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Exact Stability Regions For Linear Difference Equations With Three Parameters*

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1 Introduction

In [1], Kuruklis has obtained the necessary and sufficient conditions for the asymptotic stability of the following recurrence relation with two constant coefficients and a delay

$$f_n = af_{n-1} + bf_{n-\sigma},\ n = 1, 2, ..., \tag{1}$$

where a, b are real numbers and σ is a positive integer greater than 1. His derivation is based on tracing the zeros of the associated characteristic polynomial

$$\phi(\lambda|a, b) \equiv \lambda^\sigma - a\lambda^{\sigma-1} - b \tag{2}$$

as one of the parameters is changing. Although such a technique yields the correct conditions, the detail seems to be cumbersome. Therefore in [2], Papanicolaou gave an alternate approach which seems to be simpler than that of Kuruklis. In this note, we intend to present yet another derivation of these conditions based on relatively simple arguments. Such new approaches may likely find applications in the study of recurrence relations with several coefficients and delays.

For the sake of convenience, we will denote the maximum of the moduli of the roots of (2) by $\rho(a, b)$. It is well known that $\rho(a, b)$ is a continuous function with respect to (a, b). We will also say that a complex number z is subnormal, normal, or supernormal if $|z| < 1$, $|z| = 1$ or $|z| > 1$ respectively.

We will look for the set $\Omega(\sigma)$ of real number pairs of the form (a, b) such that $\rho(a, b) < 1$.

2 The Case Where σ Is Even

Suppose σ is even. We first observe that the stability region $\Omega(\sigma)$ is symmetric with respect to the y-axis in the x, y-plane, that is,

$$(a, b) \in \Omega(\sigma) \Leftrightarrow (-a, b) \in \Omega(\sigma).$$

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Indeed, this follows from the fact that when σ is even,

$$\phi(-\lambda|a,b) = (-\lambda)^\sigma - a(-\lambda)^{\sigma-1} - b = \lambda^\sigma + a\lambda^{\sigma-1} - b = \phi(\lambda|-a,b).$$

Therefore, we only need to characterize $\Omega(\sigma)$ in the right half plane. For this purpose, we break the right half plane into five mutually exclusive and exhaustive subregions:

$$A = \{(x,y)|\ x \geq 0, x+y \geq 1\}, \tag{3}$$

$$B = \{(x,y)|\ x = 0, -1 < y < 1\}, \tag{4}$$

$$B' = \{(x,y)|\ x = 0, y \leq -1\}, \tag{5}$$

$$C = \{(x,y)|\ y = 0, 0 < x < 1\}, \tag{6}$$

$$D = \{(x,y)|\ x > 0, y > 0, x+y < 1\}, \tag{7}$$

$$E = \{(x,y)|\ x > 0, y < 0, x+y < 1\}. \tag{8}$$

We assert that A is in the complement of $\Omega(\sigma)$. Indeed, if $(a,b) \in A$, then $a+b \geq 1$ so that $\phi(1|a,b) = 1-(a+b) \leq 0$. But since $\lim_{\lambda\to+\infty,\lambda\in R}\phi(\lambda) = +\infty$, we see that $\phi(\lambda|a,b)$ has a real root with modulus 1.

Next, we assert that B is part of $\Omega(\sigma)$. Indeed, if $(a,b) \in B$, then $a = 0$ and $-1 < b < 1$ so that $\phi(\lambda|a,b) = \lambda^\sigma - b$. Clearly, all root of ϕ are subnormal. Similarly, B' cannot be part of $\Omega(\sigma)$.

Next, we assert that C is part of $\Omega(\sigma)$. Indeed, if $(a,b) \in C$, then $\phi(\lambda|a,b) = \lambda^{\sigma-1}(\lambda - a)$. Again all roots of ϕ are subnormal.

Next, we assert that D is part of $\Omega(\sigma)$. Indeed, suppose $(a,b) \in D$, then $a > 0, b > 0$ and $a+b < 1$. If λ is a root of $\phi(\lambda|a,b) = \lambda^\sigma - a\lambda^{\sigma-\tau} - b$, then in view of $b > 0$, we have $\lambda \neq 0$. Hence $1 - a\lambda^{-1} - b\lambda^{-\sigma} = 0$ which implies $a+b < 1 \leq a\left|\lambda^{-1}\right| + b\left|\lambda^{-\sigma}\right|$. But then $a < a\left|\lambda\right|^{-1}$ or $b < b\left|\lambda\right|^{-\sigma}$. In either cases, $|\lambda| < 1$ as required.

