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Oscillation Criteria For Systems Of Difference Equations With Variable Coefficients*

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Abstract

In this paper, we obtain sufficient conditions for oscillation of all solutions of the system of difference equations with variable coefficients

$$x_i(n+1) - x_i(n) + \sum_{j=1}^{N} p_{ij}(n) x_j(n-l) = 0\,,$$

where $\{p_{ij}(n)\}$ are real sequences with $i, j = 1, 2, ..., N$ and $l \in \mathbb{Z}^+$. Furthermore, we shall establish sufficient conditions for oscillation of all solutions of the system of neutral difference equations with variable coefficients

$$\Delta\left(x_i(n) + c x_i(n-ak)\right) + \sum_{j=1}^{N} p_{ij}(n) x_j(n-l) = 0$$

where $\{p_{ij}(n)\}$ are real sequences with $i, j = 1, 2, ..., N$ and $k, l \in \mathbb{Z}^+$.

1 Introduction

In the present paper, we investigate the oscillatory properties of the system of difference equations with variable coefficients

$$x_i(n+1) - x_i(n) + \sum_{j=1}^{N} p_{ij}(n) x_j(n-l) = 0\,, \quad i = 1, 2, ..., N \tag{1}$$

where $\{p_{ij}(n)\}$ are real sequences with $i, j = 1, 2, ..., N$ and l is positive integer.

Furthermore, we shall establish sufficient conditions for the oscillation of all solutions of neutral system of difference equations with variable coefficients

$$\Delta\left(x_i(n) + c x_i(n-ak)\right) + \sum_{j=1}^{N} p_{ij}(n) x_j(n-l) = 0, \tag{2}$$

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where $\{p_{ij}(n)\}$ are real sequences with $i, j = 1, 2, ..., N$ and $a = \pm 1$ and k, l are positive integers, Δ is the first order forward difference operator, i.e, $\Delta x(n) = x(n+1) - x(n)$.

We define that a solution $x(n) = [x_1(n), x_2(n), ..., x_N(n)]^T$ of equation (1) oscillates if for some $i \in \{1, 2, ..., N\}$ and for every integer $n_0 > 0$, there exists $n > n_0$ such that $x_i(n)x_i(n+1) < 0$. A solution $x(n) = [x_1(n), x_2(n), ..., x_N(n)]^T$ is nonoscillatory if it is not eventually the trivial solution and if each component $x_i(n)$ has eventually constant signum.

Oscillation theory of difference equations has attracted many researchers. In recent years there has been much research activity concerning the oscillation of solutions of delay difference equations. For these oscillatory results, we refer to the $[1-8]$ and the references therein. In [4] Agarwal and Grace established oscillation criteria for the higher order systems of difference equations with constant coefficients. Further, in [5] Chuanxi, Kuruklis and Ladas studied oscillatory behaviour of systems of difference equations with variable coefficients. In this paper, we obtain sufficient conditions for the oscillations of all solutions of (1) and (2).

We shall need the following lemma which is given in [8] (See also [7]).

LEMMA 1. Let k be a positive integer and let $\{p_n\}$ be a sequence of non-negative real numbers such that

$$\sum_{j=0}^{k-1} p_{n+j} > 0 \qquad \text{for all large } n,$$

Assume that $\{x_n\}$ is a solution of the following difference inequalities

$$x_{n+1} - x_n + p_n x_{n-k} \leq 0\,,\ \ n = 0, 1, 2, ...$$

such that

$$x_n > 0 \ \text{ for } n \geq -k,$$

Then the difference equation

$$a_{n+1} - a_n + p_n a_{n-k} = 0,\ \ n = 0, 1, 2, ...$$

has a solution $\{a_n\}$ such that

$$0 < a_n \leq x_n \ \text{ for } n \geq -k$$

and

$$\lim_{n\to\infty} a_n = 0.$$

2 Oscillations of Equations (1) and (2)

In this section, we shall establish sufficient conditions for the oscillations of all solutions of equations (1) and (2).

