Applied Mathematics E-Notes, 5(2005), 84-112 © ISSN 1607-2510
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A Survey Of The Spectral And Differential Geometric Aspects Of The Generalized De Rham-Hodge Theory Related With Delsarte Transmutation Operators In Multidimension And Applications To Spectral And Soliton Problems Part II*

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Received 19 June 2004

Abstract

A review on spectral and differential-geometric properties of Delsarte transmutation operators in multi-dimension is given. Their differential, geometrical and topological structure in multi-dimension is analyzed, the relationships with generalized De Rham-Hodge theory of differential complexes are stated. Some applications to integrable dynamical systems theory in multi-dimension are presented.

1 Generalized De Rham-Hodge Theory Aspects And Related Delsarte-Darboux Type Binary Transformations

1.1 A differential-geometric analysis of Delsarte-Darboux type transformations that was done in Section 5 of Part 1 [23] for differential operator expressions acting in a functional space $\mathcal{H} = L_1(\mathrm{T};H)$, where $\mathrm{T} = \mathbb{R}^2$ and $H := L_2(\mathbb{R}^2;\mathbb{C}^2)$, appears to have a deep relationship with classical generalized De Rham-Hodge theory [3, 4, 5, 6] devised in the midst of the past century for a set of commuting differential operators defined, in general, on a smooth compact m-dimensional metric space M. Concerning our problem

*Mathematics Subject Classifications: 34A30, 34B05, 34B15.

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of describing the spectral structure of Delsarte-Darboux type transmutations acting in $\mathcal{H}$, we preliminarily consider following Part 1 [23] some backgrounds of the generalized de Rham-Hodge differential complex theory devised for studying transformations of differential operators. Consider a smooth metric space M being a suitably compactified form of the space $\mathbb{R}^m$, $m \in \mathbb{Z}_+$. Then one can define on $M_{\mathrm{T}} := \mathrm{T} \times M$ the standard Grassmann algebra $\Lambda(M_{\mathrm{T}}; \mathcal{H})$ of differential forms on $\mathrm{T}\times M$ and consider a generalized external Skrypnik [3, 4] anti-differentiation operator $d_{\mathcal{L}} : \Lambda(M_{\mathrm{T}}; \mathcal{H}) \rightarrow \Lambda(M_{\mathrm{T}}; \mathcal{H})$ acting as follows: for any $\beta^{(k)} \in \Lambda^k(M_{\mathrm{T}}; \mathcal{H})$, $k = \overline{0, m}$,

$$d_{\mathcal{L}}\beta^{(k)} := \sum_{j=1}^{2} dt_j \wedge \mathrm{L}_j(t; x|\partial)\beta^{(k)} \in \Lambda^{k+1}(M_{\mathrm{T}}; \mathcal{H}), \tag{1}$$

where, by definition,

$$\mathrm{L}_j(t; x|\partial) := \partial/\partial t_j - L_j(t; x|\partial) \tag{2}$$

$j = \overline{1, 2}$, are suitably defined linear differential operators in $\mathcal{H}$, commuting to each other, that is

$$[\mathrm{L}_1, \mathrm{L}_2] = 0. \tag{3}$$

We will put, in general, that differential expressions

$$L_j(t; x|\partial) := \sum_{|\alpha|=0}^{n_j(L)} a_\alpha^{(j)}(t; x)\frac{\partial^{|\alpha|}}{\partial x^\alpha}, \tag{4}$$

with coefficients $a_\alpha^{(j)} \in C^1(\mathrm{T}; C^\infty(M; End\mathbb{C}^N))$, $|\alpha| = \overline{0, n_j(L)}$ $n_j^\alpha \in \mathbb{Z}_+$, $j = \overline{0, 1}$, are some closed normal densely defined operators in the Hilbert space H for any $t \in \mathrm{T}$. It is easy to observe that the anti-differentiation of $d_{\mathcal{L}}$ defined by (1) is a generalization of the usual external anti-differentiation

$$d = \sum_{j=1}^{m} dx_j \wedge \frac{\partial}{\partial x_j} + \sum_{s=1}^{2} dt_s \wedge \frac{\partial}{\partial t_s} \tag{5}$$

for which, evidently, commutation conditions

$$[\frac{\partial}{\partial x_j}; \frac{\partial}{\partial x_k}] = 0, \quad [\frac{\partial}{\partial t_s}; \frac{\partial}{\partial t_l}] = 0, \; [\frac{\partial}{\partial x_j}; \frac{\partial}{\partial t_s}] = 0 \tag{6}$$

hold for all $j, k = \overline{1, m}$ and $s, l = \overline{1, 2}$. If now we substitute within (5) $\partial/\partial x_j \longrightarrow \mathrm{A}_j$, $\partial/\partial t_s \longrightarrow \mathrm{L}_s$, $j = \overline{1, m}$, $s = \overline{1, 2}$, we get the anti-differentiation

$$d_{\mathcal{A}} := \sum_{j=1}^{m} dx_j \wedge \mathrm{A}_j(t; x|\partial) + \sum_{j=1}^{2} dt_s \wedge \mathrm{L}_s(t; x|\partial), \tag{7}$$

where the differential expressions $\mathrm{A}_j, \mathrm{L}_S : \mathcal{H} \longrightarrow \mathcal{H}$ for all $j, k = \overline{1, m}$ and $s, l = \overline{1, 2}$, satisfy the commutation conditions $[\mathrm{A}_j, \mathrm{A}_k] = 0$, $[\mathrm{L}_s, \mathrm{L}_s] = 0$, $[\mathrm{A}_j, \mathrm{L}_s] = 0$, then the operation (7) defines on $\Lambda(M_{\mathrm{T}};\mathcal{H})$ an anti-differential with respect to which the co-chain complex

$$\mathcal{H} \longrightarrow \Lambda^0(M_{\mathrm{T}};\mathcal{H}) \xrightarrow{d_{\mathcal{A}}} \Lambda^1(M_{\mathrm{T}};\mathcal{H}) \xrightarrow{d_{\mathcal{A}}} ... \xrightarrow{d_{\mathcal{A}}} \Lambda^{\mathrm{m}+2}(M_{\mathrm{T}};\mathcal{H}) \xrightarrow{d_{\mathcal{A}}} 0 \tag{8}$$

is evidently closed, that is $d_{\mathcal{A}}d_{\mathcal{A}} \equiv 0$. As the anti-differential (1) is a particular case of (7), we obtain that the corresponding to it co-chain complex (8) is closed too.

1.2 Below we will follow ideas developed in [3, 4, 5, 6, 28]. A differential form $\beta \in \Lambda(M_{\mathrm{T}};\mathcal{H})$ will be called $d_{\mathcal{A}}$-closed if $d_{\mathcal{A}}\beta = 0$ and a form $\gamma \in \Lambda(M_{\mathrm{T}};\mathcal{H})$ will be called exact or $d_{\mathcal{A}}$-homological to zero if there exists on M_{T} such a form $\omega \in \Lambda(M_{\mathrm{T}};\mathcal{H})$ that $\gamma = d_{\mathcal{A}}\omega$.

Consider now the standard [27, 28, 8, 30] algebraic Hodge star-operation

$$* : \Lambda^{\mathrm{k}}(M_{\mathrm{T}};\mathcal{H}) \longrightarrow \Lambda^{\mathrm{m}+2-\mathrm{k}}(M_{\mathrm{T}};\mathcal{H}), \tag{9}$$

$k = \overline{0, m+2}$, as follows: if $\beta \in \Lambda^{\mathrm{k}}(M_{\mathrm{T}};\mathcal{H})$, then the form $*\beta \in \Lambda^{\mathrm{m}+2-\mathrm{k}}(M_{\mathrm{T}};\mathcal{H})$ is such that:

- $(m-k+2)$ - dimensional volume $|*\beta|$ of the form $*\beta$ equals k-dimensional volume $|\beta|$ of the form β;
- the $(m+2)$ -dimensional measure $\bar{\beta}^{\intercal} \wedge *\beta > 0$ under the fixed orientation on M_{T}.

Define also on the space $\Lambda(M_{\mathrm{T}};\mathcal{H})$ the following natural scalar product: for any $\beta, \gamma \in \Lambda^{\mathrm{k}}(M_{\mathrm{T}};\mathcal{H})$, $k = \overline{0, m}$,

$$(\beta, \gamma) := \int_{M_{\mathrm{T}}} \bar{\beta}^{\intercal} * \gamma. \tag{10}$$

Subject to the scalar product (10), one can naturally construct the corresponding Hilbert space

$$\mathcal{H}_{\Lambda}(M_{\mathrm{T}}) := \oplus_{k=0}^{m+2} \mathcal{H}_{\Lambda}^{k}(M_{\mathrm{T}}) \tag{11}$$

well suitable for our further consideration. Notice also here, that the Hodge star $*$-operation satisfies the following easily checkable property: for any $\beta, \gamma \in \mathcal{H}_{\Lambda}^{k}(M_{\mathrm{T}})$, $k = \overline{0, m}$,

$$(\beta, \gamma) = (*\beta, *\gamma), \tag{12}$$

that is the Hodge operation $* : \mathcal{H}_{\Lambda}(M_{\mathrm{T}}) \to \mathcal{H}_{\Lambda}(M_{\mathrm{T}})$ is unitary and its standard adjoint with respect to the scalar product (10) operation satisfies the condition $(*)' = (*)^{-1}$.

Denote by $d_{\mathcal{L}}^{'}$ the formally adjoint expression to the weak differential operation (1). By means of the operations $d_{\mathcal{L}}'$ and $d_{\mathcal{L}}$ in the $\mathcal{H}_{\Lambda}(M_{\mathrm{T}})$, one can naturally define [8, 27, 28, 3, 30] the generalized Laplace-Hodge-Skrypnik operator $\Delta_{\mathcal{L}} : \mathcal{H}_1(M_{\mathrm{T}}) \longrightarrow \mathcal{H}_1(M_{\mathrm{T}})$ as

$$\Delta_{\mathcal{L}} = d_{\mathcal{L}}' d_{\mathcal{L}} + d_{\mathcal{L}}' d_{\mathcal{L}}\ . \tag{13}$$

Take a form $\beta \in \mathcal{H}_\Lambda(M_T)$ satisfying the equality

$$\Delta_{\mathcal{L}}\beta = 0. \tag{14}$$

Such a form is called [3, 28, 30, 8] harmonic. One can also verify that a harmonic form $\beta \in \mathcal{H}_\Lambda(M_T)$ satisfies simultaneously the following two adjoint conditions:

$$d'_{\mathcal{L}}\beta = 0, \quad d_{\mathcal{L}}\beta = 0 \tag{15}$$

easily stemming from (13) and (14).