The subregion E is more complicated. We will need to make a further subdivision. For this purpose, let us first consider the parametric curve S defined by

$$S = \left\{(x,y)|\ x = \frac{\sin\sigma t}{\sin(\sigma-1)t}, y = \frac{-\sin t}{\sin(\sigma-1)t}, t \in (0,\pi/\sigma)\right\}. \tag{9}$$

When $\sigma = 2$, note that

$$\begin{aligned} S &= \{(x,y)|\ x = 2\cos t, y = -1, t \in (0,\pi/2)\} \\ &= \{(x,y)|\ 0 < x < 2, y = -1\}, \end{aligned}$$

so that S lies completely inside $\Omega(2)$ and separates E into two parts. The same conclusion holds for the case where $\sigma > 2$. To see this, note that $x(t) > 0$ for $t \in (0, \pi/\sigma)$,

$$\lim_{t\to 0^+} (x(t), y(t)) = \left(\frac{\sigma}{\sigma - 1}, \frac{-1}{\sigma - 1}\right),$$

$$\lim_{t\to(\pi/\sigma)^-} (x(t), y(t)) = (0, -1)$$

and the point $(\sigma/(\sigma-1), -1/(\sigma-1))$ lies on the line $x + y = 1$. Note further that

$$y'(t) = \frac{-\cos t \sin(\sigma - 1)t + (\sigma - 1)\sin t \cos(\sigma - 1)t}{\sin^2(\sigma - 1)t}.$$

If we denote the numerator on the right hand side of the above formula by $Q(t)$, then since $Q(0) = 0$ and $Q'(t) = -\sigma(\sigma - 2)\sin t \sin(\sigma - 1)t < 0$ for $t \in (0, \pi/\sigma)$, thus $Q(t) < 0$ for $t \in (0, \pi/\sigma)$. In other words, $y(t)$ decreases from the negative number $-1/(\sigma - 1)$ to the negative number -1. By similar methods, we may also show that $x(t)$ is strictly decreasing for $t \in (0, \pi/\sigma)$. This shows that S is a continuous curve which lies completely inside the region E joining the points $(\sigma/(\sigma-1), -1/(\sigma-1))$ and $(-1, 0)$, that it defines a function $y = S(x)$ whose graph is S, and that it separates E into two mutually exclusive and exhaustive parts

$$F = \{(x, y) \in E|\ y > S(x)\} \tag{10}$$

and

$$G = \{(x, y) \in E|\ y \le \min\{S(x), 1 - x\}\}. \tag{11}$$

We will show that F is part of $\Omega(\sigma)$ and G is in its complement. Once these statements can be proved, the following assertion is then clear.

THEOREM 1. Suppose σ is an even positive integer. The region of stability $\Omega(\sigma)$ is equal to the union of $B \cup C \cup D \cup F$ and that part of the plane which is symmetric to it with respect to the y-axis (see Figure 1),

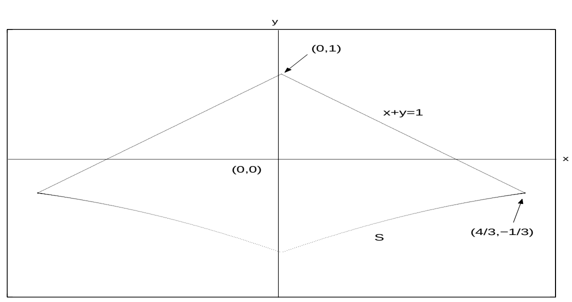

Figure 1. $\Omega(4)$ is the open region bounded inside the curves.

that is,

$$\Omega(\sigma) = \{(x,y)|\ S(|x|) < y < \min\{1+x, 1-x\}\}.$$

Here B, C, D and F are defined by (4), (6), (7) and (10) respectively, and $S(x)$ is the function whose graph is S defined by (9).

In order to show that F is part of $\Omega(\sigma)$, we first need to show the following preparatory result.

LEMMA 1. Suppose σ is even. If $a > 0$ and $b < 0$, then $\rho(a,b)$ is attained by the roots $re^{\pm it}$ of $\phi(\lambda|a,b)$ defined by (2) with phase angle $t \in [0, \pi/\sigma)$, and any other root has modulus strictly less than $\rho(a,b)$.

