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Explicit Inverses Of Several Tridiagonal Matrices*

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Abstract

Existence conditions of inverses of tri-constant-diagonal matrices with perturbed corner elements are found, and some of the important inverses are calculated to illustrate our results.

1 Introduction

In [1], explicit eigenvalues and eigenvectors are found for tridiagonal matrices of the form

$$A_n = \begin{pmatrix} b+\gamma & c & 0 & 0 & ... & 0 & 0 \\ a & b & c & 0 & ... & 0 & 0 \\ 0 & a & b & c & ... & 0 & 0 \\ ... & ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & 0 & ... & a & b+\delta \end{pmatrix}_{n\times n} \tag{1}$$

where a, b, c and γ, δ are complex numbers. In this paper, explicit inverses of these matrices will be found. Background material can be found in [3].

As in [1], we will base our investigation on the method of symbolic calculus in [2]. For this reason, we recall some terminologies used in [2]. The set of integers, the set of non-negative integers, the set of real numbers and the set of complex numbers are denoted by Z, N, R and C respectively. The number $\sqrt{-1}$ is denoted by i. We will also set $\alpha Z = \{m\alpha |\ m \in Z\}$ for $\alpha \in C$. In particular, πZ denotes the set $\{..., -2\pi, -\pi, 0, \pi, 2\pi, ...\}$. Let l^N be the set of complex sequences of the form $x = \{x_k\}_{k\in N}$ endowed with the usual linear structure. A sequence of the form $\{\alpha, 0, 0, ...\}$ is denoted by $\overline{\alpha}$ (or by α if no confusion is caused), and the sequence $\{0, 1, 0, 0, ...\}$ is denoted by $\hbar$. Given two sequences $x = \{x_k\}$ and $y = \{y_k\}$ in l^N, their convolution is denoted by $x * y$ (or xy if no confusion is caused) and is defined by

$$xy = \left\{ \sum_{k=0}^{j} x_k y_{j-k} \right\}_{j\in N}.$$

It is easily verified that $\hbar^2 = \hbar * \hbar = \{0, 0, 1, 0, 0, ...\}$ and $\hbar^n = \left\{\hbar_j^n\right\}_{j\in Z}$, $n = 1, 2, ...$, is given by $\hbar_j^n = 1$ if $n = j$ and $\hbar_j^n = 0$ otherwise. We will also set $\hbar^0 = \overline{1}$.

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In the following discussions we will assume $ac \neq 0$, and $n \geq 3$ to avoid trivial conditions.

2 Necessary Conditions For The Inverse

Let the (unique) inverse of A_n, if it exists, be denoted by

$$G_n = \left(g^{(1)}|g^{(2)}|\cdots|g^{(n)}\right) = \begin{pmatrix} g_1^{(1)} & g_1^{(2)} & \cdots & \cdots & g_1^{(n)} \\ g_2^{(1)} & g_2^{(2)} & \cdots & \cdots & g_2^{(n)} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ g_n^{(1)} & g_n^{(2)} & \cdots & \cdots & g_n^{(n)} \end{pmatrix}_{n\times n}. \quad (2)$$

Then $A_n G_n = I_n$. This when expanded, can be written as

$$\begin{aligned} ag_0^{(k)} + bg_1^{(k)} + cg_2^{(k)} &= \hbar_1^k - \gamma g_1^{(k)}, \\ ag_1^{(k)} + bg_2^{(k)} + cg_3^{(k)} &= \hbar_2^k, \\ \cdots &= \ldots, \\ ag_{n-1}^{(k)} + bg_n^{(k)} + cg_{n+1}^{(k)} &= \hbar_n^k - \delta g_n^{(k)}, \end{aligned}$$

with

$$g_0^{(k)} = g_{n+1}^{(k)} = 0, \quad k = 1, 2, \ldots, n.$$

Alternatively, we have

$$ag_{j-1}^{(k)} + bg_j^{(k)} + cg_{j+1}^{(k)} = \hbar_j^k + f_j^{(k)}, \quad j, k = 1, 2, \ldots, n, \quad (3)$$