It is easy to check that the following differential operators in $\mathcal{H}_\Lambda(M_T)$

$$d^*_{\mathcal{L}} := *d'_{\mathcal{L}}(*)^{-1} \tag{16}$$

defines also a new external anti-differential operation in $\mathcal{H}_\Lambda(M_T)$.

LEMMA 1.1. The corresponding dual to (8) co-chain complex

$$\mathcal{H} \longrightarrow \Lambda^0(M_T;\mathcal{H}) \xrightarrow{d^*_{\mathcal{L}}} \Lambda^1(M_T;\mathcal{H}) \xrightarrow{d^*_{\mathcal{L}}} \ldots \xrightarrow{d^*_{\mathcal{L}}} \Lambda^{m+2}(M_T;\mathcal{H}) \xrightarrow{d^*_{\mathcal{L}}} 0 \tag{17}$$

is exact.

A proof follows from the property $d^*_{\mathcal{L}}d^*_{\mathcal{L}} = 0$ which holds due to the definition (16).

1.3 Denote further by $\mathcal{H}^k_{\Lambda(\mathcal{L})}(M_T)$, $k = \overline{0, m+2}$, the cohomology groups of $d_{\mathcal{L}}$-closed and by $\mathcal{H}^k_{\Lambda(\mathcal{L}^*)}(M_T)$, $k = \overline{0, m+2}$, $k = \overline{0, m+2}$, the cohomology groups of $d^*_{\mathcal{L}}$-closed differential forms, respectively, and by $\mathcal{H}^k_{\Lambda(\mathcal{L}^*\mathcal{L})}(M_T)$, $k = \overline{0, m+2}$, the abelian groups of harmonic differential forms from the Hilbert sub-spaces $\mathcal{H}^k_\Lambda(M_T)$, $k = \overline{0, m+2}$. Before formulating next results, define the standard Hilbert-Schmidt rigged chain [11, 12] of positive and negative Hilbert spaces of differential forms

$$\mathcal{H}^k_{\Lambda,+}(M_T) \subset \mathcal{H}^k_\Lambda(M_T) \subset \mathcal{H}^k_{\Lambda,-}(M_T), \tag{18}$$

the corresponding hereditary rigged chains of harmonic forms:

$$\mathcal{H}^k_{\Lambda(\mathcal{L}^*\mathcal{L}),+}(M_T) \subset \mathcal{H}^k_{\Lambda(\mathcal{L}^*\mathcal{L})}(M_T) \subset \mathcal{H}^k_{\Lambda(\mathcal{L}^*\mathcal{L}),-}(M_T) \tag{19}$$

and chains of cohomology groups:

$$\begin{aligned} \mathcal{H}^k_{\Lambda(\mathcal{L}),+}(M_T) &\subset \mathcal{H}^k_{\Lambda(\mathcal{L})}(M_T) \subset \mathcal{H}^k_{\Lambda(\mathcal{L}),-}(M_T), \\ \mathcal{H}^k_{\Lambda(\mathcal{L}^*),+}(M_T) &\subset \mathcal{H}^k_{\Lambda(\mathcal{L}^*)}(M_T) \subset \mathcal{H}^k_{\Lambda(\mathcal{L}^*),-}(M_T) \end{aligned} \tag{20}$$

for all $k = \overline{0, m+2}$. Assume also that the Laplace-Hodge-Skrypnik operator (13) is reduced upon the space $\mathcal{H}^0_\Lambda(M)$. Now by reasoning similar to those in [8, 28, 30] one can formulate a little generalized [4, 5, 6, 28] De Rham-Hodge theorem.

THEOREM 1.2. The groups of harmonic forms $\mathcal{H}^k_{\Lambda,+}(M_T)$, $k = \overline{0, m+2}$, are, respectively, isomorphic to the homology groups $(H^k(M_T;\mathbb{C}))^{|\Sigma|}$, $k = \overline{0, m+2}$, where $H^k(M_T;\mathbb{C})$ is the k-th cohomology group of the manifold M_T with complex coefficients, a set $\Sigma \subset \mathbb{C}^p$, $p \in \mathbb{Z}_+$, is the set of suitable "spectral" parameters marking the linear space of independent $d^*_{\mathcal{L}}$-closed 0-forms from $\mathcal{H}^0_{\Lambda(\mathcal{L}),-}(M_{\mathrm{T}})$ and, moreover, the following direct sum decompositions

$$\begin{aligned}\mathcal{H}^k_{\Lambda,+}(M_{\mathrm{T}}) &= \mathcal{H}^k_{\Lambda(\mathcal{L}^*\mathcal{L}),+}(M_{\mathrm{T}}) \oplus \Delta_{\mathcal{L}}\mathcal{H}^k_{\Lambda,+}(M_{\mathrm{T}}) \\ &= \mathcal{H}^k_{\Lambda(\mathcal{L}^*\mathcal{L}),+}(M_{\mathrm{T}}) \oplus d_{\mathcal{L}}\mathcal{H}^{k-1}_{\Lambda,+}(M_{\mathrm{T}}) \oplus d'_{\mathcal{L}}\mathcal{H}^{k+1}_{\Lambda,+}(M_{\mathrm{T}}) \qquad (21)\end{aligned}$$

hold for any $k = \overline{0, m+2}$.

Another variant of the statement similar to that above was formulated in [3, 4] and reads as the following generalized de Rham-Hodge theorem:

THEOREM 1.3. The generalized cohomology groups $\mathcal{H}^k_{\Lambda(\mathcal{L}),+}(M_{\mathrm{T}})$, $k = \overline{0, m+2}$, are isomorphic, respectively, to the cohomology groups $(H^k(M_{\mathrm{T}}; \mathbb{C}))^{|\Sigma|}$, $k = \overline{0, m+2}$.

A proof of this theorem is based on some special sequence [3, 4, 5, 6, 7] of differential Lagrange type identities.

Define the following closed subspace

$$\mathcal{H}^*_0 := \{\varphi^{(0)}(\eta) \in \mathcal{H}^0_{\Lambda(\mathcal{L}^*),-}(M_{\mathrm{T}}) : d^*_{\mathcal{L}}\varphi^{(0)}(\eta) = 0,\ \varphi^{(0)}(\eta)|_{\Gamma},\ \eta \in \Sigma\} \qquad (22)$$

for some smooth $(m+1)$-dimensional hypersurface $\Gamma \subset M_{\mathrm{T}}$ and $\Sigma \subset (\sigma(L) \cap \bar{\sigma}(L)) \times \Sigma_\sigma \subset \mathbb{C}^p$, where $\mathcal{H}^0_{\Lambda(\mathcal{L}^*),-}(M_{\mathrm{T}})$ is, as above, a suitable Hilbert-Schmidt rigged[11, 12] zero-order cohomology group Hilbert space from the co-chain given by (20), $\sigma(L)$ and $\sigma(L^*)$ are, respectively, mutual generalized spectra of the sets of differential operators L and L^* in H at $t = 0 \in \mathrm{T}$. Thereby, the dimension $\dim \mathcal{H}^*_0 = card\ \Sigma := |\Sigma|$ is assumed to be known. The next lemma first stated by Skrypnik [3, 4] is of fundamental meaning for a proof of Theorem 1.2.

LEMMA 1.4. There exists a set of differential $(k+1)$-forms $Z^{(k+1)}[\varphi^{(0)}(\eta), d_{\mathcal{L}}\psi^{(k)}] \in \Lambda^{k+1}(M_{\mathrm{T}}; \mathbb{C})$, $k = \overline{0, m+2}$, and a set of k-forms $Z^{(k)}[\varphi^{(0)}(\eta), \psi^{(k)}] \in \Lambda^k(M_{\mathrm{T}}; \mathbb{C})$, $k = \overline{0, m+2}$, parametrized by the set $\Sigma \ni \eta$, being semilinear in $(\varphi^{(0)}(\eta), \psi^{(k)}) \in \mathcal{H}^*_0 \times \mathcal{H}^k_{\Lambda,+}(M_{\mathrm{T}})$, such that

$$Z^{(k+1)}[\varphi^{(0)}(\eta), d_{\mathcal{L}}\psi^{(k)}] = dZ^k[\phi^{(0)}(\eta\ ), \psi^{(k)}] \qquad (23)$$

for all $k = \overline{0, m+2}$ and $\eta \in \Sigma$.

PROOF. A proof is based on the following Lagrange type identity generalizing that of Part 1 and holding for any pair $(\varphi^0(\eta), \psi^(k)) \in \mathcal{H}^*_0 \times \mathcal{H}^k_{\Lambda,+}(M_{\mathrm{T}})$:

$$\begin{aligned} 0 &= \left\langle d^*_{\mathcal{L}}\phi^{(0)}(\eta), *(\psi^{(k)} \wedge \overline{\gamma})\right\rangle = \left\langle *d'_{\mathcal{L}}(*)^{-1}\varphi^{(0)}(\eta), *(\psi^{(k)} \wedge \overline{\gamma})\right\rangle \\ &= \left\langle *d'_{\mathcal{L}}(*)^{-1}\phi^{(0)}(x), \psi^{(k)} \wedge \overline{\gamma})\right\rangle \\ &= \left\langle (*)^{-1}\varphi^{(0)}(\eta), d_{\mathcal{L}}\psi^{(k)} \wedge \overline{\gamma}\right\rangle + Z^{(k+1)}[\psi^{(0)}(\eta), d_{\mathcal{L}}\psi(k)] \wedge \overline{\gamma} \\ &= \left\langle (*)_{-1}\varphi^{(0)}(\eta), d_{\mathcal{L}}\psi^{(k)} \wedge \overline{\gamma}\right\rangle + dZ^{(k)}[\varphi^{(0)}(\eta), \psi^{(k)}] \wedge \overline{\gamma}, \qquad (24)\end{aligned}$$

where $Z^{(k+1)}[\varphi^{(0)}(\eta), d_{\mathcal{L}}\psi^{(k)}] \in \Lambda^{k+1}(M_{\mathrm{T}}; \mathbb{C})$, $k = \overline{0, m+2}$, and $Z^{(k)}[\varphi^{(0)}(\eta), \psi^{(k)}] \in \Lambda^k(M_{\mathrm{T}}; \mathbb{C})$, $k = \overline{0, m+2}$, are some semilinear differential forms on M_{T} parametrized by a parameter $\lambda \in \Sigma$, and $\overline{\gamma} \in \Lambda^{m+1-k}(M_{\rm T};\mathbb{C})$ is arbitrary constant $(m+1-k)$-form. Thereby, the semilinear differential $(k+1)$-forms $Z^{(k+1)}[\varphi^{(0)}(\eta), d_{\mathcal{L}}\psi^{(k)}] \in \Lambda^{k+1}(M_{\rm T};\mathbb{C})$ and k-forms $Z^{(k)}[\varphi^{(0)}(\eta), \psi^{(k)}] \in \Lambda^{k}(M_{\rm T};\mathbb{C})$, $k = \overline{0, m+2}$, $\lambda \in \Sigma$, constructed above exactly constitute those searched for in the Lemma.

