**Applied Mathematics E-Notes, 6(2006), 26-32 ©** ISSN 1607-2510
Available free at mirror sites of http://www.math.nthu.edu.tw/∼amen/

# The Steiner Formulas For The Open Planar Homothetic Motions*

Salim Yüce†, Nuri Kuruoğlu‡

Received 18 January 2005

**Abstract**

In this paper, we present the Steiner formula for one-parameter open planar homothetic motions. Using this area formula, the generalization of the Holditch Theorem given by W. Blaschke and H. R. Müller [4, p.142] is expressed during one-parameter open planar homothetic motions. Furthermore, we obtain another formula for the swept surface area.

## 1 Introduction

Let $E$ and $E'$ be moving and fixed Euclidean planes and $\{O;\mathbf{e_1},\mathbf{e_2}\}$ and $\{O';\mathbf{e'_1},\mathbf{e'_2}\}$ be their orthonormal frames (ONFs), respectively. We suppose that $\{O';\mathbf{e'_1},\mathbf{e'_2}\}$ is fixed, whereas the vectors $\mathbf{e_1},\mathbf{e_2}$ are functions of a real parameter $t$. Then we say that $\{O;\mathbf{e_1},\mathbf{e_2}\}$ moves with respect to $\{O';\mathbf{e'_1},\mathbf{e'_2}\}$. Let $\mathbf{x}$ and $\mathbf{x'}$ be the position vectors of a point $X \in E$ with respect to the moving and fixed ONFs, respectively. By taking

$$\mathbf{OO'} = \mathbf{u} = u_1(t)\mathbf{e_1} + u_2(t)\,\mathbf{e_2}, \quad u_1(t),\, u_2(t) \in \mathbf{R},\ t \in I \subset \mathbf{R} \tag{1}$$

the motion defined by the transformation

$$\mathbf{x'} = h\,\mathbf{x} - \mathbf{u} \tag{2}$$

is called *one-parameter planar homothetic motion* with the homothetic scale $h = h(t)$ and will be denoted by $H_1 = E/E'$ for a planar homothetic motion of $E$ against $E'$ (Fig. 1).Furthermore, at the initial time $t = 0$ we consider the moving and fixed ONFs of $E$ and $E'$ are coincident. Taking $\varphi = \varphi(t)$ as the rotation angle between $\mathbf{e_1}$ and $\mathbf{e'_1}$, the equation

$$\begin{array}{l} \mathbf{e_1} = \quad \cos\varphi(t)\,\mathbf{e'_1} + \sin\varphi(t)\,\mathbf{e'_2} \\ \mathbf{e_2} = -\sin\varphi(t)\,\mathbf{e'_1} + \cos\varphi(t)\,\mathbf{e'_2} \end{array}, \quad t \in I \tag{3}$$

*Mathematics Subject Classifications: 53A17

†Ondokuz Mayıs University, Faculty of Arts and Science, Department of Mathematics, Kurupelit 55139, Samsun, Turkey

‡University of Bahçeşhir, Faculty of Arts and Science, Department of Mathematics and Computer Sciences, Bahçeşehir 34538, Istanbul, Turkey

can be written.

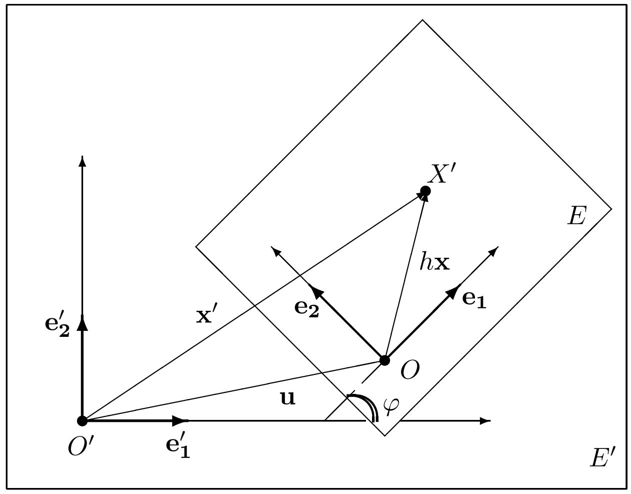


Fig. 1

During the homothetic motion $H_1$, the homothetic scale $h$, the rotation angle $\varphi$ and the vectors $\mathbf{x}$, $\mathbf{x}'$ and $\mathbf{u}$ are continuously differentiable functions of a real time parameter $t$.