where $f_1^{(k)} = -\gamma g_1^{(k)}$, $f_n^{(k)} = -\delta g_n^{(k)}$ and $f_j^{(k)} = 0$ for $j = 2, \ldots, n-1$. We may view the numbers $g_1^{(k)}, g_2^{(k)}, \ldots, g_n^{(k)}$ respectively as the first, second, ..., and the n-th term of an infinite (complex) sequence $g^{(k)} = \left\{g_j^{(k)}\right\}_{j\in N}$. For each $k \in \{1, \ldots, n\}$, let $f^{(k)} = \left\{f_j^{(k)}\right\}_{j=0}^{\infty}$ be an infinite sequence defined by

$$f_j^{(k)} = \begin{cases} -\gamma g_1^{(k)}, & j = 1, \\ -\delta g_n^{(k)}, & j = n, \\ 0, & \text{otherwise.} \end{cases}$$

Then (3) may be written as a vector equation

$$c\left\{g_{j+2}^{(k)}\right\}_{j=0}^{\infty} + b\left\{g_{j+1}^{(k)}\right\}_{j=0}^{\infty} + a\left\{g_j^{(k)}\right\}_{j=0}^{\infty} = \left\{\hbar_{j+1}^k\right\}_{j=0}^{\infty} + \left\{f_{j+1}^{(k)}\right\}_{j=0}^{\infty}, \quad k = 1, 2, \ldots, n.$$

By the same skill we have used in [1], we may reach the following

$$g^{(k)} = \frac{\left(\overline{c}g_1^{(k)} + \hbar^k + f^{(k)}\right)\hbar}{a\hbar^2 + b\hbar + \overline{c}}. \quad (4)$$

Let

$$\eta_{\pm} = \frac{-b \pm \sqrt{\xi}}{2a}$$

be the two roots of $az^2 + bz + c = 0$, where $\xi = b^2 - 4ac$. According to $\xi \neq 0$ or $\xi = 0$, there are two cases to be considered.

Case I. Suppose $\xi \neq 0$ so that η_+ and η_- are two different numbers. Since $\eta_+\eta_- = c/a \neq 0$, we may write

$$\eta_{\pm} = \frac{1}{\rho} e^{\pm i\phi}$$

for some ϕ in the strip $\{z \in C | 0 \leq \mathrm{Re} z < 2\pi\}$, where

$$\rho = \sqrt{\frac{a}{c}} \text{ and } \cos\phi = \frac{-b}{2\rho c}. \tag{5}$$

Since $\sqrt{\xi} \neq 0$, we also have $\sin\phi \neq 0$ and $\cos\phi \neq \pm 1$. Note also that $\rho^2 c^2 = ac$.

By the method of partial fractions, we may write $g^{(k)}$ in the form

$$\begin{aligned} g^{(k)} &= \frac{1}{\sqrt{\xi}} \left(\frac{1}{\eta_- - \hbar} - \frac{1}{\eta_+ - \hbar} \right) \left(c g_1^{(k)} + \hbar^k + f^{(k)} \right) \hbar \\ &= \frac{1}{\sqrt{\xi}} \left\{ \eta_-^{-(j+1)} - \eta_+^{-(j+1)} \right\}_{j=0}^{\infty} * \left(c g_1^{(k)} + \hbar^k + f^{(k)} \right) \hbar \\ &= \frac{2i}{\sqrt{\xi}} \left\{ \rho^j \sin j\phi \right\}_{j=0}^{\infty} * \left\{ c g_1^{(k)}, -\gamma g_1^{(k)}, 0, ..., 1, 0, ..., -\delta g_n^{(k)}, 0, ... \right\}. \end{aligned}$$