1.4 Based now on Lemma 1.2 one can construct the cohomology group isomorphism claimed in Theorem 1.2 formulated above. Namely, following [3, 4], let us take some singular simplicial [27, 28, 29, 30] complex $\mathcal{K}(M_{\rm T})$ of the compact metric space $M_{\rm T}$ and introduce a set of linear mappings $B_{\lambda}^{(k)} : \mathcal{H}^{k}_{\Lambda,+}M_{\rm T} \longrightarrow C_k(M_{\rm T};\mathbb{C})$, $k = \overline{0, m+2}$, $\lambda \in \Sigma$, where $C_k(M_{\rm T};\mathbb{C})$, $k = \overline{0, m+2}$, are free abelian groups over the field $\mathbb{C}$ generated, respectively, by all k-chains of singular simplexes $S^{(k)} \subset M_{\rm T}$, $k = \overline{0, m+2}$, from the simplicial complex $\mathcal{K}(M_{\rm T})$, as follows:

$$B_{\lambda}^{(k)}(\psi^{(k)}) := \sum_{S^{(k)} \in C_k(M_{\rm T};\mathbb{C}))} S^{(k)} \int_{S^{(k)}} Z^{(k)}[\varphi^{(0)}(\lambda), \psi^{(k)}] \tag{25}$$

with $\psi^{(k)} \in \mathcal{H}^{k}_{\Lambda,+}(M_{\rm T})$, $k = \overline{0, m+2}$. The following theorem [3, 4] based on mappings (25) holds.

THEOREM 1.5. The set of operators (25) parametrized by $\lambda \in \Sigma$ realizes the cohomology group isomorphism formulated in Theorem 1.2.

PROOF. A proof of this theorem one can get passing over in (25) to the corresponding cohomology $\mathcal{H}^{k}_{\Lambda(\mathcal{L}),+}(M_{\rm T})$ and homology $H_k(M_{\rm T};\mathbb{C})$ groups of $M_{\rm T}$ for every $k = \overline{0, m+2}$. If one takes an element $\psi^{(k)} := \psi^{(k)}(\mu) \in \mathcal{H}^{k}_{\Lambda(\mathcal{L}),+}(M_{\rm T})$, $k = \overline{0, m+2}$, solving the equation $d_{\mathcal{L}}\psi^{(k)}(\mu) = 0$ with $\mu \in \Sigma_k$ being some set of the related "spectral" parameters marking elements of the subspace $\mathcal{H}^{k}_{\Lambda(\mathcal{L}),-}(M_{\rm T})$, then one finds easily from (25) and identity (23) that $dZ^{(k)}[\varphi^{(0)}(\lambda),\ \psi^{(k)}(\mu)] = 0$ for all $(\lambda, \mu) \in \Sigma \times \Sigma_k$, $k = \overline{0, m+2}$. This, in particular, means due to the Poincare lemma [26, 27, 28] that there exist differential $(k-1)$-forms $\Omega^{(k-1)}[\varphi^{(0)}(\lambda), \psi^{(k)}(\mu] \in \Lambda^{k-1}(M;\mathbb{C})$, $k = \overline{0, m+2}$, such that

$$Z^{(k)}[\varphi^{(0)}(\lambda), \psi^{(k)}(\mu)] = d\Omega^{(k-1)}[\varphi^{(0)}(\lambda), \psi^{(k)}(\mu)] \tag{26}$$

for all pairs $(\varphi^{(0)}(\lambda), \psi^{(k)}(\mu)) \in \mathcal{H}_0^{*} \times \mathcal{H}^{k}_{\Lambda(\mathcal{L}),+}(M_{\rm T})$ parametrized by $(\lambda, \mu) \in \Sigma \times \Sigma_k$, $k = \overline{0, m+2}$. As a result of passing on the right hand-side of (25) to the homology groups $H_k(M_{\rm T};\mathbb{C})$, $k = \overline{0, m+2}$, one gets due to the standard Stokes theorem [26, 28, 27] that the mappings

$$B_{\lambda}^{(k)} : \mathcal{H}^{k}_{\Lambda(\mathcal{L}),+}(M_{\rm T}) \longrightarrow H_k(M_{\rm T};\mathbb{C}) \tag{27}$$

are isomorphisms for every $k = \overline{0, m+2}$ and $\lambda \in \Sigma$. Making further use of the Poincare duality [8, 27, 28] between the homology groups $H_k(M_{\rm T};\mathbb{C})$, $k = \overline{0, m+2}$, and the cohomology groups $H^k(M;\mathbb{C})$, $k = \overline{0, m+2}$, respectively, one obtains finally the statement claimed in Theorem 1.4.

2 The Spectral Structure Of Delsarte-Darboux Type Transmutation Operators In Multidimension

2.1 Take now into account that our differential operators $\mathrm{L}_j : \mathcal{H} \to \mathcal{H}$, $j = \overline{1,2}$, are of the special form (2). Assume also that differential expressions (4) are normal closed operators defined on dense subspace $D(L) \subset L_2(M; \mathbb{C}^N)$.

Then due to Theorem 1.4 one can find such a pair $(\varphi^{(0)}(\lambda), \psi^{(k)}(\mu)) \in \mathcal{H}_0^* \times \mathcal{H}^k_{\Lambda(\mathcal{L}),+}(M_{\mathrm{T}})$ parametrized by elements $(\lambda, \mu) \in \Sigma \times \Sigma_k$, for which the equality

$$B_\lambda^{(m)}(\psi^{(0)}(\mu)dx) = S^{(m)}_{(t;x)} \int_{\partial S^{(m)}_{(t;x)}} \Omega^{(m-1)}[\varphi^{(0)}(\lambda), \psi^{(0)}(\mu)dx] \tag{28}$$

holds, where $S^{(m)}_{(t;x)} \in\ H_m(M_{\mathrm{T}}; \mathbb{C})$ is some arbitrary but fixed element of parametrized by an arbitrarily chosen point $(t;x) \in M_{\mathrm{T}} \cap \partial S^{(m)}_{(t;x)}$. Consider the next integral expressions

$$\begin{aligned} \Omega_{(t;x)}(\lambda, \mu) \quad &: \quad = \int_{\sigma^{(m-1)}_{(t;x)}} \Omega^{(m-1)}[\varphi^{(0)}(\lambda), \psi^{(0)}(\mu)dx], \\ \Omega_{(t_0;x_0)}(\lambda, \mu) \quad &: \quad = \int_{\sigma^{(m-1)}_{(t_0;x_0)}} \Omega^{(m-1)}[\phi^{(0)}(\lambda), psi_{(0)}(\mu)dx], \end{aligned} \tag{29}$$

with a point $(t_0; x_0) \in M_{\mathrm{T}} \cap \partial S^{(m)}_{(t_0;x_0)}$ being taken fixed the boundaries $\sigma^{(m-1)}_{(t;x)} := \partial S^{(m)}_{t;x}$, $\sigma^{(m-1)}_{(t_0;x_0)} := \partial S^{(m)}_{t_0;x_0}$ assumed to be homological to each other as $(t;x_0) \longrightarrow (t;x) \in M_{\mathrm{T}}$, $(\lambda, \mu) \in \Sigma \times \Sigma_k$, and interpret them as the kernels [11, 12, 31] of the corresponding invertible integral operators of Hilbert-Schmidt type $\Omega_{(t;x)}, \Omega_{(t_0;x_0)} : L_2^{(\rho)}(\Sigma; \mathbb{C}) \longrightarrow L_2^{(\rho)}(\Sigma; \mathbb{C})$, where ρ is some finite Borel measure on the parameters set Σ. Define now the invertible operators expressions

$$\Omega_\pm : \psi^{(0)}(\mu) \longrightarrow \tilde{\psi}^{(0)}(\mu) \tag{30}$$

for $\psi^{(0)}(\mu)dx \in \mathcal{H}^m_{\Lambda(\mathcal{L}),+}(M_{\mathrm{T}})$ and some $\tilde{\psi}^{(0)}(\mu)dx \in \mathcal{H}^m_{\Lambda(\mathcal{L}),+}(M_{\mathrm{T}})$, $\mu \in \Sigma$, where, by definition, for any $\eta \in \Sigma$,

$$\begin{aligned} \tilde{\psi}^{(0)}(\eta) \quad &: \quad = \psi^{(0)}(\eta) \cdot \Omega^{-1}_{(t;x)} \cdot \Omega_{(t_0;x_0)} \\ &= \quad \int_\Sigma d\rho(\mu) \int_\Sigma d\rho(\xi) \psi^{(0)}(\mu) \Omega^{-1}_{(t;x)}(\mu, \xi) \Omega_{(t_0;x_0)}(\xi, \eta), \end{aligned} \tag{31}$$

being motivated by the expression (28). Namely, consider the following diagram

$$\begin{array}{ccc} \mathcal{H}^m_{\Lambda(\mathcal{L}),+}(M_{\mathrm{T}}) & \xrightarrow{\Omega_\pm} & \mathcal{H}^m_{\Lambda(\tilde{\mathcal{L}}),+}(M_{\mathrm{T}}), \\ B_\lambda^{(m)} \downarrow & \swarrow \tilde{B}_\lambda^{(m)} & \\ H_m(M_{\mathrm{T}}; \mathbb{C}) & & \end{array} \tag{32}$$

which is assumed to be commutative for some other co-chain complex

$$\mathcal{H} \longrightarrow \Lambda^0(M_{\rm T};\mathcal{H}) \xrightarrow{d_{\tilde{\mathcal{L}}}} \Lambda^1(M_{\rm T};\mathcal{H}) \xrightarrow{d_{\tilde{\mathcal{L}}}} ... \xrightarrow{d_{\tilde{\mathcal{L}}}} \Lambda^{m+2}(M_{\rm T};\mathcal{H}) \xrightarrow{d_{\tilde{\mathcal{L}}}} 0. \tag{33}$$