If there exists a smallest real number $T > 0$ such that

$$u_j(t+T) = u_j(t), \quad j = 1, 2, \quad \varphi(t+T) \quad = \varphi(t) + 2\pi\nu, \quad \nu \in \mathbf{Z}$$

$$h(t+T) = h(t), \quad h(0) = h(T) = 1, \quad \forall t \in \mathbf{R},$$

then $H_1$ is called *one-parameter closed planar homothetic motion* with the period $T$ and the number of rotations $\nu$. Otherwise, $H_1$ is called *one-parameter open planar homothetic motion*. During the homothetic motion $H_1$, to avoid the case of pure translation we assume that

$$\dot{\varphi}(t) = d\varphi/dt \neq 0.$$

If we differentiate eqs.(2) and (3) with respect to $t$, we get the *sliding velocity* of moving point $X = (x_1, x_2) \in E$ as

$$\mathbf{V}_f = \{-\dot{u}_1 + (u_2 - hx_2)\dot{\varphi} + \dot{h}x_1\}\mathbf{e_1} + \{-\dot{u}_2 + (-u_1 + hx_1)\dot{\varphi} + \dot{h}x_2\}\mathbf{e_2}. \tag{4}$$

If $\mathbf{V}_f = 0$ (i.e. for the points that are fixed in both $E$ and $E'$), then we obtain

$$p_1 = \frac{\dot{h}(\dot{u}_1 - u_2\dot{\varphi}) + h\dot{\varphi}\,(\dot{u}_2 + u_1\dot{\varphi})}{\dot{h}^2 + (h\dot{\varphi})^2}, \; p_2 = \frac{\dot{h}(\dot{u}_2 + u_1\dot{\varphi}) - h\dot{\varphi}\,(\dot{u}_1 - u_2\dot{\varphi})}{\dot{h}^2 + (h\dot{\varphi})^2} \tag{5}$$

where the point $P=(p_1,p_2)$ is called the *rotation pole* or *center of the instantaneous rotation* of the homothetic motion $H_1$. Also the set of the pole points on $E$ and $E'$ are called *the moving* and *fixed pole curves* and denoted by $(P)$ and $(P')$, respectively. Using the pole point we can rewrite eq.(4) as

$$\mathbf{V}_f=\{(x_1-p_1)\dot{h}-(x_2-p_2)h\dot{\varphi}\}\mathbf{e_1}+\{(x_1-p_1)h\dot{\varphi}+(x_2-p_2)\dot{h}\}\mathbf{e_2}.$$

# 2 The Steiner Formulas For The Open Planar Homothetic Motion

## 2.1 I.

We will study surface area swept out by the segment $\mathbf{Q'X}$, which is occurred by a fixed point $X=(x_1,x_2)\in E$ and the fixed point $Q'\in E'$, under the open homothetic motion $H_1$: If $H_1$ is restricted to time interval $[t_1,t_2]$, then, the segment $\mathbf{Q'X^t}$ $(t\in I=[t_1,t_2])$ sweeps a surface with the orientated area

$$F_X^{Q'}=\frac{1}{2}\int_{t_1}^{t_2}[\mathbf{x'}-\mathbf{q'},d\mathbf{x'}], \tag{6}$$

where the symbol $[\alpha,\beta]$ is used instead of the area of parallelogram constituted by the vectors $\alpha$ and $\beta$.From the sliding velocity of a fixed point $X=(x_1,x_2)\in E$ with respect to $E'$, we have

$$d\mathbf{x'}=\{(x_1-p_1)dh-(x_2-p_2)hd\varphi\}\mathbf{e_1}+\{(x_1-p_1)hd\varphi+(x_2-p_2)dh\}\mathbf{e_2}. \tag{7}$$

If we substitute eqs.(2), (5) and (7) into eq. (6), then we find

$$\begin{aligned}2F_X^{Q'} \;=\;& (x_1^2+x_2^2)\int_{t_1}^{t_2}h^2d\varphi-x_1\int_{t_1}^{t_2}(q_1+p_1)\,h^2d\varphi-x_2\int_{t_1}^{t_2}(q_2+p_2)h^2d\varphi\\&+\int_{t_1}^{t_2}[(q_1p_1+q_2p_2)h^2d\varphi+(q_1p_2-q_2p_1)hdh]\\&+x_1\int_{t_1}^{t_2}(q_2-p_2)h\,dh+x_2\int_{t_1}^{t_2}(-q_1+p_1)h\,dh\end{aligned} \tag{8}$$

If $X=O$ $(x_1=x_2=0)$ is taken, then for the swept surface area of the segment $\mathbf{Q'O}$, we get

$$2F_O^{Q'}=\int_{t_1}^{t_2}[(q_1p_1+q_2p_2)h^2d\varphi+(q_1p_2-q_2p_1)hdh]. \tag{9}$$

