(3) blows up at a finite time T.

We remark that the life span T is estimated by $0 < T \leq \frac{L(0)^{1-\theta}}{c_{13}(\theta-1)}$, where $L(t)$ and c_{13} are given in (20) and (30) respectively, and θ is some positive constant given in the following proof.

PROOF. (i) For $0 \leq E(0) < E_1$, we set

$$H(t) = E_2 - E(t),\ t \geq 0, \tag{8}$$

where $E_2 = \frac{E(0)+E_1}{2}$. By (7), we see that $H'(t) \geq 0$. Thus we obtain

$$H(t) \geq H(0) = E_2 - E(0) > 0,\ t \geq 0. \tag{9}$$

Let

$$A(t) = \int_\Omega u u_t dx + \frac{1}{2}\|\nabla u\|_2^2. \tag{10}$$

By differentiating (10) and then using (1), we obtain

$$\begin{aligned} A'(t) &= \|u_t\|_2^2 - \|\nabla u\|_m^m - \int_\Omega |u_t|^{l-2} u_t u dx \\ &\quad - \sum_{i=1}^N \int_\Omega |u_{x_i t}|^{\gamma-2} u_{x_i t} u_{x_i} dx + \|u\|_p^p. \end{aligned} \tag{11}$$

Hence, by (4), we obtain from (11)

$$\begin{aligned} A'(t) &= a_1 \|u_t\|_2^2 + a_2 \|\nabla u(t)\|_m^m - \int_\Omega |u_t|^{l-2} u_t u dx \\ &\quad - \sum_{i=1}^N \int_\Omega |u_{x_i t}|^{\gamma-2} u_{x_i t} u_{x_i} dx + pH(t) - pE_2. \end{aligned} \quad (12)$$

where $a_1 = 1 + \frac{p}{2}$ and $a_2 = \frac{p}{m} - 1$. We observe that $a_i > 0$ for $i = 1, 2$. Moreover

$$\begin{aligned} a_2 \|\nabla u(t)\|_m^m - pE_2 &= a_2 \frac{\lambda_2^m - \lambda_1^m}{\lambda_2^m} \|\nabla u(t)\|_m^m + a_2 \lambda_1^m \frac{\|\nabla u(t)\|_m^m}{\lambda_2^m} - pE_2 \\ &\geq c_1 \|\nabla u(t)\|_m^m + c_2, \end{aligned} \quad (13)$$

where the last inequality is obtained by Lemma 2 (ii), λ_2 is given in Lemma 2, $c_1 = a_2 \frac{\lambda_2^m - \lambda_1^m}{\lambda_2^m}$ and $c_2 = a_2 \lambda_1^m - pE_2$. By Lemma 2 (ii), we have $c_1 > 0$ and by (9), we see that $c_2 > 0$. Thus, by (13), we arrive at

$$\begin{aligned} A'(t) &\geq a_1 \|u_t\|_2^2 + c_1 \|\nabla u(t)\|_m^m - \int_\Omega |u_t|^{l-2} u_t u dx \\ &\quad - \sum_{i=1}^N \int_\Omega |u_{x_i t}|^{\gamma-2} u_{x_i t} u_{x_i} dx + pH(t). \end{aligned} \quad (14)$$

On the other hand, by using Hölder inequality twice, we have

$$\left| \int_\Omega |u_t|^{l-2} u_t u dx \right| \leq c_3 \|u\|_p^{1-\frac{p}{l}} \|u\|_p^{\frac{p}{l}} \|u_t\|_l^{l-1}, \quad (15)$$

where $c_3 = (\mathrm{vol}(\Omega))^{\frac{p-l}{lp}}$. Note that, from (8) and (6), we get

$$H(t) \leq E_1 - \frac{1}{m} \|\nabla u\|_m^m + \frac{1}{p} \|u\|_p^p \leq E_1 - \frac{1}{m} \lambda_1^m + \frac{1}{p} \|u\|_p^p,$$

where the last inequality is derived by Lemma 2(ii). Thus, by (9) and Sobolev embedding theorem, we see that

$$0 < H(0) \leq H(t) \leq \frac{1}{p} \|u\|_p^p \leq k_0 \|\nabla u\|_m^p, \ t \geq 0, \quad (16)$$

where $k_0 = \frac{B_1^p}{p}$. Then, using (16) and Hölder inequality, we have from (15)

$$\left| \int_\Omega |u_t|^{l-2} u_t u dx \right| \leq c_4 \|u\|_p^{\frac{p}{l}} H(t)^{\frac{1}{p} - \frac{1}{l}} \|u_t\|_l^{l-1}.$$