Here, by definition, the generalized anti-differentiation is

$$d_{\tilde{\mathcal{L}}} := \sum_{j=1}^{2} dt_j \wedge \tilde{\rm L}_j(t;x|\partial) \tag{34}$$

with

$$\begin{aligned} \tilde{\rm L}_j &= \partial/\partial t_j - \tilde{L}_j(t;x|\partial), \\ \tilde{L}_j(t;x|\partial) &:= \sum_{|\alpha|=0}^{n_j(\tilde{L})} \tilde{a}_\alpha^{(j)}(t;x)\frac{\partial^{|\alpha|}}{\partial x^\alpha}, \end{aligned} \tag{35}$$

where coefficients $\tilde{a}_\alpha^{(j)} \in C^1({\rm T};C^\infty(M;{\rm End}\mathbb{C}^N)$, $|\alpha| = \overline{0,n_j(\tilde{L})}$, $n_j(\tilde{L}) := n_j(L) \in \mathbb{Z}_+$, $j = \overline{1,2}$. The corresponding isomorphisms $\tilde{B}_\lambda^{(m)} : \mathcal{H}^m_{\Lambda(\mathcal{L}),+}(M_{\rm T}) \longrightarrow H_m(M_{\rm T};\mathbb{C})$, $\lambda \in \Sigma$, act, by definition, as follows:

$$\tilde{B}_\lambda^{(m)}(\tilde{\psi}^{(0)}(\mu)dx) = S^{(m)}_{(t;x)} \int_{\partial S^{(m)}_{(t;x)}} \tilde{\Omega}^{(m-1)}[\tilde{\varphi}^{(0)}(\lambda),\tilde{\psi}^{(0)}(\mu)dx], \tag{36}$$

where $\tilde{\varphi}^{(0)}(\lambda) \in \tilde{\mathcal{H}}_0^* \subset \mathcal{H}^0_{\Lambda(\mathcal{L}^*),-}(M_{\rm T})$, $\lambda \in (\sigma(\tilde{L}) \cap \bar{\sigma}(\tilde{L}^*)) \times \Sigma_\sigma$,

$$\tilde{\mathcal{H}}_0^* := \{\tilde{\varphi}^{(0)}(\lambda) \in \mathcal{H}^m_{\Lambda(\mathcal{L}^*),-}(M_{\rm T}) : d^*_{\tilde{\mathcal{L}}}\tilde{\varphi}^{(0)}(x) = 0, \tilde{\varphi}^{(0)}(\lambda)|_{\tilde{\Gamma}} = 0, \lambda \in \Sigma\} \tag{37}$$

for some hypersurface $\tilde{\Gamma} \subset M_{\rm T}$. Respectively, one defines the following closed subspace

$$\tilde{\mathcal{H}}_0 := \{\tilde{\psi}^{(0)}(\mu) \in \mathcal{H}^0_{\Lambda(\mathcal{L}^*),-}(M_{\rm T}) : d^*_{\tilde{\mathcal{L}}}\tilde{\psi}^{(0)}(\lambda) = 0, \tilde{\psi}^{(0)}(\mu)|_{\tilde{\Gamma}} = 0, \mu \in \Sigma\} \tag{38}$$

for the hyperspace $\tilde{\Gamma} \subset M_{\rm T}$, introduced above.

Suppose now that the elements (31) belong to the closed subspace (38), that is

$$d_{\tilde{\mathcal{L}}}\tilde{\psi}^{(0)}(\mu) = 0. \tag{39}$$

Define similarly to (38) a closed subspace $\tilde{\mathcal{H}}_0^* \subset \mathcal{H}^m_{\Lambda(\mathcal{L}^*),-}(M_{\rm T})$ as follows:

$$\mathcal{H}_0 := \{\psi^{(0)}(\lambda) \in \mathcal{H}^0_{\Lambda(\mathcal{L}^*),-}(M_{\rm T}) : d_{\mathcal{L}}\psi^{(0)}(\lambda) = 0, \psi^{(0)}(\lambda)|_{\Gamma} = 0, \lambda \in \Sigma\} \tag{40}$$

for all $\mu \in \Sigma$. Then due to the commutativity of the diagram (32) there exist the corresponding two invertible mappings

$$\Omega_\pm : \mathcal{H}_0 \to \tilde{\mathcal{H}}_0, \tag{41}$$

depending on ways of their extending over the whole Hilbert space $\mathcal{H}^m_{\Lambda,-}(M_{\rm T})$. Extend now operators (41) upon the whole Hilbert space $\mathcal{H}^m_{\Lambda,-}(M_{\rm T})$ by means of the standard method [23, 22] of variation of constants, taking into account that for kernels

$\Omega_{(t;x)}(\lambda,\mu), \Omega_{(t_0;x_0)}(\lambda,\mu) \in L_2^{(p)}(\Sigma;\mathbb{C}) \otimes L_2^{(p)}(\Sigma;\mathbb{C}),\ \lambda,\mu \in \Sigma$, one can write down the following relationships:

$$\begin{aligned}
&\Omega_{(t;x)}(\lambda,\mu) - \Omega_{(t_0;x_0)}(\lambda,\mu)\\
=&\int_{\partial S^{(m)}_{(t;x)}} \Omega^{(m-1)}[\varphi^{(0)}(x),\psi^{(0)}(\mu)dx] - \int_{\partial S^{(m)}_{(t_0;x_0)}} \Omega^{(m-1)}[\varphi^{(0)}(\lambda),\psi^{(0)}(\mu)dx]\\
=&\int_{S^{(m)}_{\pm}(\sigma^{(m-1)}_{(t;x)},\sigma^{(m-1)}_{(t_0;x_0)})} d\Omega^{(m-1)}[\varphi^{(0)}(\lambda),\psi^{(0)}(\mu)dx]\\
=&\int_{S^{(m)}_{\pm}(\sigma^{(m-1)}_{(t;x)},\sigma^{(m-1)}_{(t_0;x_0)})} Z^{(m)}[\varphi^{(0)}(\lambda),\psi^{(0)}(\mu)dx], \qquad (42)
\end{aligned}$$

where, by definition, m-dimensional open surfaces $S^{(m)}_{\pm}(\sigma^{(m-1)}_{(t;x)},\sigma^{(m-1)}_{(t_0;x_0)}) \subset M_{\mathrm{T}}$ are spanned smoothly without self-intersection between two homological cycles $\sigma^{(m-1)}_{(t;x)} = \partial S^{(m)}_{(t;x)}$ and $\sigma^{(m-1)}_{(t_0;x_0)} = \partial S^{(m)}_{(t_0;x_0)} \in C_{m-1}(M_{\mathrm{T}};\mathbb{C})$ in such a way that the boundary $\partial(S^{(m)}_{+}(\sigma^{(m-1)}_{(t_0;x_0)},\sigma^{(m-1)}_{(t_0;x_0)}) \cup S^{(m)}_{-}(\sigma^{(m-1)}_{(t;x)},\sigma^{(m-1)}_{(t_0;x_0)})) = \oslash$. Making use of the relationship (42), one can thereby find easily the following integral operator expressions in $\mathcal{H}_-$:

$$\Omega_{\pm} = \mathbf{1} - \int_{\Sigma} d\rho(\eta)\tilde{\psi}^{(0)}(\xi)\Omega^{-1}_{(t_0;x_0)}(\xi,\eta) \times \int_{S^{(m)}_{\pm}(\sigma^{(m-1)}_{(t;x)},\sigma^{(m-1)}_{(t_0;x_0)})} Z^{(m)}[\varphi^{(0)}(\eta),(\cdot)dx] \qquad (43)$$

defined for fixed pairs $(\varphi^{(0)}(\xi),\psi^{(0)}(\eta)) \in \mathcal{H}_0^* \times \mathcal{H}_0$ and $(\tilde{\varphi}^{(0)}(\xi),\tilde{\psi}^{(0)}(\mu)) \in \tilde{\mathcal{H}}_0^* \times \tilde{\mathcal{H}}_0$, $\lambda,\mu \in \Sigma$, being bounded invertible operators of Volterra type [17, 18, 13, 31] on the whole Hilbert space $\mathcal{H}$. Moreover, for the differential operators $\tilde{\mathrm{L}}_j : \mathcal{H} \longrightarrow \mathcal{H},\ j = \overline{1,2}$, one can get easily the following expressions

$$\tilde{\mathrm{L}}_j = \Omega_{\pm}\mathrm{L}_j\Omega_{\pm}^{-1} \qquad (44)$$

for $j = \overline{1,2}$, where the left hand-side of (44) does not depend on sings "$\pm$" of the right-hand sides. Thereby, the Volterrian integral operators (43) are the Delsarte-Darboux transmutation operators, mapping a given set $\mathcal{L}$ of differential operators into a new set $\tilde{\mathcal{L}}$ of differential operators transformed via the Delsarte expressions (44).