Then we have, by evaluating the convolution in the expression for $g^{(k)}$:

$$\begin{aligned} g_j^{(k)} &= \frac{2i}{\sqrt{\xi}} \Big\{ c g_1^{(k)} \rho^j \sin j\phi - \gamma g_1^{(k)} \rho^{j-1} \sin(j-1)\phi \\ &\quad + H(j-k)\rho^{j-k} \sin(j-k)\phi - H(j-n)\delta g_n^{(k)} \rho^{j-n} \sin(j-n)\phi \Big\} \end{aligned} \tag{6}$$

for $j \geq 1$, where $H(x)$ is the Heaviside function defined by $H(x) = 1$ if $x \geq 0$ and $H(x) = 0$ if $x < 0$. In particular,

$$\frac{\sqrt{\xi}}{2i} g_n^{(k)} = c g_1^{(k)} \rho^n \sin n\phi - \gamma g_1^{(k)} \rho^{n-1} \sin(n-1)\phi + \rho^{n-k} \sin(n-k)\phi$$

and

$$\begin{aligned} \frac{\sqrt{\xi}}{2i} g_{n+1}^{(k)} &= c g_1^{(k)} \rho^{n+1} \sin(n+1)\phi - \gamma g_1^{(k)} \rho^n \sin n\phi \\ &\quad + \rho^{n+1-k} \sin(n+1-k)\phi - \delta g_n^{(k)} \rho \sin\phi \\ &= 0. \end{aligned}$$

If the inverse exists, then $g_1^{(k)}$ and $g_n^{(k)}$ form a **unique** solution pair and hence we must have

$$\begin{aligned}\Delta &= \begin{vmatrix} -c\rho^n \sin n\phi + \gamma\rho^{n-1}\sin(n-1)\phi & \rho c\sin\phi \\ -c\rho^{n+1}\sin(n+1)\phi + \gamma\rho^n \sin n\phi & \delta\rho\sin\phi \end{vmatrix} \\ &= \rho^n\left(ac\sin(n+1)\phi - (\gamma+\delta)\rho c\sin n\phi + \gamma\delta\sin(n-1)\phi\right)\sin\phi \neq 0. \qquad (7)\end{aligned}$$

Furthermore, if $\Delta \neq 0$, then

$$g_1^{(k)} = \frac{\Delta_1}{\Delta}, \qquad (8)$$

where

$$\begin{aligned}\Delta_1 &= \begin{vmatrix} \rho^{n-k}\sin(n-k)\phi & \rho c\sin\phi \\ \rho^{n+1-k}\sin(n+1-k)\phi & \delta\rho\sin\phi \end{vmatrix} \\ &= \rho^{n+1-k}\left(\delta\sin(n-k)\phi - \rho c\sin(n+1-k)\phi\right)\sin\phi \qquad (9)\end{aligned}$$

By substituting these into (6), we obtain

$$\begin{aligned}g_j^{(k)} &= \frac{\rho^{j-k}}{\rho c\sin\phi}\left\{\frac{(\rho c\sin j\phi - \gamma\sin(j-1)\phi)(\delta\sin(n-k)\phi - \rho c\sin(n+1-k)\phi)}{ac\sin(n+1)\phi - (\gamma+\delta)\rho c\sin n\phi + \gamma\delta\sin(n-1)\phi}\right. \\ &\quad \left. + H(j-k)\sin(j-k)\phi\right\} \qquad (10)\end{aligned}$$

for $1 \leq j \leq n$.

Case II. Suppose $\xi = 0$ so that $\eta_\pm$ are two equal roots. In this case, $b^2 = 4ac$, and from (4),

$$\begin{aligned}g^{(k)} &= \frac{\left(\bar{c}g_1^{(k)} + \hbar^k + f^{(k)}\right)\hbar}{c\left(1 - 2\left(\frac{-b}{2c}\right)\hbar + \left(\frac{-b}{2c}\hbar\right)^2\right)} = \frac{1}{\rho c}\frac{\rho\hbar}{(1-\rho\hbar)^2}\left(cg_1^{(k)} + \hbar^k + f^{(k)}\right) \\ &= \frac{1}{\rho c}\left\{j\rho^j\right\}_{j=0}^{\infty} * \left\{cg_1^{(k)}, -\gamma g_1^{(k)}, 0, ..., 1, 0, ..., -\delta g_n^{(k)}, 0, ...\right\},\end{aligned}$$

where $\rho = \dfrac{-b}{2c}$. The j-th term of $g^{(k)}$ now is

$$\begin{aligned}g_j^{(k)} &= \frac{1}{\rho c}\left\{cg_1^{(k)} j\rho^j - \gamma g_1^{(k)}(j-1)\rho^{j-1}\right. \\ &\quad \left. + H(j-k)(j-k)\rho^{j-k} - H(j-n)\delta g_n^{(k)}(j-n)\rho^{j-n}\right\}. \qquad (11)\end{aligned}$$

