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Bounded Oscillation Of Higher Order Neutral Differential Equations With Oscillating Coefficients*

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Abstract

Sufficient conditions are established for the oscillation of bounded solutions of *n-th* order neutral type nonlinear differential equations with oscillating coefficients of the form

$$[x(t) + p(t)x(\tau(t))]^{(n)} + f(t, x(t), x(\sigma(t))) = s(t)$$

1 Introduction

In the last few years many authors have studied the oscillation theory of neutral differential equations of higher orders. The oscillation of solutions of neutral differential equations is of both theoretical and practical interest. We note that such equations appear in networks containing lossless transmission lines and in the study of vibrating masses attached to an elastic bar (see J. Hale [3]).

In this paper we are interested in the oscillation of bounded solutions of *n-th* order neutral type differential equations of the form

$$[x(t) + p(t)x(\tau(t))]^{(n)} + f(t, x(t), x(\sigma(t))) = s(t), \tag{1}$$

where $n \geq 2$, and the following conditions are always assumed to hold:

(H1) $p(t) \in C(R_+, R)$ is an oscillatory function, with $\lim_{t\to\infty} p(t) = 0$, where $R_+ = [0, \infty)$;

(H2) $\tau(t)$, $\sigma(t) \in C(R_+, R)$, $\sigma(t) \leq t$, and $\lim_{t\to\infty} \tau(t) = \lim_{t\to\infty} \sigma(t) = \infty$;

(H3) $f : R_+ \times R \times R \to R$ is continuous, and $yf(t, x, y) > 0$ for $xy > 0$;

(H4) There exists an oscillatory function $r(t) \in C^n(R_+, R)$, such that $r^{(n)}(t) = s(t)$, and $\lim_{t\to\infty} r(t) = 0$.

In this paper we aim to establish a new criteria for the oscillation of bounded solutions of equation (1) by using the arguments developed by Zafer in [1]. Zafer in [1] established conditions for the oscillation of (1) when the $0 \leq p(t) < 1$. While we consider the case when $p(t)$ is an oscillating function.

As is customary, a solution $x(t)$ of equation (1) is said to be oscillatory if it has arbitrarily large zeros. Otherwise it is called nonoscillatory.

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2 Auxiliary Lemmas

The following lemmas will be needed in the proof of our main results. The first lemma is due to Kiguradze.

LEMMA 1. ([2] p. 193). Let $y(t)$ be an n times differentiable function on R_+ of constant sign, $y^{(n)}(t)$ be of constant sign and not identically equal to zero in any interval $[t_0,\infty)$, $t_0 \geq 0$, and $y(t)y^{(n)}(t) \leq 0$. Then:

i. There exists a $t_1 \geq t_0$ such that $y^{(k)}(t)$, $k = 1, ..., n-1$, is of constant sign on $[t_1,\infty)$,

ii. There exists an integer l, $0 \leq l \leq n-1$, with $n-l$ odd, such that

$$y(t)y^{(k)}(t) > 0, k = 0, 1, ..., l, \quad t \geq t_1, \tag{2}$$

$$(-1)^{n+k-1}y(t)y^{(k)}(t) > 0, k = l+1, ..., n-1, \quad t \geq t_1, \tag{3}$$

and

iii.

$$|y(t)| \geq \frac{(t-t_1)^{n-1}}{(n-1)\dots(n-l)}\left|y^{(n-1)}(2^{n-l-1}t)\right|, t \geq t_1. \tag{4}$$

LEMMA 2. [4]. Let $n \geq 3$ be an odd integer, $\beta(t) \in C(R_+, R_+)$, $0 < \beta(t) \leq \beta_0$, and $y \in C^n(R_+, R)$ such that $(-1)^i y^{(i)}(t) > 0$, $0 \leq i \leq n-1$, and $y^{(n)}(t) \leq 0$. Then

$$y(t-\beta(t)) \geq \frac{(\beta(t))^{n-1}}{(n-1)!}y^{(n-1)}(t), \quad \text{for} \quad t \geq \beta_0. \tag{5}$$

3 Main Results

We have two main results.

THEOREM 1. Assume that $\phi(t)$ is a nonnegative continuous function on R_+, and that $w(t) > 0$ for $t > 0$ is continuous and nondecreasing on R_+ with:

$$|f(t,x,y)| \geq \phi(t)w\left(\frac{|y|}{[\sigma(t)]^{n-1}}\right) \tag{6}$$

and

$$\int_0^{\pm\alpha}\frac{dx}{w(x)} < \infty \text{ for every } \alpha > 0. \tag{7}$$

If n is even and

$$\int^{\infty}\phi(t)dt = \infty, \tag{8}$$

then every bounded solution $x(t)$ of equation (1) is oscillatory.