2.2 Suppose now that all of differential operators $L_j(t;x|\partial)$, $j = \overline{1,2}$, considered above do not depend on the variable $t \in \mathrm{T}$. Then, evidently, one can take

$$\begin{array}{rcl}
\mathcal{H}_0 & : & =\{\psi_\mu^{(0)}(\xi)\in L_{2.-}(M;\mathbb{C}^N): L_j\psi_\mu^{(0)}(\xi)=\mu_j\psi_\mu^{(0)}(\xi),\\
j & = & \overline{1,2},\ \psi_\mu^{(0)}(\xi)|_{\tilde{\Gamma}}=0, \mu=(\mu_1,\mu_2)\in\sigma(\tilde{L})\cap\overline{\sigma}(L^*), \xi\in\Sigma_\sigma\}\\
\tilde{\mathcal{H}}_0 & : & =\{\tilde{\psi}_\mu^{(0)}(\xi)\in L_{2.-}(M;\mathbb{C}^N): \tilde{L}_j\tilde{\psi}_\mu^{(0)}(\xi)=\mu_j\tilde{\psi}_\mu^{(0)}(\xi),\\
j & = & \overline{1,2},\ \tilde{\psi}_\mu^{(0)}(\xi)|_{\tilde{\Gamma}}=0, \mu=(\mu_1,\mu_2)\in\sigma(\tilde{L})\cap\overline{\sigma}(L^*), \xi\in\Sigma_\sigma\}\\
\mathcal{H}_0^* & : & =\{\varphi_\lambda^{(0)}(\eta)\in L_{2.-}(M;\mathbb{C}^N): L_j^*\varphi_\lambda^{(0)}(\eta)=\bar{\lambda}_j\varphi_\lambda^{(0)}(\eta), j=\overline{1,2},\\
\varphi_\lambda^{(0)}(\eta)|_{\tilde{\Gamma}} & = & 0, \lambda=(\lambda_1,\lambda_2)\in\sigma(\tilde{L})\cap\overline{\sigma}(L^*), \eta\in\Sigma_\sigma\}\\
\tilde{\mathcal{H}}_0^* & : & =\{\tilde{\varphi}_\lambda^{(0)}(\eta)\in L_{2.-}(M;\mathbb{C}^N): \tilde{L}_j^*\tilde{\varphi}_\lambda^{(0)}(\eta)=\bar{\lambda}_j\varphi_\lambda^{(0)}(\eta),\\
j & = & \overline{1,2},\ \tilde{\varphi}_\lambda^{(0)}(\eta)|_{\tilde{\Gamma}}=0, \lambda=(\lambda_1,\lambda_2)\in\sigma(\tilde{L})\cap\overline{\sigma}(L^*), \eta\in\Sigma_\sigma\}
\end{array} \tag{45}$$

and construct the corresponding Delsarte-Darboux transmutation operators

$$\begin{array}{rcl}
\Omega_\pm & = & 1-\int_{\sigma(\tilde{L})\cap\overline{\sigma}(L^*)} d\rho_\sigma(\lambda)\int_{\Sigma_\sigma\times\Sigma_\sigma} d\rho_{\Sigma_\sigma}(\xi)d\rho_{\Sigma_\sigma}(\eta)\\
& & \times\int_{S_\pm^{(m)}\sigma^{(m-1)}_{(t_0;x_0)},\sigma^{(m-1)}_{(t_0;x_0)}} dx\tilde{\psi}_\lambda^{(0)}(\xi)\Omega_{x_0}^{-1}(\lambda;\xi;\eta)\bar{\varphi}_\lambda^{(0),\intercal}(\eta)(\cdot)
\end{array} \tag{46}$$

acting already in the Hilbert space $L_{2,+}(M;\mathbb{C}^N)$, where for any $(\lambda;\xi,\eta)\in(\sigma(\tilde{L})\cap\overline{\sigma}(L^*)\times\Sigma_\sigma^2$ kernels,

$$\Omega_{(x_0)}(\lambda;\xi,\eta):=\int_{\sigma_{x_0}^{(m-1)}}\Omega^{(m-1)}[\varphi_\lambda^{(0)}(\xi),\psi_\lambda^{(0)}(\eta)dx] \tag{47}$$

for $(\xi,\eta)\in\Sigma_\sigma^2$ and every $\lambda\in\sigma(\tilde{L})\ \cap\overline{\sigma}(L^*)$ belong to $L_2^{(\rho)}(\Sigma_\sigma;\mathbb{C})\otimes L_2^{(\rho)}(\Sigma_\sigma;\mathbb{C})$. Moreover, as $\partial\Omega_\pm/\partial t_j=0,\ j=\overline{1,2},$ one gets easily the set of differential expressions

$$\tilde{L}_j(x|\partial):=\Omega_\pm L_j(x|\partial)\Omega_\pm^{-1} \tag{48}$$

$j=\overline{1,2}$, also commuting, evidently, to each other.

The Volterrian operators (46) possess some additional properties. Namely, define the following Fredholm type integral operator in H :

$$\Omega:=\Omega_+^{-1}\Omega_-, \tag{49}$$

which can be written in the form

$$\Omega=\mathbf{1}+\Phi(\Omega), \tag{50}$$

where the operator $\Phi(\Omega)\in\mathcal{B}_\infty(H)$ is compact. Moreover, due to the relationships (48) one gets easily that the following commutator conditions

$$[\Omega,L_j]=0 \tag{51}$$

hold for $j = \overline{1,2}$.

Denote now by $\hat{\Phi}(\Omega) \in H_- \otimes H_-$ and $\hat{K}_+(\Omega)$, $\hat{K}_-(\Omega) \in H_- \otimes H_-$ the kernels corresponding [11, 12] to operators $\Phi(\Omega) \in \mathcal{B}_\infty(H)$ and $\Omega_\pm - \mathbf{1} \in \mathcal{B}_\infty(H)$. Then due to the fact that *supp* $K_+(\Omega) \cap suppK_-(\Omega) = \sigma_x^{(m-1)} \cup \sigma_{x_0}^{(m-1)}$, one gets from (49) and (50) the well known Gelfand-Levitan-Marchenko linear integral equation

$$\hat{K}_+(\Omega) + \Phi(\hat{\Omega}) + \hat{K}_+(\Omega)_+ \cdot \hat{\Phi}(\Omega) = \hat{K}_-(\Omega), \tag{52}$$

allowing to find the factorizing the Fredholmian operator (49) kernel $\hat{K}_+(\Omega)(x;y) \in H_- \otimes H_-$ for all $y \in suppK_+(\Omega)$. The conditions (51) can be rewritten suitably as follows:

$$(L_{j,ext} \otimes \mathbf{1})\hat{\Phi}(\Omega) = (1 \otimes L^*_{j,ext})\hat{\Phi}(\Omega) \tag{53}$$

for $j = \overline{1,2}$, where $L_{j,ext} \in \mathcal{L}(H_-)$, $j = \overline{1,2}$, and their adjoint $L^*_{j,ext} \in \mathcal{L}(H_-)$, $j = \overline{1,2}$, are the corresponding extensions [11, 20, 23, 12] of the differential operators L_j and $L^*_j \in \mathcal{L}(H)$, $j = \overline{1,2}$.

Concerning the relationships (48) one can write down [11, 23] kernel conditions similar to (53):

$$(\tilde{L}_{j,ext} \otimes \mathbf{1})\hat{K}_\pm(\Omega) = (\mathbf{1} \otimes L^*_{j,ext})\hat{K}_\pm(\Omega), \tag{54}$$

where as above, $\tilde{L}_{j,ext} \in \mathcal{L}(H_-)$, $j = \overline{1,2}$, are the corresponding rigging extensions of the differential operators $\tilde{L}_j \in \mathcal{L}(H)$, $j = \overline{1,2}$.

2.3 Proceed now to analyzing the question about the general differential and spectral structure of transformed operator expression (44). It is evident that the above conditions (52) and (53) imposed on the kernels $\hat{K}_\pm(\Omega) \in \mathcal{H}_- \otimes \mathcal{H}_-$ of Delsarte-Darboux transmutation operators are necessary for the operator expressions (44) to be really differential. Consider the important question whether these conditions are also sufficient?

For studying this question let us consider Volterrian operators (43) and (46) with kernels satisfying the conditions (52) and (53), assuming that suitable oriented surfaces $S_\pm^{(m)}(\sigma_{(t;x)^{(m-1)}}, \sigma_{(t_0;x_0)^{(m-1)}}) \in C_m(M_\mathrm{T};\mathbb{C})$ are given as follows:

$$\begin{aligned} S_+^{(m)}(\sigma_{(t;x)^{(m-1)}}, \sigma_{(t_0;x_0)^{(m-1)}}) &= \{(t';x') \in M_\mathrm{T} : t' = P(t;x|x'),\ t \in \mathrm{T}\}, \\ S_-^{(m)}(\sigma_{(t;x)^{(m-1)}}, \sigma_{(t_0;x_0)^{(m-1)}}) &= \{(t';x') \in M_\mathrm{T} : t' = P(t;x|x') \in \mathrm{T}\backslash[t_0,t]\}, \end{aligned} \tag{55}$$

where a mapping $P \in C^\infty(M_\mathrm{T} \times M;\mathrm{T})$ is smooth and such that the boundaries $\partial S_\pm^{(m)}(\sigma_{(t;x)}^{(m-1)}, \sigma_{(t_0;x_0)}^{(m-1)}) = \pm(\sigma_{(t;x)}^{(m-1)} - \sigma_{(t_0;x_0)}^{(m-1)})$ with cycles $\sigma_{(t;x)}^{(m-1)}$ and $\sigma_{(t_0;x_0)}^{(m-1)} \in \mathcal{K}(M_\mathrm{T})$ being homological to each other for any choice of points $(t_0;x_0)$ and $(t;x) \in M_\mathrm{T}$. Then one can see by means of some simple but cumbersome calculations, based on considerations from [33] and [9], that the resulting expressions on the right hand-sides of

$$\tilde{\mathrm{L}} = \mathrm{L} + [\mathrm{K}_\pm(\Omega), \mathrm{L}] \cdot \Omega_\pm^{-1} \tag{56}$$

are exactly equal to each other differential ones if such there was the expression for an operator $\mathrm{L} \in \mathcal{L}(\mathcal{H})$.