THEOREM 1. Let $\{p_{ij}(n)\}$ be real sequences with $i, j = 1, 2, ..., N$ and let l be a positive integer. If every solution of the equation

$$z(n+1) - z(n) + p(n)z(n-l) = 0 \tag{3}$$

oscillates, where

$$p(n) = \min_{1 \leq i \leq N} \left\{ p_{ii}(n) - \sum_{j=1,\, j \neq i}^{N} |p_{ji}(n)| \right\} > 0, \tag{4}$$

then every solution of (1) oscillates.

PROOF. Assume that equation (1) has a nonoscillatory and eventually positive solution $x(n) = [x_1(n), x_2(n), ..., x_N(n)]^T$. Then, there exists an integer $n_0 \geq 0$ such that $x_i(n) > 0$ for $n \geq n_0,\ i = 1, 2, ..., N$. If we let

$$w(n) = \sum_{j=1}^{N} x_j(n),$$

then

$$\begin{aligned} w(n+1) - w(n) &= -\sum_{i=1}^{N} p_{ii}(n)x_i(n-l) - \sum_{i=1}^{N} \sum_{j=1,\, j \neq i}^{N} p_{ij}(n)x_j(n-l) \\ &\leq -\sum_{i=1}^{N} p_{ii}(n)x_i(n-l) + \sum_{i=1}^{N} \sum_{j=1,\, j \neq i}^{N} |p_{ji}(n)|\, x_i(n-l). \end{aligned}$$

Therefore, from the above inequality that we find the following

$$w(n+1) - w(n) + \sum_{i=1}^{N} \left[p_{ii}(n) - \sum_{j=1,\, j \neq i}^{N} |p_{ji}(n)| \right] x_i(n-l) \leq 0$$

or

$$w(n+1) - w(n) + p(n)w(n-l) \leq 0\ , \qquad n \geq n_1 \geq n_0. \tag{5}$$

By the eventually positivity of $x_1(n), x_2(n), ..., x_N(n)$, we conclude that $w(n)$ is eventually positive. Then by Lemma 1, we see that

$$z(n+1) - z(n) + p(n)z(n-l) = 0$$

has a positive solution $\{z(n)\}$ for $n \geq n_1$, which contradicts to our hypothesis and completes the proof.

REMARK 1. It is shown in [8] that, if

$$\liminf_{n \to \infty} \left[\frac{1}{k} \sum_{i=n-k}^{n-1} p(i) \right] > \frac{k^k}{(k+1)^{k+1}},$$

then every solution of equation (3) oscillates.

Thus, we have the following corollary is immediate.

COROLLARY 1. Let $p(n)$ be as in (4) and let l be a positive integer. If

$$\liminf_{n\to\infty}\left[\frac{1}{k}\sum_{i=n-k}^{n-1}p(i)\right] > \frac{k^k}{(k+1)^{k+1}}$$

holds, then all solutions of equation (1) oscillate.

THEOREM 2. Assume that $\{p_{ij}(n)\}$ be real sequences with $i,j = 1,2,...,N$, $a = -1$ and that k,l are positive integers. Suppose also that $0 \le c < 1$. If every solution of the equation

$$z(n+1) - z(n) + (1-c)\,p(n)z(n-l) = 0 \tag{6}$$

is oscillatory, where $p(n)$ is defined in (4), then every solution of equation (2) oscillates.

THEOREM 3. Assume that $\{p_{ij}(n)\}$ be real sequences with $i,j = 1,2,...,N$, $a = 1$ and that k,l $(l > k)$ are positive integers. Suppose also that $c > 1$. If every solution of the equation

$$z(n+1) - z(n) + \left(\frac{1-c}{c^2}\right)p(n)z\,(n-(l-k)) = 0 \tag{7}$$

oscillates, where $p(n)$ is defined in (4), then every solution of equation (2) oscillates.