PROOF. Assume that equation (1) has a bounded non-oscillatory solution $x(t)$. Without loss of generality, we may assume that $x(t)$ is eventually positive (the proof is similar when $x(t)$ is eventually negative). That is, let $x(t) > 0$, $x(\tau(t)) > 0$ and $x(\sigma(t)) > 0$ for $t \geq t_0 \geq 0$. Set

$$z(t) = x(t) + p(t)\, x(\tau(t)) - r(t). \tag{9}$$

From (1) and (9) we have

$$z^{(n)}(t) = -f(t, x(t), x(\sigma(t))) < 0. \tag{10}$$

Thus $z^{(n)}(t) < 0$. It follows that $z^{(i)}(t)$ $(i = 0, 1, ..., n-1)$ is strictly monotonic and of constant sign eventually. Since $p(t)$ and $r(t)$ are oscillating functions, $\lim_{t\to\infty} p(t) = \lim_{t\to\infty} r(t) = 0$, and $x(t)$ is bounded, there exists a $t_1 \geq t_0$ such that $z(t) > 0$ for $t \geq t_1$. Since $x(t)$ is bounded, and by using the facts that $\lim_{t\to\infty} p(t) = \lim_{t\to\infty} r(t) = 0$, then it follows from (9) that there is a $t_2 \geq t_1$, such that $z(t)$ is also bounded for $t \geq t_2$.

Now, by applying Lemma 1, there exist a $t_3 \geq t_2$ and an integer l with $n - l$ odd such that (2), (3), and (4) are satisfied by $z(t)$ for $t \geq t_3$. Since n is even, and $z(t)$ is bounded it follows that $l = 1$ (otherwise $z(t)$ is not bounded). And from (2) we have $z'(t) > 0$, so $z(t)$ is increasing.

Since $x(t)$ is bounded, by (H1) it follows that $\lim_{t\to\infty} p(t)\, x(\tau(t)) = 0$. Then using this fact with $\lim_{t\to\infty} r(t) = 0$, and by (9) there exists a $t_4 \geq t_3$ such that

$$x(t) \geq \lambda z(t), \quad t \geq t_4, \tag{11}$$

where λ is some number in $(0, 1)$.

From (4), and the fact that $z(t)$ is increasing, we have

$$z(t) \geq z\left(2^{2-n} t\right) \geq \frac{2^{(2-n)(n-1)}}{(n-1)} (t - t_5)^{n-1} z^{(n-1)}(t), \quad t \geq t_5 = 2^{n-2} t_3.$$

Therefore, by choosing $t_6 > t_5$, arbitrarily large, we have

$$z(t) \geq c t^{n-1} z^{(n-1)}(t), \quad t \geq t_6, \tag{12}$$

where $c > 0$ is an appropriate constant dependent upon n.

Let $t_7 \geq t_6$ be such that $\sigma(t) \geq t_6$ for all $t \geq t_7$. From (11), (12) and the decreasing character of $z^{(n-1)}(t)$ we then have

$$\frac{x(\sigma(t))}{[\sigma(t)]^{n-1}} \geq \lambda c z^{(n-1)}(t), \quad t \geq t_7. \tag{13}$$

Using (6), and (13) then it follows from (10) that

$$z^{(n)}(t) + \phi(t)\, w\left(\lambda c z^{(n-1)}(t)\right) \leq 0 \tag{14}$$

Setting $u(t) = \lambda c z^{(n-1)}(t)$,and integrating (14) divided by $w(u(t))$ from t_7 to t, we obtain

$$\lambda c \int_{t_7}^{t} \phi(v)\,dv \leq \int_{u(t)}^{u(t_7)} \frac{ds}{w(s)}. \tag{15}$$

Since $u'(t) < 0$, $u(t)$ is decreasing. And since $u(t) > 0$, it follows that $\lim_{t\to\infty} u(t) = L \geq 0$. If $L \neq 0$, then by (15) we must have

$$\int_{t_7}^{\infty} \phi(t)\,dt < \infty, \tag{16}$$

which is contrary to (8). In the case when $L = 0$, letting $t \to \infty$ in (15) and using (7), we again obtain (16). The proof is complete.

In the next theorem besides conditions (H1)-(H4) we further assume that:

(H5) $0 < t - \sigma(t) \leq \sigma_0$, where σ_0 is positive constant.

THEOREM 2. Assume that $\phi(t)$ is a nonnegative continuous function on R_+, and that $w(t) > 0$ for $t > 0$ is continuous and nondecreasing on R_+ with:

$$|f(t,x,y)| \geq \phi(t) w\left(\frac{|y|}{[t-\sigma(t)]^{n-1}}\right) \tag{17}$$

and

$$\int_0^{\pm\alpha} \frac{dx}{w(x)} < \infty \text{ for every } \alpha > 0. \tag{18}$$

If n is odd and

$$\int^{\infty} \phi(t)dt = \infty, \tag{19}$$

then every bounded solution $x(t)$ of equation (1) is oscillatory.