A similar procedure leads to the necessary condition

$$\Delta = \rho^n\left(ac(n+1) - \rho c(\gamma+\delta)n + \gamma\delta(n-1)\right) \neq 0. \qquad (12)$$

Furthermore, if $\Delta \neq 0$, then

$$g_1^{(k)} = \frac{\Delta_1}{\Delta}, \qquad (13)$$

where

$$\Delta_1 = \rho^{n+1-k} \left(\delta \left(n-k \right) - \rho c \left(n+1-k \right) \right). \tag{14}$$

By substituting these into (11), we obtain

$$\begin{aligned} g_j^{(k)} &= \frac{\rho^{j-k}}{\rho c} \left\{ \frac{(\rho c j - \gamma(j-1)) \left(\delta \left(n-k \right) - \rho c \left(n+1-k \right) \right)}{ac \left(n+1 \right) - \rho c \left(\gamma + \delta \right) n + \gamma \delta \left(n-1 \right)} \right. \\ & \left. + H \left(j-k \right) \left(j-k \right) \right\}. \end{aligned} \tag{15}$$

THEOREM 1. Let the inverse of the matrix A_n be $G_n = \left(g^{(1)} | g^{(2)} | \cdots | g^{(n)} \right)$. If $b^2 - 4ac \neq 0$, then the necessary and sufficient condition for G_n to exist is that (7) holds for some $\phi \in \{ z \in C | 0 \leq \mathrm{Re} z < 2\pi \}$ that satisfies (5). Furthermore, if the inverse exists, then $g_j^{(k)}$ are given by (10). If $b^2 - 4ac = 0$, then the necessary and sufficient condition for G_n to exist is that (12) holds. Furthermore, if the inverse exists, then $g_j^{(k)}$ are given by (15).

We remark that sufficient conditions for the existence of the inverse of A_n are added in the above result. This is valid since the above arguments leading to necessary condition of Theorem 1 can be reversed. We remark also that since $\cos z$ is 2π-periodic, the restriction $\phi \in \{ z \in C | 0 \leq \mathrm{Re} z < 2\pi \}$ can be relaxed to $\phi \in C$.

3 Inverses Of Some Special Toeplitz Matrices

We may now apply Theorem 1 for finding the inverses of several special tridiagonal matrices. For motivation, consider the case where $\gamma = \delta = \rho c$ in A_n.

THEOREM 2. Suppose $\gamma = \delta = \rho c$ in the matrix A_n.

(i) Suppose $b^2 \neq 4ac$. Then the inverse G_n of A_n given by (2) exists if, and only if, $\cos \phi = -b/2\rho c$ for some $\phi \in C$ and $\sin n\phi \neq 0$. Furthermore, if it exists, then

$$g_j^{(k)} = \frac{\rho^{j-k} \left(\cos \left(n+1-j-k \right) \phi + \cos \left(n - |j-k| \right) \phi \right)}{2\rho c \sin \phi \sin n\phi}. \tag{16}$$

(ii) Suppose $b^2 = 4ac$, then the matrix is singular and the inverse does not exist.

PROOF. Suppose the inverse G_n exists and is of the form (2). If $b^2 \neq 4ac$, then substituting $\gamma = \delta = \rho c$ into (7), we necessarily have

$$\cos \phi = -b/2\rho c, \ \ \phi \in C$$

and

$$\Delta = 2\rho^n ac \sin n\phi \left(\cos \phi - 1 \right) \sin \phi \neq 0.$$

If $\sin n\phi \neq 0$, then the inverse exists, and by (10) we have

$$\begin{aligned} g_j^{(k)} &= \frac{\rho^{j-k}}{\rho c \sin \phi} \left(\frac{\cos \left(j - \frac{1}{2} \right) \phi \cos \left(n-k+\frac{1}{2} \right) \phi}{\sin n\phi} + H \left(j-k \right) \sin \left(j-k \right) \phi \right) \\ &= \frac{\rho^{j-k}}{2\rho c \sin \phi \sin n\phi} \times \begin{cases} \cos \left(n-k-j+1 \right) \phi + \cos \left(n-k+j \right) \phi, & j < k \\ \cos \left(n-k-j+1 \right) \phi + \cos \left(n-j+k \right) \phi, & j \geq k \end{cases}, \end{aligned}$$

which is (16).