PROOF. It is easily checked, by looking at the derivative of ϕ, that ϕ as a function of a real variable is strictly decreasing on $(-\infty, 0)$ and $\phi(0|a,b) = -b > 0$. Thus ϕ does not have any negative roots. We may therefore let $\lambda_{\pm 1} = r_1 e^{\pm it_1}$, $\lambda_{\pm 2} = r_2 e^{\pm it_2}, ..., \lambda_{\pm\sigma/2} = r_{\sigma/2} e^{\pm it_{\sigma/2}}$, where $t_1, ..., t_{\sigma/2} \in [0, \pi)$, be all the roots of $\phi(\lambda|a,b)$. Our plan is as follows. We will try to localize the roots in the r,t-plane. Since $b \neq 0$, we have $r_1, ..., r_{\sigma/2} > 0$. Therefore, we can concentrate our attention to the region $\Phi = \{(r,t)|\ r > 0, 0 \le t < \pi\}$ of the r,t-plane.

First of all, consider the positive r-axis in Φ. From $\phi'(\lambda) = \lambda^{\sigma-2}\{\sigma\lambda - a(\sigma-1)\}$ we see that ϕ as a function of a real variable is decreasing on $(-\infty, \bar{\xi})$ and then increasing on $(\bar{\xi}, +\infty)$, where

$$\bar{\xi} = a(\sigma-1)/\sigma,$$

and the absolute minimum is

$$\phi\left(\bar{\xi}|a,b\right) = -\frac{1}{\sigma-1}\bar{\xi}^{\sigma} - b.$$

Let

$$\phi_* = -\frac{1}{\sigma-1}\bar{\xi}^{\sigma}. \tag{12}$$

Then if $b < \phi_*$, ϕ does not have any real roots, if $\phi_* = b$, ϕ has exactly one real root and if $b > \phi_*$, ϕ has two distinct real roots ξ_1 and ξ_2 such that

$$0 < \xi_1 < \bar{\xi} < \xi_2.$$

In other words, depending on b and ϕ_*, there may be no roots, one root, or two roots of ϕ in the positive r-axis of Φ.

Next, consider the interior of Φ. If re^{it}, where $t \in (0,\pi)$, is a root of ϕ, then from $\phi(re^{it}|a,b) = 0$, we have

$$r^{\sigma}\cos\sigma t - ar^{\sigma-1}\cos(\sigma-1)t = b, \tag{13}$$

$$r^{\sigma}\sin\sigma t - ar^{\sigma-1}\sin(\sigma-1)t = 0. \tag{14}$$

Since $r > 0$, we see from (14) that

$$r\sin\sigma t = a\sin(\sigma-1)t. \tag{15}$$

If we multiply (13) by $\sin(\sigma-1)t$ and then replace $a\sin(\sigma-1)t$ by $r\sin\sigma t$ in the resulting equation, we see that

$$b\sin(\sigma-1)t = -r^\sigma \sin t. \tag{16}$$

Furthermore, by squaring both sides of (13) and (14) and adding the resulting equations, we obtain

$$\cos t = \frac{r^{2\sigma} + a^2 r^{2(\sigma-1)} - b^2}{2ar^{2\sigma-1}}. \tag{17}$$

By reversing the previous arguments, we may easily derive (13) and (14) from (15), (16) and (17). It follows that the solutions of (15), (16) and (17) in the interior of Φ give rise to the same set of roots of ϕ in the interior of Φ.

Since $a > 0$ and $b < 0$, if $t \in (0,\pi)$ is a solution of (15) and (16), then

$$\begin{aligned} b\sin(\sigma-1)t &= -r^\sigma \sin t < 0, \\ a\sin(\sigma-1)t &= r\sin\sigma t > 0. \end{aligned}$$

Hence $\sin(\sigma-1)t > 0$ and $\sin\sigma t > 0$, so that

$$t \in \bigcup_{j=1,2,\ldots,\sigma/2} \left(\frac{2(j-1)\pi}{\sigma-1}, \frac{(2j-1)\pi}{\sigma} \right).$$

By analyzing the function $a\sin(\sigma-1)t/\sin\sigma t$, we easily see that the implicit relation (15) defines exactly $\sigma/2$ strictly increasing functions $f_1(r), ..., f_{\sigma/2}(r)$ such that the domain of f_1 is $(\bar{\xi},\infty)$ and its range is contained in $(0,\pi/\sigma)$, and the domains of $f_2, ..., f_{\sigma/2}$ are all equal to $(0,\infty)$ and the corresponding ranges are contained in

$$\left(\frac{2(j-1)\pi}{\sigma-1}, \frac{(2j-1)\pi}{\sigma} \right),\ j = 2,3,...,\sigma/2,$$

respectively.