Hence by Young's inequality, we obtain

$$\left| \int_\Omega |u_t|^{l-2} u_t u dx \right| \leq c_4 \left(\varepsilon_1^l \|u\|_p^p + \varepsilon_1^{-l'} H'(t) \right) H(t)^{-\alpha_1}, \quad (17)$$

where $\alpha_1 = \frac{1}{l} - \frac{1}{p} > 0$, $\varepsilon_1 > 0$, $l' = \frac{l}{l-1}$ and $c_4 = c_3 p^{-\alpha_1}$. Letting $0 < \alpha < \alpha_1$ and by (16), we see that

$$\left| \int_\Omega |u_t|^{l-2} u_t u dx \right| \le c_4 \left(\varepsilon_1^l H(0)^{-\alpha_1} \|u\|_p^p + \varepsilon_1^{-l'} H(0)^{\alpha-\alpha_1} H(t)^{-\alpha} H'(t) \right). \tag{18}$$

Similarly, as in deriving (18), we also have

$$\begin{aligned} & \left| \sum_{i=1}^N \int_\Omega |u_{x_i t}|^{\gamma-2} u_{x_i t} u_{x_i} dx \right| \\ \le \quad & c_5 \left(\varepsilon_2^\gamma H(0)^{-\alpha_2} \|\nabla u\|_m^m + \varepsilon_2^{-\gamma'} H(0)^{\alpha-\alpha_2} H(t)^{-\alpha} H'(t) \right), \end{aligned} \tag{19}$$

where $0 < \alpha < \alpha_2$, $\alpha_2 = \frac{m-\gamma}{p\gamma} > 0$, $\varepsilon_2 > 0$, $\gamma' = \frac{\gamma}{\gamma-1}$ and $c_5 = (\mathrm{vol}(\Omega))^{\frac{m-\gamma}{m\gamma}} k_0^{-\alpha_2}$. In order to satisfy both (18) and (19), we choose $0 < \alpha < \min\{\alpha_1, \alpha_2\}$.

Now, we define

$$L(t) = H(t)^{1-\alpha} + \delta_1 A(t), \ t \ge 0, \tag{20}$$

where δ_1 is a positive constant to be specified later. By differentiating (20), and then by (18), (19) and (15), we see that

$$\begin{aligned} L'(t) \quad \ge \quad & \left[1 - \alpha - \delta_1 \left(c_4 \varepsilon_1^{-l'} H(0)^{\alpha-\alpha_1} + c_5 \varepsilon_2^{-\gamma'} H(0)^{\alpha-\alpha_2} \right) \right] H(t)^{-\alpha} H'(t) \\ & + \delta_1 \left[a_1 \|u_t\|_2^2 + pH(t) \right] + \delta_1 \left(c_1 - c_5 \varepsilon_2^\gamma H(0)^{-\alpha_2} \right) \|\nabla u(t)\|_m^m \\ & - \delta_1 c_4 \varepsilon_1^l H(0)^{-\alpha_1} \|u\|_p^p. \end{aligned} \tag{21}$$

Letting $a_3 = \min\{a_1, \frac{mc_1}{3}, \frac{p}{2}\}$ and decomposing $\delta_1 pH(t)$ in (21) by $\delta_1 pH(t) = 2a_3\delta_1 H(t) + (p - 2a_3)\delta_1 H(t)$, and by (8) and (4), we obtain

$$\begin{aligned} L'(t) \quad \ge \quad & \left[1 - \alpha - \delta_1 \left(c_4 \varepsilon_1^{-l'} H(0)^{\alpha-\alpha_1} + c_5 \varepsilon_2^{-\gamma'} H(0)^{\alpha-\alpha_2} \right) \right] H(t)^{-\alpha} H'(t) \\ & + \delta_1 \left[\frac{2a_3}{p} - c_4 \varepsilon_1^l H(0)^{-\alpha_1} \right] \|u\|_p^p + \delta_1 (a_1 - a_3) \|u_t\|_2^2 \\ & + \delta_1 (c_1 - c_5 \varepsilon_2^\gamma H(0)^{-\alpha_2} - \frac{2a_3}{m}) \|\nabla u(t)\|_m^m + (p - 2a_3)\delta_1 H(t). \end{aligned} \tag{22}$$

Now, we choose $\varepsilon_1, \varepsilon_2 > 0$ small enough such that $\varepsilon_1^l \le \frac{a_3}{pc_4} H(0)^{\alpha_1}$, $\varepsilon_2^\gamma \le \frac{a_3}{2mc_5} H(0)^{\alpha_2}$ and

$$0 < \delta_1 < \frac{(1-\alpha)}{2} \left(c_4 \varepsilon_1^{-l'} H(0)^{\alpha-\alpha_1} + c_5 \varepsilon_2^{-\gamma'} H(0)^{\alpha-\alpha_2} \right)^{-1}.$$