Once we have found $g_j^{(k)}$, then we may reverse the arguments leading to Theorem 1 and conclude that $\left(g^{(1)}|\cdots|g^{(n)}\right)$ is the inverse of A_n. On the other hand, if $\sin n\phi = 0$, then $\Delta = 0$ and by Theorem 1, the inverse of A_n does not exist.

Suppose $b^2 = 4ac$. By substituting $\gamma = \delta = \rho c$ into (12), we have

$$\Delta = \rho^n ac\,(n + 1 - 2n + n - 1) = 0,$$

the inverse does not exist. The proof is complete.

We may follow the same arguments to show the following for the case where $\gamma = \delta = -\rho c$

THEOREM 3. Suppose $\gamma = \delta = -\rho c$ in the matrix A_n.

(i) Suppose $b^2 \neq 4ac$. Then the inverse G_n of A_n given by (2) exists if, and only if, $\cos\phi = -b/2\rho c$ for some $\phi \in C$ and $\sin n\phi \neq 0$. Furthermore, if it exists, then,

$$g_j^{(k)} = \frac{-\rho^{j-k}\left(\cos\left(n+1-j-k\right)\phi - \cos\left(n-|j-k|\right)\phi\right)}{2\rho c\sin\phi\sin n\phi}. \tag{17}$$

(ii) Suppose $b^2 = 4ac$, then the inverse G_n of A_n given by (2) exists, and

$$g_j^{(k)} = \frac{-\rho^{j-k}}{4n\rho c} \times \begin{cases} (2j-1)(2n+1-2k), & j < k \\ (2k-1)(2n+1-2j), & j \geq k \end{cases}. \tag{18}$$

Theoretically, we can obtain the explicit formulas for the perturbed tridiagonal Toeplitz matrices (1) for arbitrary γ and δ by (10) and (15), though in most cases the formulas may be intrinsically complicated in forms. However, if γ and δ are some special values such as 0 or $\pm\sqrt{ac}$, the formulas are generally elegant in forms, especially in the case $b^2 = 4ac$. In the following we present some derived results in this aspects. The derivation process are simple and are similar to that given above for $\gamma = \delta = \rho c$.

For the sake of simplicity, we will set

$$\Gamma^{\pm} = \frac{-\rho^{j-k}}{\rho c\sin\phi\left(\sin\left(n+1\right)\phi \pm \sin n\phi\right)}$$

in Theorems 4,5,6 and 7.

THEOREM 4. Suppose $\gamma = \rho c$ and $\delta = 0$ in the matrix A_n.

(i) Suppose $b^2 \neq 4ac$. Then the inverse G_n of A_n given by (2) exists if, and only if, $\cos\phi = -b/2\rho c$ for some $\phi \in C$ and $\sin\left(n+1\right)\phi - \sin n\phi \neq 0$. Furthermore, if it exists, then

$$g_j^{(k)} = \Gamma^{-} \times \begin{cases} \left(\sin j\phi - \sin\left(j-1\right)\phi\right)\sin\left(n+1-k\right)\phi, & j < k \\ \left(\sin k\phi - \sin\left(k-1\right)\phi\right)\sin\left(n+1-j\right)\phi, & j \geq k \end{cases}. \tag{19}$$

(ii) Suppose $b^2 = 4ac$. Then the inverse G_n of A_n given by (2) exists, and

$$g_j^{(k)} = \frac{-\rho^{j-k}}{\rho c} \times \begin{cases} n+1-k, & j < k \\ n+1-j, & j \geq k \end{cases}. \tag{20}$$

THEOREM 5. Suppose $\gamma = -\rho c$ and $\delta = 0$ in the matrix A_n.