Next, we consider the implicit relation (17). For the sake of convenience, let us set

$$H(r) = \frac{r^{2\sigma} + a^2 r^{2(\sigma-1)} - b^2}{2ar^{2\sigma-1}},$$

$$\phi(r) = r^\sigma - ar^{\sigma-1} - b,$$

$$\phi_1(r) = r^\sigma + ar^{\sigma-1} + b,$$

$$\phi_2(r) = r^\sigma - ar^{\sigma-1} + b,$$

and

$$\phi_3(r) = r^\sigma + ar^{\sigma-1} - b$$

for $r > 0$. Then it is easily checked that

$$\phi(r) = \phi_2(r) - 2b,\ \phi_3(r) = \phi_1(r) - 2b,\ \phi_3(r) = \phi(r) + 2ar^{\sigma-1},\ \phi_1(r) = \phi_2(r) + 2ar^{\sigma-1},$$

$$\phi_1(r)\phi_3(r) = (r^\sigma + ar^{\sigma-1})^2 - b^2 = (r^\sigma - ar^{\sigma-1})^2 - b^2 + 4ar^{2\sigma-1} = \phi_2(r)\phi(r) + 4ar^{2\sigma-1},$$

and

$$H(r) = \frac{1}{2ar^{2\sigma-1}}\phi_2(r)\phi(r) + 1 = \frac{1}{2ar^{2\sigma-1}}\phi_1(r)\phi_3(r) - 1. \tag{18}$$

Since $\phi_1(0) = b < 0$, $\phi_1(+\infty) = +\infty$ and $\phi_1'(r) = r^{\sigma-2}(\sigma r + (\sigma-1)a) > 0$ for $r > 0$, $\phi_1(r)$ has a unique positive root r_1. Since $\phi_1(r) = \phi_2(r) + 2ar^{\sigma-1} > \phi_2(r)$, $\phi_2(r)$ has a unique positive root r_2 and $r_1 < r_2$. Furthermore, since $\phi_2(r) = \phi(r) + 2b$, ϕ_2 is decreasing on $(0, \bar{\xi})$ and increasing on $(\bar{\xi}, +\infty)$ and hence $r_2 > \max\{r_1, \bar{\xi}\}$.

When $b < \phi_*$, $\phi(r) > 0$ for $r > 0$. Thus $\phi_3(r) = \phi(r) + 2ar^{\sigma-1} > \phi(r) > 0$ for $r > 0$, and $\phi(r)\phi_2(r) < 0 < \phi_1(r)\phi_3(r) = \phi_2(r)\phi(r) + 4ar^{2\sigma-1}$ for $r \in (r_1, r_2)$. It follows that

$$-1 < \frac{1}{2ar^{2\sigma-1}}\phi_2(r)\phi(r) + 1 = H(r) < 1 \text{ for } r \in (r_1, r_2).$$

Hence

$$g_-(r) = (\arccos \circ H)(r),\ r \in (r_1, r_2),$$

is a well defined function on (r_1, r_2), and $\lim_{r\to r_1^+} g_-(r) = \pi$, $\lim_{r\to r_2^-} g_-(r) = 0$.

If we now consider the points of intersection of the graphs of $f_1, f_2, ..., f_{\sigma/2}$ and the graph of g_-, we see that the point of intersection of the graph of g_- and f_1 has the largest horizontal coordinate (see Figure 2).

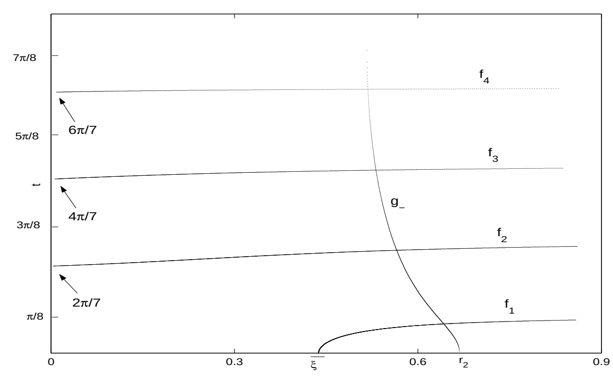

Figure 2. $x \sin \sigma y = a \sin(\sigma - 1)y, \cos y = H(x)$ where $\sigma = 8, a = 1/2, b = -0.01$.