THEOREM 4. Assume that $\{p_{ij}(n)\}$ be real sequences with $i,j = 1,2,...,N$, $a = 1$ and that k,l are positive integers. Suppose also that $c = 1$. If every solution of the equation

$$z(n+1) - z(n) + \frac{1}{2}p(n)z(n-l) = 0 \tag{8}$$

oscillates, where $p(n)$ is defined in (4), then every solution of equation (2) oscillates.

THEOREM 5. Assume that $\{p_{ij}(n)\}$ be real sequences with $i,j = 1,2,...,N$, $a = -1$ and that k,l are positive integers. Suppose also that $-1 \le c < 0$. If every solution of the equation

$$z(n+1) - z(n) + p(n)z(n-l) = 0 \tag{9}$$

oscillates, where $p(n)$ is defined in (4), then every solution of equation (2) oscillates.

PROOF OF THEOREMS 2-5. Suppose that $x(n) = [x_1(n), x_2(n), ..., x_N(n)]^T$ be a nonoscillatory and eventually positive solution of (2), $a = \pm 1$. Then, there exists an integer $n_0 \ge 0$ such that $x_i(n) > 0$ for $n \ge n_0,\ i = 1,2,...,N$. We let

$$z(n) = \sum_{i=1}^{N}x_i(n) + c\sum_{i=1}^{N}x_i(n-ak) \tag{10}$$

Then, we have

$$z(n+1) - z(n) + \sum_{i=1}^{N}\sum_{j=1}^{N} p_{ij}(n)x_j(n-l) = 0.$$

So, as in Theorem 1, we have for $n \geq n_1 \geq n_0$

$$z(n+1) - z(n) + p(n)w(n-l) \leq 0. \tag{11}$$

It is clear that $\{z(n)\}$ and $\{w(n)\}$ are positive sequences, we see from (10) that if $a = -1$ and $0 \leq c < 1$, then eventually $z(n) = w(n) + cw(n+k)$, and we get eventually,

$$w(n) = z(n) - cw(n+k) \geq z(n) - cz(n+k) \geq (1-c)z(n),$$

therefore, we get eventually,

$$w(n-l) \geq (1-c)z(n-l). \tag{12}$$

If $a = 1$ and $c > 1$, then,

$$\begin{aligned} w(n) &= \frac{1}{c}\left(z(n+k) - w(n+k)\right) \\ &= \frac{1}{c}z(n+k) - \frac{1}{c^2}\left(z(n+2k) - w(n+2k)\right) \\ &\geq \frac{1}{c}z(n+k) - \frac{1}{c^2}z(n+k) \\ &= \left(\frac{c-1}{c^2}\right)z(n+k), \end{aligned}$$

therefore, by using above inequlity, we get eventually,

$$w(n-l) \geq \left(\frac{c-1}{c^2}\right)z(n-(l-k)). \tag{13}$$

Now, we take the $a = -1$ and $c = 1$. Then, by (10) eventually,

$$z(n) = w(n) + w(n+k),$$

so eventually,

$$\begin{aligned} w(n) &= z(n) - w(n+k) \\ &\geq z(n) - w(n) \end{aligned}$$

and we have eventually,

$$w(n) \geq \frac{1}{2}z(n). \tag{14}$$

Now, we take the $a = -1$ and $-1 \leq c < 0$. Then, by (10) eventually,

$$z(n) = w(n) + cw(n+k)$$

and we have eventually,

$$w(n) = z(n) - cw(n+k)$$

and so eventually,

$$w(n) \geq z(n)$$

and we have eventually,

$$w(n-l) \geq z(n-l) \tag{15}$$

Next, from the above we have the following
(*i*) If $a = -1$ and $0 \leq c < 1$, then, by (11) and (12), we obtain eventually,