Concerning the inverse operators $\Omega_{\pm}^{-1} \in \mathcal{L}(\mathcal{H})$ present in (56), one can notice here that due to the functional symmetry between closed subspaces $\mathcal{H}_0$ and $\tilde{\mathcal{H}}_0 \subset \tilde{\mathcal{H}}_-$, the defining relationships (41) and (31) are reversible, that is, there exist the inverse operator mappings $\Omega_{\pm}^{-1} : \tilde{\mathcal{H}}_0 \to \mathcal{H}_0$, such that

$$\Omega_{\pm}^{-1} : \tilde{\psi}^{(0)}(\lambda) \longrightarrow \psi^{(0)}(\lambda) := \tilde{\psi}^{(0)}(\lambda) \cdot \tilde{\Omega}_{(t;x)}^{-1} \tilde{\Omega}_{(t;x)} \tag{57}$$

for some suitable kernels $\tilde{\Omega}_{(t;x)}(\lambda, \mu)$ and $\tilde{\Omega}_{(t_0;x_0)}(\lambda, \mu) \in L_2^{(\rho)}(\Sigma; \mathbb{C}) \otimes L_2^{(\rho)}(\Sigma; \mathbb{C})$, related naturally with the transformed differential expression $\tilde{\mathrm{L}} \in \mathcal{L}(\mathcal{H})$. Thereby, due to the expressions (57) one can write down similar to (46) the following inverse integral operators:

$$\begin{aligned} \Omega_{\pm}^{-1} \quad &= \quad \mathbf{1} - \int_{\Sigma} d\rho(\xi) \int_{\Sigma} d\rho(\eta) \psi^{(0)}(\xi) \tilde{\Omega}_{t_0;x_0}^{-1}(\xi, \eta) \\ &\quad \times \int_{S_{\pm}^{(m)}(\sigma_{(t;x)}^{(m-1)}, \sigma_{(t_0;x_0)}^{(m-1)})} \tilde{Z}^{(m)}[\tilde{\varphi}^{(0)}(\eta), (\cdot) dx] \end{aligned} \tag{58}$$

defined for fixed pairs $(\tilde{\varphi}^{(0)}(\xi), \tilde{\psi}^{(0)}(\eta)) \in \tilde{\mathcal{H}}_0^* \times \tilde{\mathcal{H}}_0$ and $(\varphi^{(0)}(\xi), \psi^{(0)}(\eta)) \in \mathcal{H}_0^* \times \mathcal{H}_0$, $\xi, \eta \in \Sigma$, and being bounded invertible operators of Volterra type in the whole Hilbert space $\mathcal{H}$. In particular, the compatibility conditions $\Omega_{\pm} \Omega_{\pm}^{-1} = \mathbf{1} = \Omega_{\pm}^{-1} \Omega_{\pm}$ must be fulfilled identically in $\mathcal{H}$, involving some restrictions identifying measures ρ and Σ and possible asymptotic conditions of coefficient functions of the differential expression $\mathrm{L} \in \mathcal{L}$. Such kinds of restrictions were already mentioned before in [36, 37, 38], where in particular the relationships with the local and nonlocal Riemann problems were discussed.

2.4 Within the framework of the general construction presented above one can give a natural interpretation of so called Backlund transformations for coefficient functions of a given differential operator expression $\mathrm{L} \in \mathcal{L}(\mathcal{H})$. Namely, following the symbolic considerations in [40], we reinterpret the approach devised there for constructing the Backlund transformations making use of the techniques based on the theory of Delsarte transmutation operators. Let us define two different Delsarte-Darboux transformed differential operator expressions

$$\mathrm{L}_1 = \Omega_{1,\pm} \mathrm{L} \Omega_{1,\pm}^{-1}, \qquad \mathrm{L}_2 = \Omega_{2,\pm} \mathrm{L} \Omega_{2,\pm}^{-1}, \tag{59}$$

where $\Omega_{1,+}, \Omega_{2,-} \in \mathcal{L}(\mathcal{H})$ are some Delsarte transmutation Volterrian operators in $\mathcal{H}$ with Borel spectral measures ρ_1 and ρ_2 on Σ, such that the following conditions

$$\Omega_{1,+}^{-1} \Omega_{1,-} = \Omega = \Omega_{2,+}^{-1} \Omega_{2,-} \tag{60}$$

hold. Making use now of the conditions (59) and relationships (60) one finds easily that the operator $\mathrm{B} := \Omega_{2,-} \Omega_{1,+}^{-1} \in \mathcal{L}(\mathcal{H})$ satisfies the following operator equations:

$$\mathrm{L}_2 \mathrm{B} = \mathrm{B} \mathrm{L}_1, \quad \Omega_{2,\pm} \mathrm{B} = \mathrm{B} \Omega_{1,\pm}, \tag{61}$$

which motivate the next definition.

DEFINITION 2.1. An invertible symbolic mapping $\mathrm{B} : \mathcal{L}(\mathcal{H}) \longrightarrow \mathcal{L}\ (\mathcal{H})$ will be called a Darboux-Backlund transformation of an operator $\mathrm{L}_1 \in \mathcal{L}(\mathcal{H})$ into the operator $\mathrm{L}_2 \in \mathcal{L}(\mathcal{H})$ if there holds the condition

$$[\mathrm{QB}, \mathrm{L}_1] = 0 \tag{62}$$

for some linear differential expression $\mathrm{Q} \in \mathcal{L}(\mathcal{H})$.

The condition (62) can be realized as follows. Take any differential expression $\mathrm{q} \in \mathcal{L}(\mathcal{H})$. satisfying the symbolic equation

$$[\mathrm{qB}, \mathrm{L}] = 0. \tag{63}$$

Then, making use of the transformations like (59), from (60) one finds that

$$[\mathrm{QB}, \mathrm{L}_1] = 0, \tag{64}$$

where owing to (61),

$$\mathrm{QB} := \Omega_{1,+} q \mathrm{B} \Omega_{1,+}^{-1} = \Omega_{1,+} \mathrm{q} \Omega_{2,+}^{-1} \mathrm{B}. \tag{65}$$

Therefore, the expression $\mathrm{Q} = \Omega_{1,+} q \Omega_{2,+}^{-1}$ proves to be really differential one too owing to the conditions (61).

The consideration above related with the symbolic mapping $\mathrm{B} : \mathcal{L}(\mathcal{H}) \rightarrow \mathcal{L}(\mathcal{H})$ gives rise to an effective tool of constructing self-Backlund transformations for coefficients of differential operator expressions $\mathrm{L}_1, \mathrm{L}_2 \in \mathcal{L}(\mathcal{H})$ having many applications [14, 34, 25, 31, 22] in spectral and soliton theories.

2.5 Return now back to studying the structure Delsarte-Darboux transformations for a polynomial differential operators pencil

$$\mathrm{L}(\lambda; x|\partial) := \sum_{j=0}^{n(L)} L_j(x|\partial)\lambda^j, \tag{66}$$

where $n(L) \in \mathbb{Z}_+$ and $\lambda \in \mathbb{C}$ is a complex-valued parameter. It is asked to find the corresponding to (66) Delsarte-Darboux transformations $\Omega_{\lambda,\pm} \in \mathcal{L}(H)$, $\lambda \in \mathbb{C}$, such that for some polynomial differential operators pencil $\tilde{\mathrm{L}}(\lambda; x|\partial) \in \mathcal{L}(\mathcal{H})$ the following Delsarte-Lions [2] transmutation condition

$$\tilde{\mathrm{L}}\Omega_{\lambda,\pm} = \Omega_{\lambda,\pm}\mathrm{L} \tag{67}$$

holds for almost all $\lambda \in \mathbb{C}$. For such transformations $\Omega_{\lambda\pm} \in \mathcal{L}(H)$ to be found, let us consider a parameter $\tau \in \mathbb{R}$ dependent differential operator $\mathrm{L}_\tau(x|\partial) \in \mathcal{L}(\mathcal{H}_\tau)$, where

$$\mathrm{L}_\tau(x|\partial) := \sum_{j=0}^{n(L)} L_j(x|\partial)\partial^j/\partial\tau^j, \tag{68}$$

acting in the functional space $\mathcal{H}_\tau = C^{q(L)}(\mathbb{R}_\tau;\mathcal{H})$ for some $q(L) \in \mathbb{Z}_+$. Then one can easily construct the corresponding Delsarte-Darboux transformations $\Omega_{\tau,\pm} \in \mathcal{L}(\mathcal{H}_\tau)$ of Volterra type for some differential operator expression

$$\tilde{\mathrm{L}}_\tau(x|\partial) := \sum_{j=0}^{n(L)} \tilde{L}_j(x|\partial)\partial^j/\partial\tau^j, \tag{69}$$

if the following Delsarte-Lions [2] transmutation conditions

$$\tilde{\mathrm{L}}_\tau\Omega_{\tau,\pm} = \Omega_{\tau,\pm}\mathrm{L}_\tau \tag{70}$$

hold in $\mathcal{H}_\tau$. Thus, making use of the results obtained above, one can write down that

$$\begin{aligned}\Omega_{\tau,\pm} \;=\; & \mathbf{1} - \int_\Sigma d\rho_\Sigma(\xi)\int_\Sigma d\rho_\Sigma(\eta)\tilde{\psi}_\tau^{(0)}(\lambda;\xi)\Omega^{-1}_{(\tau_0;x_0)}(\lambda;\xi,\eta)\\ & \times\int_{S_\pm^{(m)}(\sigma^{(m-1)}_{(\tau;x)},\sigma^{(m-1)}_{(\tau_0;x_0)})} Z^{(m)}[\varphi_\tau^{(0)}(\lambda;\eta),(\cdot)dx]\end{aligned} \tag{71}$$

defined by means of the following closed subspaces $\mathcal{H}_{\tau,0} \subset \mathcal{H}_{\tau,-}$ and $\mathcal{H}^*_{\tau,0} \subset \mathcal{H}^*_{\tau,-}$:

$$\begin{aligned}\mathcal{H}_{\tau,0} \quad &: \;= \{\psi_\tau^{(0)}(\lambda;\xi) \in \mathcal{H}_{\tau,-} : \mathrm{L}_\tau\psi_\tau^{(0)}(\lambda;\xi) = 0,\\ \psi_\tau^{(0)}(\lambda;\xi)|_{\tau=0} \quad &= \quad \psi^{(0)}(\lambda;\xi) \in \mathcal{H},\ \mathrm{L}\psi^{(0)}(\lambda;\xi) = 0,\\ \psi^{(0)}(\lambda;\xi)|_\Gamma \quad &= \quad 0,\ \lambda \in \mathbb{C},\ \xi \in \Sigma\},\end{aligned}$$

$$\begin{aligned}\mathcal{H}^*_{\tau,0} \quad &: \;= \{\varphi_\tau^{(0)}(\lambda;\eta) \in \mathcal{H}^*_{\tau,-} : \mathrm{L}_\tau\varphi_\tau^{(0)}(\lambda;\eta) = 0,\\ \varphi_\tau^{(0)}(\lambda;\eta)|_{\tau=0} \quad &= \quad \varphi^{(0)}(\lambda;\eta) \in \mathcal{H}^*,\ \mathrm{L}\varphi^{(0)}(\lambda;\eta) = 0,\\ \varphi^{(0)}(\lambda;\eta)|_\Gamma \quad &= \quad 0,\ \lambda \in \mathbb{C},\ \eta \in \Sigma\}.\end{aligned} \tag{72}$$