(i) Suppose $b^2 \neq 4ac$. Then the inverse G_n of A_n given by (2) exists if, and only if, $\cos\phi = -b/2\rho c$ for some $\phi \in C$ and $\sin(n+1)\phi + \sin n\phi \neq 0$. Furthermore, if it exists, then

$$g_j^{(k)} = \Gamma^+ \times \begin{cases} (\sin j\phi + \sin(j-1)\phi)\sin(n+1-k)\phi, & j < k \\ (\sin k\phi + \sin(k-1)\phi)\sin(n+1-j)\phi, & j \geq k \end{cases}. \tag{21}$$

(ii) Suppose $b^2 = 4ac$. Then the inverse G_n of A_n given by (2) exists, and

$$g_j^{(k)} = \frac{-\rho^{j-k}}{\rho c\,(2n+1)} \times \begin{cases} (2j-1)(n+1-k), & j < k \\ (2k-1)(n+1-j), & j \geq k \end{cases}. \tag{22}$$

THEOREM 6. Suppose $\gamma = 0$ and $\delta = \rho c$ in the matrix A_n.

(i) Suppose $b^2 \neq 4ac$. Then the inverse G_n of A_n given by (2) exists if, and only if, $\cos\phi = -b/2\rho c$ for some $\phi \in C$ and $\rho^n \sin(n+1)\phi - \sin\phi \neq 0$. Furthermore, if it exists, then

$$g_j^{(k)} = -\Gamma^- \times \begin{cases} \sin j\phi\,(\sin(n-k) - \sin(n+1-k)\phi), & j < k \\ \sin k\phi\,(\sin(n-j) - \sin(n+1-j)\phi), & j \geq k \end{cases}. \tag{23}$$

(ii) Suppose $b^2 = 4ac$. Then the inverse G_n of A_n given by (2) exists, and

$$g_j^{(k)} = \frac{-\rho^{j-k}}{\rho c} \times \begin{cases} j, & j < k \\ k, & j \geq k \end{cases}. \tag{24}$$

THEOREM 7. Suppose $\gamma = 0$ and $\delta = -\rho c$ in the matrix A_n

(i) Suppose $b^2 \neq 4ac$. Then the inverse G_n of A_n given by (2) exists if, and only if, $\cos\phi = -b/2\rho c$ for some $\phi \in C$ and $\sin(n+1)\phi + \sin n\phi \neq 0$. Furthermore, if it exists, then

$$g_j^{(k)} = \Gamma^+ \times \begin{cases} \sin j\phi\,(\sin(n-k) + \sin(n+1-k)\phi) & j < k \\ \sin k\phi\,(\sin(n-j) + \sin(n+1-j)\phi) & j \geq k \end{cases}. \tag{25}$$

(ii) Suppose $b^2 = 4ac$. Then the inverse G_n of A_n given by (2) exists, and

$$g_j^{(k)} = \frac{-\rho^{j-k}}{\rho c\,(2n+1)} \times \begin{cases} j\,(2n+1-2k), & j < k \\ k\,(2n+1-2j), & j \geq k \end{cases}. \tag{26}$$

THEOREM 8. Suppose $\gamma = -\delta = \rho c$ in the matrix A_n.

(i) Suppose $b^2 \neq 4ac$. Then the inverse G_n of A_n given by (2) exists if, and only if, $\cos\phi = -b/2\rho c$ for some $\phi \in C$ and $\cos n\phi \neq 0$. Furthermore, if it exists, then,

$$g_j^{(k)} = \frac{-\rho^{j-k}(\sin(n+1-j-k)\phi + \sin(n-|j-k|)\phi)}{2\rho c \sin\phi \cos n\phi}. \tag{27}$$

(ii) Suppose $b^2 = 4ac$. Then the inverse G_n of A_n given by (2) exists, and

$$g_j^{(k)} = \frac{-\rho^{j-k}}{2\rho c} \left\{ \begin{array}{ll} 2n+1-2k, & j < k \\ 2n+1-2j, & j \geq k \end{array} \right. . \tag{28}$$

THEOREM 9. Suppose $\gamma = -\ \delta = -\rho c$ in the matrix A_n.