Moreover, since $\dot{\varphi}(t) \neq 0$ and $\dot{\varphi}(t)$ is a continuous function, we can say that $\dot{\varphi}(t) < 0$ or $\dot{\varphi}(t) > 0$, that is, $\dot{\varphi}(t)$ has the same sign in everywhere in the closed interval $[t_1, t_2]$. Hence using the mean value theorem of integral-calculus for time interval $[t_1, t_2]$, there exists at least one point $t_0 \in [t_1, t_2]$ such that

$$\int_{t_1}^{t_2} h^2 d\varphi = \int_{t_1}^{t_2} h^2 \dot{\varphi}\, dt = h^2(t_0)\delta, \tag{10}$$

where $\delta = \int_{t_1}^{t_2} d\varphi$ is *total rotation angle (Gesamtdrehwinkel)* of the motion. The Steiner point $S = (s_1, s_2)$,which is the center of gravity of the moving pole curve, is given by

$$s_j = \frac{\int_{t_1}^{t_2} h^2 p_j d\varphi}{\int_{t_1}^{t_2} h^2 d\varphi}, \; j = 1, 2. \tag{11}$$

Then from eqs.(8), (9) and (10), we get

$$F_X^{Q'} = F_O^{Q'} + h^2(t_0)\frac{\delta}{2}(x_1^2 + x_2^2 - 2\, a_1\, x_1 - 2\, a_2\, x_2) + \mu_1 x_1 + \mu_2 x_2, \tag{12}$$

such that

$$2h^2(t_0)a_j\delta = \int_{t_1}^{t_2} (q_j + p_j)h^2 d\varphi, \;\; \mu_1 = \frac{1}{2}\int_{t_1}^{t_2} (q_2 - p_2)h\, dh, \;\; \mu_2 = \frac{1}{2}\int_{t_1}^{t_2} (-q_1 + p_1)h\, dh.$$

Eq.(12) is called the *Steiner formula* for the open planar homothetic motion $H_1$.

So, using eq. (12), we can give the following theorems without proof.

THEOREM 1. During the open homothetic motion $H_1$ , all the fixed points $X = (x_1, x_2) \in E$ which have equal surface area $F_X^{Q'}$ lie on the same circle with the center

$$C = (s_1 - \frac{\mu_1}{h^2(t_0)\delta}, \; s_2 - \frac{\mu_2}{h^2(t_0)\delta})$$

in the moving plane $E$.

**Special case 1.** In the case of homothetic scale $h \equiv 1$, from eq. (12), we get

$$F_X^{Q'} = F_O + \frac{\delta}{2}(x_1^2 + x_2^2 - 2s_1x_1 - 2s_2x_2)$$

which was given by Blaschke and Müller [4, p. 117]. If $H_1$ is the closed planar homothetic motion ($\delta = 2\pi\, \nu$), then from eq. (12), we get

$$F_X = F_O + h^2(t_0)\pi\nu(x_1^2 + x_2^2 - 2s_1x_1 - 2s_2x_2) + \mu_1 x_1 + \mu_2 x_2,$$

which was given by Tutar and Kuruoğlu [1].

THEOREM 2**.** Let $A$, $B$ and $X$ be three collinear points in $E$ and $Q'$ be a fixed point in $E'$ . During the open homothetic motion $H_1$, for orientated areas $F_A^{Q'}, F_B^{Q'}$ and $F_X^{Q'}$ of surfaces swept out by the segments $\mathbf{Q'A}, \mathbf{Q'B}$ and $\mathbf{Q'X}$, respectively, we get

$$F_X^{Q'} = [aF_B^{Q'} + bF_A^{Q'}]/(a+b) - h^2(t_0)ab\delta/2. \tag{13}$$

**Special case 2.** In the case of homothetic scale $h \equiv 1$, from eq. (13), we have

$$F_X^{Q'} = [aF_B^{Q'} + bF_A^{Q'}]/(a+b) - ab\delta/2,$$

which was given by Pottmann [2]. If $H_1$ is the closed planar homothetic motion ($\delta = 2\pi\ \nu$), then we obtain

$$F_X = [aF_B + bF_A]/(a+b) - h^2(t_0)\,\pi\,\nu\,ab,$$

which was given by Kuruoğlu and Yüce [3].

Moreover, if we choose another point instead of the fixed point $Q' \in E'$ on the fixed plane $E'$, then eq. (13) is also valid.