Recalling now that our operators $L_j \in \mathcal{L}(\mathcal{H})$, $j = \overline{0,r(L)}$, do not depend on the parameter $\tau \in \mathbb{R}$, one can derive easily from (71)

$$\begin{aligned}\Omega_\pm \;=\; & \mathbf{1} - \int_\Sigma d\rho_\Sigma(\xi)\int_\Sigma d\rho_\Sigma(\eta)\tilde{\psi}^{(0)}(\lambda;\xi)\Omega^{-1}_{(x_0)}(\lambda;\xi,\eta)\\ & \times\int_{S_\pm^{(m)}(\sigma^{(m-1)}_{(x)},\sigma^{(m-1)}_{(x_0)})} Z_0^{(m)}[\varphi^{(0)}(\lambda;\eta),(\cdot)dx],\end{aligned} \tag{73}$$

where we put $\sigma_x^{(m-1)} := \sigma^{(m-1)}_{(\tau_0;x)}$, $\sigma_{x_0}^{(m-1)} := \sigma^{(m-1)}_{(\tau_0;x_0)} \in C_{m-1}(\mathbb{R}^m;\mathbb{C})$ and

$$Z_0^{(m)}[\varphi^{(0)}(\lambda;\eta),\psi^{(0)}dx] := Z^{(m)}[\varphi_\tau^{(0)}(\lambda;\eta),\psi_\tau^{(0)}dx]|_{d\tau=0}. \tag{74}$$

The corresponding to (73) closed subspaces $\mathcal{H}_0 \in \mathcal{H}_-$ and $\mathcal{H}_0^* \in \mathcal{H}_-^*$ are given as follows:

$$\mathcal{H}_0 := \{\psi^{(0)}(\lambda;\xi) \in \mathcal{H}_- : \mathrm{L}\psi^{(0)}(\lambda;\xi) = 0,\ \psi^{(0)}(\lambda;\xi)|_\Gamma = 0,\ \lambda \in \mathbb{C},\ \xi \in \Sigma\}, \tag{75}$$

$$\mathcal{H}^*_{\tau,0} := \{\varphi^{(0)}(\lambda;\eta) \in \mathcal{H}^*_- : \mathrm{L}\varphi^{(0)}(\lambda;\eta) = 0,\ \varphi^{(0)}(\lambda;\eta)|_\Gamma = 0,\ \lambda \in \mathbb{C},\ \eta \in \Sigma\}.$$

Thereby, making use of the expressions (73) one can construct the Delsarte-Darboux transformed linear differential pencil $\tilde{\mathrm{L}} \in \mathcal{L}(\mathcal{H})$, whose coefficients are related with those of the pencil $\mathrm{L} \in \mathcal{L}(\mathcal{H})$ via some Backlund type relationships useful for applications (see [22, 19, 41, 42, 37]) in the soliton theory.

3 Delsarte-Darboux Type Transmutation Operators For Special Multi-Dimensional Expressions And Their Applications

3.1 A perturbed self-adjoint Laplace operator in $\mathbb{R}^n$. Consider the Laplace operator $-\Delta_m$ in $H := L(\mathbb{R}^m;\mathbb{C})$ perturbed by the multiplication operator on a function $q \in W_2^2(\mathbb{R}^m;\mathbb{C})$, that is the operator

$$L(x|\partial) := -\Delta_m + q(x), \tag{76}$$

where $x \in \mathbb{R}^m$. The operator (76) is self-adjoint in H. Applying the results from Section 1 to the differential expression (76) in the Hilbert space H, one can write down the following invertible Delsarte-Darboux transmutation operators:

$$\begin{aligned}\Omega_\pm \quad = \quad & \mathbf{1} - \int_{\sigma(L)} d\rho_\sigma(\xi) \int_{\sigma(L)} d\rho_\sigma(\xi) \int_{\Sigma_\sigma} d\rho_{\Sigma_\sigma}(\xi) \int_{\Sigma_\sigma} d\rho_{\Sigma_\sigma}(\eta) \\ & \times \tilde{\psi}^{(0)}(\lambda;\xi)\Omega^{-1}_{(x_0)}(\lambda;\xi,\eta) \int^{(0)}_{S^{(m)}_\pm(\sigma^{(m-1)}_{(x)},\sigma^{(m-1)}_{(x_0)})} dy \bar{\varphi}^{(0)\intercal}(\lambda;\eta), (\cdot),\end{aligned} \tag{77}$$

where $\sigma_x^{(m-1)} \in \mathcal{K}(\mathbb{R}^m)$ is some closed and in general non-compact simplicial hypersurface in $\mathbb{R}^m$ parametrized by a running point $x \in \sigma_x^{(m-1)}$, and $\sigma_{x_0}^{(m-1)} \in \mathcal{K}(\mathbb{R}^m)$ is a suitable homological to $\sigma_x^{(m-1)}$ simplicial hypersurface in $\mathbb{R}^m$ parametrized by a point $x_0 \in \sigma_{x_0}^{(m-1)}$. There exist exactly two m-dimensional subspaces spanning them, say $S_\pm^{(m)}(\sigma_x^{(m-1)},\sigma_{x_0}^{(m-1)}) \in \mathcal{K}(\mathbb{R}^m)$, such that

$$S_+^{(m)}(\sigma_x^{(m-1)},\sigma_{x_0}^{(m-1)}) \cup\ S_-^{(m)}(\sigma_x^{(m-1)},\sigma_{x_0}^{(m-1)}) = \mathbb{R}^m.$$

Taking into account these subspaces, one can rewrite down compactly the Delsarte-Darboux transmutation operators (77) for (76):

$$\Omega_\pm = \mathbf{1} + \int_{S_\pm^{(m)}(\sigma_x^{(m-1)},\sigma_{x_0}^{(m-1)})} dy \hat{K}_\pm(\Omega)(x;y)(\cdot), \tag{78}$$

where, as before, $x \in \sigma_x^{(m-1)}$ and kernels $\hat{K}_\pm(\Omega) \in H_- \otimes H_-$ satisfy the equations (54), or equivalently,

$$-\Delta_m(x;\partial)\hat{K}_\pm(\Omega)(x;y) + \Delta_m(y;\partial)\hat{K}_\pm(\Omega)(x;y) = (q(y) - \tilde{q}(x))\hat{K}_\pm(\Omega)(x;y) \tag{79}$$

for all $x, y \in \mathrm{supp}\hat{K}_{\pm}(\Omega)$. Take for simplicity, a non-compact closed simplicial hypersurface $\sigma_x^{(m-1)} = \sigma_{x,\gamma}^{(m-1)} := \{y \in \mathbb{R}^m : \langle x - y, \gamma\rangle = 0\}$ and the degenerate simplicial cycle $\sigma_{x_0}^{(m-1)} := x_0 = \infty \in \mathbb{R}^m$, where $\gamma \in \mathbb{S}^{m-1}$ is an arbitrary normal unite vector, $||\gamma|| = 0$. Then, evidently,

$$S_{\pm}^{(m)}(\sigma_{x,\gamma}^{(m-1)}, \sigma_{\infty}^{(m-1)}) := S_{\pm\gamma,x}^{(m)} = \{y \in \mathbb{R}^m : \langle x - y, \pm\gamma\rangle \ \geq 0\} \tag{80}$$

and our transmutation operators (78) take the form

$$\Omega_{\pm\gamma} = \mathbf{1} + \int_{S_{\pm\gamma,x}^{(m)}} dy \hat{K}_{\pm\gamma}(\Omega)(x; y)(\cdot), \tag{81}$$

where $\mathrm{supp}\hat{K}_{\pm\gamma}(\Omega) = S_{\pm\gamma,x}^{(m)}$, $S_{+\gamma,x}^{(m)} \cap S_{-\gamma,x}^{(m)} = \sigma_{x,\gamma}^{(m-1)} \cup \sigma_{\infty}^{(m-1)}$ and $S_{+\gamma,x}^{(m)} \cup S_{-\gamma,x}^{(m)} = \mathbb{R}^m$ for any direction $\gamma \in \mathbb{S}^{m-1}$.

The invertible transmutation Volterrian operators like (81) were constructed before by Faddeev [9] for the self-adjoint perturbed Laplace operator (76) in $\mathbb{R}^3$. He called them [9] transformation operators with a Volterrian direction $\gamma \in \mathbb{S}^{m-1}$. It is easy to see that Faddeev's expressions (81) are very special cases of the general expressions (78) obtained above.

Define now making use of (78) the following Fredholmian operator in the Hilbert space H :

$$\Omega := (\mathbf{1} + K_+(\Omega))^{-1}(\mathbf{1} + K_-(\Omega)) = \mathbf{1} + \Phi(\Omega) \tag{82}$$

with the compact part $\Phi(\Omega) \in \mathcal{B}_\infty(H)$. Then the commutation equality

$$[L, \Phi(\Omega)] = \mathbf{0} \tag{83}$$

together with the Gelfand-Levitan-Marchenko equation

$$K_+(\Omega) + \hat{\Phi}(\Omega) + \hat{K}_+(\Omega) \cdot \hat{\Phi}(\Omega) = \hat{K}_-(\Omega) \tag{84}$$

for the corresponding kernels $\hat{K}_{\pm}(\Omega)$ and $\hat{\Phi}(\Omega) \in H_- \otimes H_-$ hold.

In [9] there was thoroughly analyzed the spectral structure of kernels $\hat{K}_{\pm}(\Omega) \in H_- \otimes H_-$ in (81) making use of the analytical properties of the corresponding Green functions of the operator (76). As one can see from (77), these properties depend strongly both on the structure of the spectral measures ρ_σ on $\sigma(L)$ and ρ_{Σ_σ} on Σ_σ and on analytical behavior of the kernel $\Omega_\infty(\lambda; \xi, \eta) \in L_2^{(\rho)}(\Sigma_\sigma; \mathbb{C}) \otimes L_2^{(\rho)}(\Sigma_\sigma; \mathbb{C})$, $\xi, \eta \in \Sigma_\sigma$, for all $\lambda \in \sigma(L)$. In [9] there was stated for any direction $\gamma \in \mathbb{S}^{m-1}$ the dependence of kernels $\hat{K}_{\pm}(\Omega) \in H_- \otimes H_-$ on the regularized determinant of the resolvent $R_\mu(L) \in \mathcal{B}(H)$, $\mu \in \mathbb{C}\backslash\sigma(L)$ is a regular point, for the operator (76). This dependence can also be clarified if we make use of the approach from Section 2.