(i) Suppose $b^2 \neq 4ac$. Then the inverse G_n of A_n given by (2) exists if, and only if, $\cos\phi = -b/2\rho c$ for some $\phi \in C$ and $\cos n\phi \neq 0$. Furthermore, if it exists, then

$$g_j^{(k)} = \frac{\rho^{j-k}\left(\sin\left(n+1-j-k\right)\phi - \sin\left(n-|j-k|\right)\phi\right)}{2\rho c \sin\phi \cos n\phi}. \tag{29}$$

(ii) Suppose $b^2 = 4ac$. Then the inverse G_n of A_n given by (2) exists, and

$$g_j^{(k)} = \frac{-\rho^{j-k}}{2\rho c} \times \left\{ \begin{array}{ll} 2j-1, & j < k \\ 2k-1, & j \geq k \end{array} \right. . \tag{30}$$

4 Examples

Matrices of the form (1) when $a = c = 1$ and $b = -2$ are often encountered in mathematical models involving discrete heat equations. In this case, the formulas become very simple. Here are some numerical examples for $n = 5$.

EXAMPLE 1. Suppose $\gamma = \rho c = 1, \delta = 0$. Then by (20), we have

$$\begin{pmatrix} -1 & 1 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 & -2 \end{pmatrix}^{-1} = -\begin{pmatrix} 5 & 4 & 3 & 2 & 1 \\ 4 & 4 & 3 & 2 & 1 \\ 3 & 3 & 3 & 2 & 1 \\ 2 & 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}. \tag{31}$$

Suppose $\gamma = -\rho c = -1, \delta = 0$, then by (22), we have

$$\begin{pmatrix} -3 & 1 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 & -2 \end{pmatrix}^{-1} = \frac{-1}{11}\begin{pmatrix} 5 & 4 & 3 & 2 & 1 \\ 4 & 12 & 9 & 6 & 3 \\ 3 & 9 & 15 & 10 & 5 \\ 2 & 6 & 10 & 14 & 7 \\ 1 & 3 & 5 & 7 & 9 \end{pmatrix}. \tag{32}$$

Suppose $\gamma = \delta = -\rho c = -1$, then by (18), we have

$$\begin{pmatrix} -3 & 1 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 & -3 \end{pmatrix}^{-1} = \frac{-1}{20}\begin{pmatrix} 9 & 7 & 5 & 3 & 1 \\ 7 & 21 & 15 & 9 & 3 \\ 5 & 15 & 25 & 15 & 5 \\ 3 & 9 & 15 & 21 & 7 \\ 1 & 3 & 5 & 7 & 9 \end{pmatrix}$$

Suppose $\gamma = \delta = \rho c = 1$, then by Theorem 2, the matrix is singular.

EXAMPLE 2. Consider the following perturbed Toeplitz matrix

$$A_4 = \begin{pmatrix} -1+\gamma & -i & 0 & 0 \\ i & -1 & -i & 0 \\ 0 & i & -1 & -i \\ 0 & 0 & i & -1+\delta \end{pmatrix}.$$

Since $a = i, c = \overline{a} = -i, b = -1$ so that $b^2 \neq 4ac$, $\rho = i$, $\rho c = 1$ and $\phi = \cos^{-1}\frac{1}{2} = \frac{\pi}{3}$. Suppose $\gamma = \rho c = 1, \delta = 0$. Then by Theorem 2, since $\sin\frac{5\pi}{3} - \sin\frac{4\pi}{3} = 0$, we see that the matrix

$$\begin{pmatrix} 0 & -i & 0 & 0 \\ i & -1 & -i & 0 \\ 0 & i & -1 & -i \\ 0 & 0 & i & -1 \end{pmatrix}$$

is singular and the inverse does not exist.