Next, suppose $b = \phi_*$. Then ϕ has the the unique root $\bar{\xi} = (\sigma-1)a/\sigma$ in $(0, +\infty)$. Since $\phi_1(\bar{\xi}) = \phi_1(\bar{\xi}) + \phi(\bar{\xi}) = 2\bar{\xi}^\sigma > 0$, we see that $r_1 < \bar{\xi} < r_2$. As in the previous case, we may see that $H(r) \in (-1, +1)$ for $r \in (r_1, \bar{\xi})$, so that

$$g(r) = (\arccos \circ H)(r),\ r \in (r_1, \bar{\xi})$$

is a well defined function on $(r_1, \bar{\xi})$, and

$$\lim_{r \to r_1^+} g(r) = \pi, \ \lim_{r \to \bar{\xi}^-} g(r) = 0.$$

If we now consider the points of intersection of the graphs of $f_1, f_2, ..., f_{\sigma/2}$ and the graph of g, we see that they are only those from the intersection of the graph of g and $f_2, f_3, ...f_{\sigma/2}$, and their first coordinates are all strictly less than the real root $\bar{\xi}$ of ϕ.

Next, suppose $b > \phi_*$. Then ϕ has exactly two distinct roots ξ_1 and ξ_2 such that $0 < \xi_1 < \bar{\xi} < \xi_2$. Since

$$\phi_1(\xi_1) = \phi_1(\xi_1) + \phi(\xi_1) = 2\xi_1^\sigma > 0$$

and

$$\phi_2(\xi_1) = \phi_2(\xi_1) + \phi(\xi_1) = 2b < 0,$$

we see that $r_1 < \xi_1 < \bar{\xi} < \xi_2 < r_2$. As in the previous case, we may see that

$$g_+(r) = (\arccos \circ H)(r), \ r \in (r_1, \xi_1)$$

is a well defined function on (r_1, ξ_1), and

$$\lim_{r \to r_1^+} g_+(r) = \pi, \ \lim_{r \to \xi_1} g_+(r) = 0.$$

If we now consider the points of intersection of the graphs of $f_1, f_2, ..., f_{\sigma/2}$ and the graph of g_+, we see that they are only those from the intersection of the graph of g and $f_2, f_3, ...f_{\sigma/2}$, and their first coordinates are all strictly less than the real root ξ_1 of ϕ (see Figure 3).

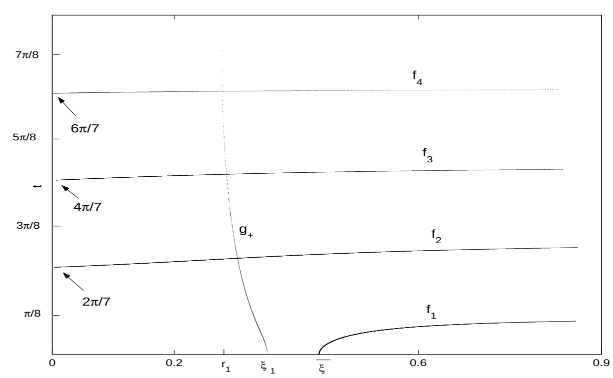

Figure 3. $x \sin \sigma y = a \sin(\sigma - 1)y, \cos y = H(x)$ where $\sigma = 8, a = 1/2, b = -0.0001$.

In all cases, the root among $\lambda_{+1}, ..., \lambda_{+\sigma/2}$ with the largest modulus is attained only by the root which has phase angle $t \in [0, \pi/\sigma)$. The proof is complete.

We now show that F is part of $\Omega(\sigma)$. Let $(a,b) \in F$. Suppose to the contrary that $\rho(a,b) \geq 1$. Then since $\rho(0,0) = 0$, on a continuous path connecting $(0,0)$ and (a,b) and lying inside F, by means of continuity, there would exist a point (α,β) such that

$$\rho(a,b) = 1 = \left|e^{\pm it}\right|,\ t \in [0,\pi),$$

where e^{it} is a root of $\phi(\lambda|a,b)$. In view of Lemma 1, we may assume that $t \in [0,\pi/\sigma)$. If $t = 0$, then $e^{it} = 1$ is a root of $\phi(\lambda|a,b)$, that is, $\beta = 1+\alpha$. This is contrary to the definition of (α,β). If $t \in (0,\pi/\sigma)$, then e^{it} is a root of $\phi(\lambda|a,b)$, that is,

$$\begin{aligned} \cos\sigma t - \alpha\cos(\sigma-1)t &= \beta, \\ \sin\sigma t - \alpha\sin(\sigma-1)t &= 0, \end{aligned}$$

or

$$\alpha = \frac{\sin\sigma t}{\sin(\sigma-1)t},\ \beta = \frac{-\sin t}{\sin(\sigma-1)t}.$$

This says that the point $(\alpha,\beta) \in S$, which is contrary to the definition of F. The proof of our assertion is complete.