Then (22) becomes

$$L'(t) \ge c_6 \delta_1 \left(\|u\|_p^p + \|u_t\|_2^2 + H(t) + \|\nabla u\|_m^m \right), \tag{23}$$

here $c_6 = \min\left\{\frac{a_3}{p},\ a_1 - a_3,\ \frac{a_3}{2m},\ p - 2a_3\right\}$. Thus $L(t)$ is a nondecreasing function for $t \geq 0$. Letting δ_1 be small enough in (20), then we have $L(0) > 0$. Hence $L(t) > 0$ for $t \geq 0$. Now set $\theta = \frac{1}{1-\alpha}$. Since $\alpha < \min\{\alpha_1, \alpha_2\} < 1$, it is evident that $1 < \theta < \frac{1}{1-\min\{\alpha_1,\alpha_2\}}$. By Young's inequality and Hölder inequality, it follows that

$$L(t)^{\theta} \leq c_7 \left[H(t) + \left(\int_{\Omega} u_t u dx \right)^{\theta} + \left(\|\nabla u\|_2^2 \right)^{\theta} \right], \tag{24}$$

where $c_7 = 2^{2(\theta-1)} \max\{1, \delta_1^{\theta}\}$. On the other hand, for $p > 2$ and using Hölder inequality and Young's inequality, we obtain

$$\left(\int_{\Omega} u_t u dx \right)^{\theta} \leq c_8 \|u_t\|_2^{\theta} \|u\|_p^{\theta} \leq c_9 \left(\|u\|_p^{\mu\theta} + \|u_t\|_2^{\nu\theta} \right), \tag{25}$$

where $c_8 = (\mathrm{vol}(\Omega))^{\frac{\theta(p-2)}{2p}}$, $\frac{1}{\mu} + \frac{1}{\nu} = 1$ and $c_9 = c_9(c_8, \mu, \nu) > 0$. Now choose $\alpha \in \left(0,\ \min(\alpha_1,\ \alpha_2,\ \frac{1}{2} - \frac{1}{p},\ 1 - \frac{2}{m})\right)$ and take $\nu = 2(1-\alpha)$ to get $\mu\theta = \frac{2}{1-2\alpha} < p$. Note that, from (16), we see that

$$\left(\frac{1}{pH(0)} \right)^{\frac{1}{p}} \|u\|_p \geq 1 \text{ and } \left(\frac{k_0}{H(0)} \right)^{\frac{1}{p}} \|\nabla u\|_m \geq 1. \tag{26}$$

Thus, from (26), we obtain

$$\|u\|_p^{\mu\theta} = \|u\|_p^{\frac{2}{1-2\alpha}} \leq c_{10} \|u\|_p^p, \tag{27}$$

and by Hölder inequality, we also get

$$\|\nabla u\|_2^{2\theta} \leq c_{11} \|\nabla u\|_m^m, \tag{28}$$

where $c_{10} = \left(\frac{1}{pH(0)} \right)^{1 - \frac{2}{p(1-2\alpha)}}$ and $c_{11} = (\mathrm{vol}(\Omega))^{\frac{(m-2)\theta}{m}} \left(\frac{k_0}{H(0)} \right)^{\frac{m-2\theta}{p}}$. Consequently by (25), (27) and (28), we have from (24)

$$L(t)^{\theta} \leq c_{12} \left[H(t) + \|u\|_p^p + \|\nabla u\|_m^m + \|u_t\|_2^2 \right], \tag{29}$$

here c_{12} is some positive constant. From (29) and (23), we get

$$L'(t) \geq c_{13} L(t)^{\theta},\ t \geq 0, \tag{30}$$

here $c_{13} = \frac{c_6 \delta_1}{c_{12}}$. An integration of (30) over $(0, t)$ then yields

$$L(t) \geq \left(L(0)^{1-\theta} - c_{13}(\theta - 1) t \right)^{-\frac{1}{\theta-1}}. \tag{31}$$

Since $L(0) > 0$, (31) shows that L becomes infinite in a finite time $T \leq T^* = \frac{L(0)^{1-\theta}}{c_{13}(\theta-1)}$.

(ii) For $E(0) < 0$, we set $H(t) = -E(t)$, instead of (8). Then, applying the same arguments as in part (i), we have our result.

REMARK. If $\beta_i = 0$ for $i = 1, ..., N$, then (1)-(3) becomes

$$\begin{aligned} &u_{tt} - \Delta u_t - \sum_{i=1}^{N} \frac{\partial}{\partial x_i} \sigma_i \left(u_{x_i}\right) + h(u_t) = f(u) \text{ in } \Omega \times [0, \infty), \\ &u(x, 0) = u_0(x), u_t(x, 0) = u_1(x), x \in \Omega, \\ &u(x, t) = 0, x \in \partial\Omega, t \geq 0. \end{aligned} \tag{32}$$

The nonexistence of global solution of (32) can be shown by using arguments similar to those in the proof of Theorem 2.

THEOREM 3. Suppose that the assumptions of Theorem 1 hold and that $p > \max\{m, l\}$. If $E(0) < 0$, or, $0 \leq E(0) < E_1$ and $\|\nabla u_0\|_m > \lambda_1$, then the local solution of the problem (32) blows up at a finite time T.

Indeed, we first define the energy function $E(t)$ for problem (32) as in (4) . Repeating the arguments of the proof of Theorem 2, dropping $\sum_{i=1}^{N} \int_\Omega |u_{x_i t}|^{\gamma-2} u_{x_i t} u_{x_i} dx$ in (11), (12) and (14), $\|\nabla u_t\|_\gamma^\gamma$ in (7) and letting $c_5 = 0$ in (21), then, we can easily get the conclusion of Theorem 3.

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