Suppose $\gamma = -1, \delta = 0$. By Theorem 5, since $\sin\frac{\pi}{3}\left(\sin\frac{5\pi}{3} + \sin\frac{4\pi}{3}\right) = -\frac{3}{2} \neq 0$, hence the inverse exists and, by (21), we have

$$g_j^{(k)} = \frac{2\,(i)^{j-k}}{3} \times \begin{cases} \left(\sin\frac{j\pi}{3} + \sin\left(\frac{(j-1)\,\pi}{3}\right)\right)\sin\left(\frac{(5-k)\,\pi}{3}\right) & j < k \\ \left(\sin\frac{k\pi}{3} + \sin\left(\frac{(k-1)\,\pi}{3}\right)\right)\sin\left(\frac{(5-j)\,\pi}{3}\right) & j \geq k \end{cases},$$

which gives

$$\begin{pmatrix} -2 & -i & 0 & 0 \\ i & -1 & -i & 0 \\ 0 & i & -1 & -i \\ 0 & 0 & i & -1 \end{pmatrix}^{-1} = \frac{1}{2}\begin{pmatrix} -1 & 0 & -1 & i \\ 0 & 0 & -2i & -2 \\ -1 & 2i & 1 & -i \\ -i & -2 & i & -1 \end{pmatrix}. \tag{33}$$

Suppose $\gamma = 0, \delta = -1$. By Theorem 7, since $\sin\frac{\pi}{3}\left(\sin\frac{5\pi}{3} + \sin\frac{4\pi}{3}\right) = -\frac{3}{2} \neq 0$, hence the inverse exists and, by (25), we have

$$g_j^{(k)} = \frac{2\,(i)^{j-k}}{3} \times \begin{cases} \sin\frac{j\pi}{3}\left(\sin\left(\frac{(4-k)\,\pi}{3}\right) + \sin\left(\frac{(5-k)\,\pi}{3}\right)\right) & j < k \\ \sin\frac{k\pi}{3}\left(\sin\left(\frac{(4-j)\,\pi}{3}\right) + \sin\left(\frac{(5-j)\,\pi}{3}\right)\right) & j \geq k \end{cases},$$

which gives

$$\begin{pmatrix} -1 & -i & 0 & 0 \\ i & -1 & -i & 0 \\ 0 & i & -1 & -i \\ 0 & 0 & i & -2 \end{pmatrix}^{-1} = \frac{1}{2}\begin{pmatrix} -1 & -i & -2 & i \\ i & 1 & -2i & -1 \\ -2 & 2i & 0 & 0 \\ -i & -1 & 0 & -1 \end{pmatrix}.$$

This may also be obtained by (33) and the property of symmetry.

EXAMPLE 3. Direct application of Theorem 1 when $\gamma, \delta \neq \rho c$ or 0 is also possible, as we have pointed out in the last section. For example, suppose $\gamma = i, \delta = -i$ in the matrix

$$A_4 = \begin{pmatrix} 2+\gamma & -i & 0 & 0 \\ i & 2 & -i & 0 \\ 0 & i & 2 & -i \\ 0 & 0 & i & 2+\delta \end{pmatrix},$$

where $n = 4, a = i, c = -i$ and $b = 2$ so that $b^2 = 4ac$ and $\rho = -b/2c = -i$ and $\rho c = -1$. Then by (12)

$$\Delta = (-i)^4 (5+3) = 8 \neq 0.$$

Hence the inverse exists, and by (15) of Theorem 1 with $\rho = -i$, we have

$$\begin{aligned} g_j^{(k)} &= -(-i)^{j-k} \left(\frac{(-j - i(j-1))(-i(4-k) + (5-k))}{(4+1)+(4-1)} + H(j-k)(j-k) \right) \\ &= \frac{-(i)^{k-j}}{8} \times \begin{cases} 2jk - 9j - k + 4 + i(5-j-k) & j < k \\ 2jk - 9k - j + 4 + i(5-j-k) & j \geq k \end{cases}, \end{aligned}$$

which gives

$$\begin{pmatrix} 2+i & -i & 0 & 0 \\ i & 2 & -i & 0 \\ 0 & i & 2 & -i \\ 0 & 0 & i & 2-i \end{pmatrix}^{-1} = \frac{-1}{8} \begin{pmatrix} -4+3i & -2-3i & 2-i & i \\ 2+3i & -8+i & -5i & 2+i \\ 2-i & 5i & -8-i & 2-3i \\ -i & 2+i & -2+3i & -4-3i \end{pmatrix}.$$

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