$$z(n+1) - z(n) + (1-c)p(n)z(n-l) \leq 0,$$

(*ii*) If $a = 1$ and $c > 1$, then, by (11) and (13), we obtain eventually,

$$z(n+1) - z(n) + \left(\frac{c-1}{c^2}\right) p(n)z(n-(l-k)) \leq 0,$$

(*iii*) If $a = -1$ and $c = 1$, then, by (11) and (14), we obtain eventually,

$$z(n+1) - z(n) + \frac{1}{2}p(n)z(n-l) \leq 0.$$

(*iv*) If $a = -1$ and $-1 \leq c < 0$, then, by (11) and (15), we obtain eventually,

$$z(n+1) - z(n) + p(n)z(n-l) \leq 0.$$

Thus, the rest of the proof is a slight modification of the proof of Theorem 1.

The following corollaries are immediate.

COROLLARY 2. Let $\{p_{ij}(n)\}$ be real sequences with $i, j = 1, 2, ..., N$, k and l be positive integers, $a = -1$ and $0 \leq c < 1$, if

$$\liminf_{n\to\infty} \left[\frac{1}{k}\sum_{i=n-k}^{n-1} p(i)\right] > \frac{1}{(1-c)}\frac{k^k}{(k+1)^{k+1}},$$

where $p(n)$ is defined in (4), then every solution of (2) oscillates.

COROLLARY 3. Let $\{p_{ij}(n)\}$ be real sequences with $i, j = 1, 2, ..., N$, k and l be positive integers, $a = 1$ and $c > 1$, if

$$\liminf_{n\to\infty} \left[\frac{1}{k}\sum_{i=n-k}^{n-1} p(i)\right] > \left(\frac{c^2}{c-1}\right)\frac{(l-k)^{l-k}}{(l-k+1)^{l-k+1}}$$

where $p(n)$ is defined in (4), then every solution of (2) oscillates.

COROLLARY 4. Let $\{p_{ij}(n)\}$ be real sequences with $i, j = 1, 2, ..., N$, k and l be positive integers, $a = 1$ and $c = 1$, if

$$\liminf_{n\to\infty}\left[\frac{1}{k}\sum_{i=n-k}^{n-1} p(i)\right] > 2\frac{k^k}{(k+1)^{k+1}}$$

where $p(n)$ is defined in (4), then every solution of (2) oscillates.

COROLLARY 5. Let $\{p_{ij}(n)\}$ be real sequences with $i, j = 1, 2, ..., N$, k and l be positive integers, $a = -1$ and $-1 \leq c < 0$, if

$$\liminf_{n\to\infty}\left[\frac{1}{k}\sum_{i=n-k}^{n-1} p(i)\right] > \frac{k^k}{(k+1)^{k+1}},$$

where $p(n)$ is defined in (4), then every solution of (2) oscillates.

References

[1] R. P. Agarwal, Difference Equations and Inequalities, Marcel Dekker, New York, 2000.

[2] R. P. Agarwal and P.J.Y. Wong, Advanced Topics in Difference Equations, Kluwer, Dordrecht, 1997.

[3] R. P. Agarwal, S.R. Grace and D. O'Regan, Oscillation Theory for Difference and Functional Differential Equations, Kluwer Academic Publishers, The Netherlands, 2000.

[4] R. P. Agarwal, S.R. Grace, The oscillation of systems of difference equations, Appl. Math. Lett., 13(2000), 1–7.

[5] C. X. Qian, S. A. Kuruklis and G. Ladas, Oscillations of systems of difference equations with variable coefficients, J. Math. Phy. Sci., 25(1)(1991), 1–12.

[6] L. H. Erbe and B. G. Zhang, Oscillation of discrete analogues of delay equations, Differential Integral Equations, 2(3)(1989), 300–309.

[7] I. Györi and G. Ladas, Oscillation Theory of Delay Differential Equations with Appllications, Oxford University Press, Oxford, 1991.

[8] G. Ladas, Ch. G. Philos and Y. G. Sficas, Sharp Conditions for the oscillation of delay difference equations, Journal of Applied Mathematics and Simulation, 2(2)(1989), 101–111.