## 2.2 II.

Under the open homothetic motion $H_1$, we now calculate the area $F_X^P$ of surface swept by the pole ray $\mathbf{PX}$. If we divide the area element *df* of the swept surface into "*partial triangle*" as shown in Fig. 2, then from Fig. 3,

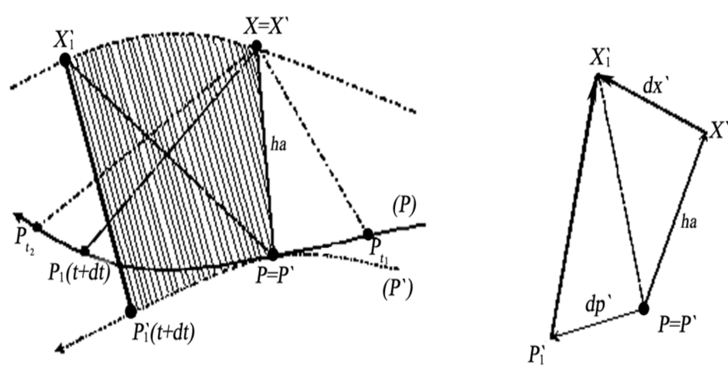


Fig. 2 and Fig. 3

we can write

$$df = \frac{1}{2}[\mathbf{x}' - \mathbf{p}', d\mathbf{x}'] + \frac{1}{2}[\mathbf{P'_1 X'}_1, d\mathbf{p}']$$

or

$$df = \frac{1}{2}[\mathbf{x}' - \mathbf{p}', d\mathbf{x}'] + \frac{1}{2}[\mathbf{x}' + d\mathbf{x}' - \mathbf{p}' - d\mathbf{p}', d\mathbf{p}'].$$

Since $[d\mathbf{x}', d\mathbf{p}'] = 0$ and $[d\mathbf{p}', d\mathbf{p}'] = 0$, we have

$$df = \frac{1}{2}[\mathbf{x}' - \mathbf{p}', d\mathbf{x}'] + \frac{1}{2}[\mathbf{x}' - \mathbf{p}', d\mathbf{p}']. \tag{14}$$

If we denote $|\mathbf{x} - \mathbf{p}| = a$, then we get

$$[\mathbf{x}' - \mathbf{p}', d\mathbf{x}'] = h^2 a^2 d\varphi. \tag{15}$$

Also, since $\mathbf{x}' - \mathbf{p}' = h(\mathbf{x} - \mathbf{p})$ and $d\mathbf{p}' = hd\mathbf{p}$, we get

$$\frac{1}{2}[\mathbf{x}' - \mathbf{p}', d\mathbf{p}'] = \frac{1}{2}h^2[\mathbf{p} - \mathbf{x}, d\mathbf{p}] = h^2(-d\Delta_P), \tag{16}$$

where $d\Delta_p$ is the area of "*infinitesimal triangle*" swept out by the pole ray **PX** on the moving plane $E$.

If we substitute eqs.(15) and (16) into eq. (14), we get

$$df = \frac{1}{2}h^2 a^2 d\varphi - h^2 d\Delta_P. \tag{17}$$

If we integrate the eq.(17) for $t \in [t_1, t_2]$, then we obtain

$$F_X^P = \frac{1}{2}\int_{t_1}^{t_2} h^2(t) a^2(t) d\varphi(t) - \int_{t_1}^{t_2} h^2(t)\ d\Delta_p(t). \tag{18}$$

Using the mean value theorem of integral-calculus for the interval $[t_1, t_2]$, there exists at least one point $t_0 \in [t_1, t_2]$ such that

$$\int_{t_1}^{t_2} h^2(t) a^2(t) d\varphi(t) = h^2(t_0) \int_{t_1}^{t_2} a^2(t) d\varphi(t) \tag{19}$$

and

$$\int_{t_1}^{t_2} h^2(t)\, d\Delta_p(t) = h^2(t_0)\Delta_P. \tag{20}$$

If we substitute eqs. (19) and (20) into eq (18), we get

$$F_X^P = h^2(t_0)\{\frac{1}{2}\int_{t_1}^{t_2} a^2(t) d\varphi(t) - \Delta_P\}, \tag{21}$$

where $\Delta_P$ is the area of triangle bounded by the pole rays $\mathbf{P_{t_1}X}$,$\mathbf{P_{t_2}X}$ of the moving plane $E$ and the arc segment between the points $P_{t_1}$,$P_{t_2}$ of the moving pole curve $(P)$.

**Special case 3.** In the case of the homothetic scale $h \equiv 1$, we get

$$F_X^P = \frac{1}{2}\int_{t_1}^{t_2} a^2(t) d\varphi(t) - \Delta_P$$

which was given by Blaschke and Müller [4,p. 118].

## References

[1] A. Tutar and N. Kuruoğlu, The Steiner formula and the Holditch theorem for the homothetic motions on the planar kinematics, Mech. Mach. Theory, 34 (1999),1-6.

[2] H. Pottmann, Zum Satz von Holditch in der euklidischen Ebene, El. Math., 41 (1986), 1-6.

[3] N. Kuruoğlu and S. Yüce, The generalized Holditch theorem for the homothetic motions on the planar kinematics, Czech Math. J., 54(2) (2004), 337-340.

[4] W. Blaschke and H. R. Müller, Ebene Kinematik, Oldenbourg, München, 1956.