PROOF. Assume that equation (1) has a bounded non-oscillatory solution $x(t)$. Without loss of generality, we may assume that $x(t)$ is eventually positive (the proof is similar when $x(t)$ is eventually negative). That is, let $x(t) > 0$, $x(\tau(t)) > 0$ and $x(\sigma(t)) > 0$ for $t \geq t_0 \geq 0$. Set $z(t)$ as in (9). Then from (1) and (9) we have (10). From (10) $z^{(n)}(t) < 0$. It follows that $z^{(i)}(t)$ $(i = 0, 1, ..., n-1)$ is strictly monotonic and of constant sign eventually. Since $p(t)$ and $r(t)$ are oscillating functions, $\lim_{t\to\infty} p(t) = \lim_{t\to\infty} r(t) = 0$, and $x(t)$ is bounded, there exists a $t_1 \geq t_0$ such that $z(t) > 0$ for $t \geq t_1$. Since $x(t)$ is bounded, and by using $\lim_{t\to\infty} p(t) = \lim_{t\to\infty} r(t) = 0$, then it follows from (9) that there is a $t_2 \geq t_1$, such that $z(t)$ is also bounded for $t \geq t_2$.

Now, by applying Lemma 1, there exist a $t_3 \geq t_2$ and an integer l with $n - l$ odd such that (2), and (3) are satisfied by $z(t)$ for $t \geq t_3$. Since n is odd and $z(t)$ is bounded then $l = 0$ (otherwise $z(t)$ is not bounded). Hence from relations (2), and (3) we have

$$(-1)^k z^{(k)}(t) > 0, \quad k = 0, 1, ..., n-1. \tag{20}$$

Since n is odd, from (H5) and (20) follows that we can apply Lemma 2. Using Lemma 2, we have

$$z(\sigma(t)) = z(t-(t-\sigma(t))) \geq \frac{[t-\sigma(t)]^{n-1}}{(n-1)!} z^{(n-1)}(t), \quad t \geq t_3 + \sigma_0.$$

Hence,

$$z(\sigma(t)) \geq \frac{[t-\sigma(t)]^{n-1}}{(n-1)!} z^{(n-1)}(t), \quad t \geq t_3 + \sigma_0. \tag{21}$$

Since $x(t)$ is bounded, by (H1) it follows that $\lim_{t\to\infty} p(t)\, x(\tau(t)) = 0$. Then using this fact with $\lim_{t\to\infty} r(t) = 0$, and by (9) there exists a $t_4 \geq t_3$ such that

$$x(t) \geq \lambda z(t), \quad t \geq t_4, \tag{22}$$

where λ is some number in $(0,1)$.

Let $t_5 \geq \max\{t_4, t_3 + \sigma_0\}$ be such that $\sigma(t) \geq t_5$ for all $t \geq t_5$. From (21), and (22) we have

$$\frac{x(\sigma(t))}{[t-\sigma(t)]^{n-1}} \geq \frac{\lambda z^{(n-1)}(t)}{(n-1)!}, \quad t \geq t_5. \tag{23}$$

Using (17), and (23), then it follows from (10) that

$$z^{(n)}(t) + \phi(t)\, w\left(\lambda c z^{(n-1)}(t)\right) \leq 0, \tag{24}$$

where $c = \frac{1}{(n-1)!}$. The remainder of the proof is similar to that of Theorem 1 and we omit it.

4 Examples

We consider two examples.

EXAMPLE 1. Consider the equation

$$\begin{aligned} &\left[x(t) + 4e^{-\frac{t+\pi}{2}} \sin\frac{t}{2}\, x\left(\frac{t-\pi}{2}\right)\right]'' + e^{\frac{t-\pi}{6}} \left(4 + 2e^{\frac{-5}{6}t} \cos^{\frac{2}{3}} t\right) x^{\frac{1}{3}} \left(t - \frac{\pi}{2}\right) \\ = \;& -4e^{-\frac{t}{6}} \cos^{\frac{1}{3}} t. \end{aligned}$$

By setting $w(x) = x^{\frac{1}{3}}$, $\phi(t) = \left(t - \frac{\pi}{2}\right)^{\frac{1}{3}} e^{\frac{t-\pi}{6}} \left(4 + 2e^{\frac{-5}{6}t} \cos^{\frac{2}{3}} t\right)$, we can see that all conditions of Theorem 1 are satisfied. Thus every bounded solution of the above equation is oscillatory. In fact, $x(t) = e^{-t} \sin t$ is such a solution.

EXAMPLE 2. Consider the equation

$$\begin{aligned} &\left[x(t) + 4e^{-\frac{t+\pi}{2}} \sin\frac{t}{2}\, x\left(\frac{t-\pi}{2}\right)\right]''' + e^{\frac{t-2\pi}{6}} \left(4 - 2e^{\frac{-5}{6}t} \sin^{\frac{2}{3}} t\right) x^{\frac{1}{3}} (t - \pi) \\ = \;& -4e^{-\frac{t}{6}} \sin^{\frac{1}{3}} t - 2e^{-t} \cos t. \end{aligned}$$

By setting $w(x) = x^{\frac{1}{3}}$, $\phi(t) = \pi^{\frac{2}{3}} e^{\frac{t-2\pi}{6}} \left(4 - 2e^{\frac{-5}{6}t} \sin^{\frac{2}{3}} t\right)$, we can see that all conditions of Theorem 2 are satisfied. Hence every bounded solution is oscillatory. In fact, $x(t) = e^{-t} \sin t$ is such a solution.

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