Finally, we show that G is in the complement of $\Omega(\sigma)$. Indeed, if $(a,b) \in S$, then $\rho(a,b) = 1$. If $(a,b) \in G\backslash S$ but $\rho(a,b) < 1$. Then since $\rho(0,-2) = 2^{1/2} > 1$. There is some point $(\alpha,\beta) \in G$ such that $\rho(a,b) = 1$. By means of the same arguments above, a contradiction will be reached. The proof of Theorem 1 is now complete.

3 The Case Where σ Is Odd

As in the last Section, we may show that the stability region $\Omega(\sigma)$, when σ is odd, is symmetric with respect to the origin. Therefore, we only need to characterize $\Omega(\sigma)$ in the right half plane. We break the half plane into five mutually exclusive and exhaustive subregions A, B, B', C, D and E defined by (3), (4), (5), (6), (7) and (8) respectively. Furthermore, we may show that A and B' are in the complement of $\Omega(\sigma)$, and that B, C and D are parts of $\Omega(\sigma)$. As before, the subregion E is more complicated. Fortunately, if we define S as in (9), then the same arguments in the last section show that $x(t) > 0$ for $t \in (0,\pi/\sigma)$,

$$\lim_{t\to 0^+}(x(t),y(t)) = \left(\frac{\sigma}{\sigma-1},\frac{-1}{\sigma-1}\right),$$

$$\lim_{t\to(\pi/\sigma)^-}(x(t),y(t)) = (0,-1),$$

and S is a continuous curve which lies completely inside the region E, joins the point $(\sigma/(\sigma-1),-1/(\sigma-1))$ and $(0,-1)$, defines a function $y = S(x)$ whose graph is S, and separates E into two mutually exclusive and exhaustive parts

$$F = \{(x,y) \in E|\ y > S(x)\}$$

and

$$G = \{(x,y) \in E |\ y \leq \min\{S(x), 1-x\}\}.$$

LEMMA 2. Suppose σ is odd, $\sigma \geq 3$, $a > 0$ and $b < 0$. Then $\rho(a,b)$ is achieved by the roots $re^{\pm it}$ of $\phi(\lambda|a,b)$ defined by (2) with phase angle $t \in [0, \pi/\sigma)$, and any other root of $\phi(\lambda|a,b)$ has modulus less than $\rho(a,b)$.

The proof is similar to that of Lemma 1 except that ϕ is now strictly increasing on $(-\infty, 0)$ and $\phi(0|a,b) = -b > 0$ so that $\phi(\lambda|a,b)$ has exactly one negative root $\tilde{\lambda}$. We may therefore let $\lambda_{\pm 1} = r_1 e^{\pm it_1}$, $\lambda_{\pm 2} = r_2 e^{\pm it_2}, ..., \lambda_{\pm(\sigma-1)/2} = r_{(\sigma-1)/2} e^{\pm it_{(\sigma-1)/2}}$, where $t_1, ..., t_{(\sigma-1)/2} \in [0, \pi)$, be the rest of the roots of $\phi(\lambda|a,b)$. The rest of the proof is the same as in Lemma 1 and is thus omitted.

Once Lemma 2 is available, we may repeat the arguments in the previous section to show that F is part of $\Omega(\sigma)$, and G is not. This fact then immediately leads us to the following result.

THEOREM 2. Suppose σ is odd. The region of stability $\Omega(\sigma)$ is equal to the union of $B \cup C \cup D \cup F$ and that part of the plane which is symmetric to it with respect to the origin, where B, C, D and F are defined by (4), (6), (7) and (10) respectively (see Figure 5).

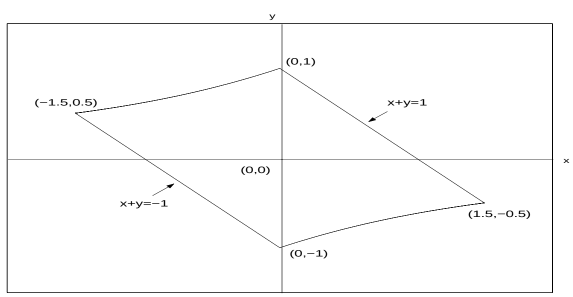

Figure 4. $\Omega(3)$ is the open region bounded by black solid curves.

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