3.2 A two-dimensional Dirac type operator. Let us define in $H := L_2(\mathbb{R}^2; \mathbb{C}^2)$ a two-dimensional Dirac type operator

$$\tilde{\mathrm{L}}_1(x; \partial) := \begin{pmatrix} \partial/\partial x_1 & \tilde{u}_1(x) \\ \tilde{u}_2(x) & \partial/\partial x_2 \end{pmatrix}, \tag{85}$$

where $x := (x_1, x_2) \in \mathbb{R}^2$, and coefficients $\tilde{u}_j \in W_2^1(\mathbb{R}^2; \mathbb{C})$, $j = \overline{1,2}$. the transformation properties of the operator (85) were studied [15] thoroughly by L.P. Nizhnik [15]. In particular, he constructed some special class of the Delsarte-Darboux transmutation operators in the form

$$\Omega_{\pm} = \mathbf{1} + \int_{S_{\pm}^{(2)}(\sigma_x^{(1)}, \sigma_{\infty}^{(1)})} dy \hat{K}_{\pm}(\Omega)(x;y)(\cdot), \tag{86}$$

where for two orthonormal vectors γ_1 and $\gamma_2 \in \mathbb{S}^1$, $||\gamma_1|| = 1 = ||\gamma_2||$,

$$\begin{aligned} S_{+}^{(2)}(\sigma_x^{(1)}, \sigma_{\infty}^{(1)}) \quad &: \quad = \{y \in \mathbb{R}^2 : \langle x - y, \gamma_1 \rangle \geq 0\} \\ \cap\{y \quad &\in \quad \mathbb{R}^2 : \langle x - y, \gamma_2 \rangle \geq 0\}, \\ S_{-}^{(2)}(\sigma_x^{(1)}, \sigma_{\infty}^{(1)}) \quad &: \quad = \{y \in \mathbb{R}^2 : \langle x - y, \gamma_1 \rangle \leq 0\} \\ \cup\{y \quad &\in \quad \mathbb{R}^2 : \langle x - y, \gamma_2 \rangle \leq 0\}. \end{aligned} \tag{87}$$

In the case when $< x, \gamma_j >= x_j \in \mathbb{R}$, $j = \overline{1,2}$, the corresponding kernel

$$\hat{K}_{+}(\Omega) = \begin{pmatrix} K_{+,11}^{(1)} \delta_{<y-x,\gamma_1>} + K_{+,11}^{(0)}(x;y) & K_{+,12}^{(1)} \delta_{<y-x,\gamma_2>} + K_{+,12}^{(0)}) \\ K_{+,21}^{(1)} \delta_{<y-x,\gamma_1>} + K_{+,21}^{(0)}(x;y) & K_{+,22}^{(1)} \delta_{<y-x,\gamma_2>} + K_{+,22}^{(0)} \end{pmatrix} \tag{88}$$

is Dirac delta-function singular, being, in part, localized on half-lines $\langle y - x, \gamma_2 \rangle = 0$ and $\langle y - x, \gamma_1 \rangle = 0$, with regular all coefficients $K_{+,ij}^{(l)} \in C^1(\mathbb{R}^2 \times \mathbb{R}^2; \mathbb{C})$ for all $i, j = \overline{1,2}$ and $l = \overline{0,1}$. Such a property of the transmutation kernels for the perturbed Laplace operator (76) was also observed in [9], where it was motivated by the necessary condition for the transformed operator $\tilde{L}(x; \partial) \in \mathcal{L}(H)$ to be differential. As one can check, the same reason of the existence of singularities holds in (88).

Let us now consider the general expression like (78) for the corresponding hypersurfaces $S_{\pm}^{(2)}(\sigma_x^{(1)}, \sigma_{\infty}^{(1)})$ spanning between a closed non-compact smooth cycle $\sigma_x^{(1)} \in \mathcal{K}(\mathbb{R}^2)$ and the infinite point $\sigma_{\infty}^{(1)} := \infty \in \mathcal{K}(\mathbb{R}^2)$. A running point $x \in \sigma_x^{(1)}$ is taken arbitrary but, as usual, fixed. The kernels $\hat{K}_{\pm}(\Omega) \in H_- \times H_-$ in (86) satisfy the standard conditions (53) and (54), that is

$$(\tilde{\mathrm{L}}_{1,ext} \otimes \mathbf{1}) \hat{K}_{\pm}(\Omega) = (\mathbf{1} \otimes \mathrm{L}_{1,ext}^{*}) \hat{K}_{\pm}(\Omega), \ [\mathrm{L}_1, \Phi(\Omega)] = 0, \tag{89}$$

for some matrix differential Dirac type operator $\mathrm{L}_1 \in \mathcal{L}(H)$ of the form (76). Together with this Dirac operator the following matrix second order differential operator

$$\tilde{\mathrm{L}}_2(x; \partial) := \mathbf{1} \frac{\partial}{\partial t} + \begin{pmatrix} \frac{\partial^2}{\partial x_1^2} \pm \frac{\partial^2}{\partial x_2^2} - \tilde{v}_2 & -2 \frac{\partial \tilde{u}_1}{\partial x_2} \\ -2 \frac{\partial \tilde{u}_2}{\partial x_1} & \frac{\partial^2}{\partial x_1^2} \pm \frac{\partial^2}{\partial x_2^2} - \tilde{v}_1 \end{pmatrix} \tag{90}$$

in the parametric space $\mathcal{H} := C^1(\mathbb{R}; H)$ was studied in [15, 16] for which there was developed scattering theory and given its application for constructing soliton-like exact solutions to the so called Davey-Stewartson nonlinear dynamical system in partial derivatives. The latter was based on the fact that two operators $\tilde{\mathrm{L}}_1$ and $\tilde{\mathrm{L}}_2 \in \mathcal{L}(H)$ are commuting to each other.

Namely, consider the Volterrian operators $\Omega_\pm \in \mathcal{L}(\mathcal{H})$ realizing the following Delsarte-Darboux transmutations:

$$\tilde{\mathrm{L}}_1\Omega_\pm = \Omega_\pm \mathrm{L}_1, \quad \tilde{\mathrm{L}}_2\Omega_\pm = \Omega_\pm \mathrm{L}_2. \tag{91}$$

Here we put

$$\begin{aligned} \mathrm{L}_1(x;\partial) \ &: = \begin{pmatrix} \partial/\partial x_1 & 0 \\ 0 & \partial/\partial x_2 \end{pmatrix}, \\ \mathrm{L}_2(x;\partial) \ &: = \mathbf{1}\frac{\partial}{\partial t} + \begin{pmatrix} \frac{\partial^2}{\partial x_1^2} \pm \frac{\partial^2}{\partial x_2^2} - \alpha_2(x_2) & 0 \\ 0 & \frac{\partial^2}{\partial x_1^2} \pm \frac{\partial^2}{\partial x_2^2} - \alpha_1(x_1) \end{pmatrix}, \end{aligned} \tag{92}$$

where $\alpha_j \in W_2^1(\mathbb{R};\mathbb{C})$, $j = \overline{1,2}$, are some given functions. It is evident that operators (92) are commuting to each other. Then, if the operators $\Omega_\pm \in \mathcal{L}(H)$ exist and satisfy (91), the following commutation condition

$$[\tilde{\mathrm{L}}_1, \tilde{\mathrm{L}}_2] = 0 \tag{93}$$

holds, that there was exactly claimed above and effectively exploited before in [15, 16].

Recall now that for the operators $\Omega_\pm \in \mathcal{L}(H)$ to exist they must satisfy additionally the kernel conditions (89) and

$$(\tilde{\mathrm{L}}_{2,ext} \otimes \mathbf{1})\hat{K}_\pm(\Omega) = (\mathbf{1} \otimes \mathrm{L}^*_{2,ext})\hat{K}_\pm(\Omega), \ [\mathrm{L}_2, \Phi(\Omega)] = 0, \tag{94}$$

where, as before, the operator $\Phi(\Omega) \in \mathcal{B}_\infty(H)$ is defined by (82) as

$$\Omega := \mathbf{1} + \Phi(\Omega). \tag{95}$$

Owing to the evident commutation condition (93) the set of equations (89) and (94) is compatible giving rise to the expression like (86), where the kernel $\hat{K}_+(\Omega) \in H_- \otimes H_-$ satisfies the set of differential equations generalizing those from [15, 16]:

$$\begin{aligned} \frac{\partial K_{+,11}}{\partial x_1} + \frac{\partial K_{+,11}}{\partial y_1} + \tilde{u}_1 K_{+,21} \ &= \ 0, \ \ \frac{\partial K_{+,12}}{\partial x_1} + \frac{\partial K_{+,12}}{\partial y_1} + \tilde{u}_1 K_{+,22} = 0, \\ \frac{\partial K_{+,21}}{\partial x_2} + \frac{\partial K_{+,21}}{\partial x_1} + \tilde{u}_2 K_{+,11} \ &= \ 0, \ \ \frac{\partial K_{+,22}}{\partial x_2} + \frac{\partial K_{+,22}}{\partial y_2} + \tilde{u}_2 K_{+,12} = 0, \end{aligned}$$

$$\begin{aligned} \pm\frac{\partial \tilde{u}_1}{\partial x_2} K_{+,21} \ &= \ \frac{\partial K_{+,11}}{\partial t} + [(\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial y_1^2}) \pm (\frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial y_2^2})]K_{+,11} \\ &\quad + (\alpha_2(x_2) - \tilde{v}_2(x))K_{+,11}, \\ \pm\frac{\partial \tilde{u}_1}{\partial x_2} K_{+,21} \ &= \ \frac{\partial K_{+,22}}{\partial t} + [(\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial y_1^2}) \pm (\frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial y_2^2})]K_{+,22} \\ &\quad + (\alpha_1(x_1) - \tilde{v}_1(x))K_{+,22}, \\ \mp 2\frac{\partial \tilde{u}_1}{\partial x_2} K_{+,22} \ &= \ \frac{\partial K_{+,12}}{\partial t} + [(\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial y_1^2}) \pm (\frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial y_2^2})]K_{+,12} \\ &\quad + (\alpha_1(x_1) - \tilde{v}_2(x))K_{+,22}, \\ 2\frac{\partial \tilde{u}_2}{\partial x_1} K_{+,22} \ &= \ \frac{\partial K_{+,21}}{\partial t} + [(\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial y_1^2}) \pm (\frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial y_2^2})]K_{+,21} \\ &\quad + (\alpha_2(x_2) - \tilde{v}_1(x))K_{+,11}. \end{aligned} \tag{96}$$

Moreover, the following conditions

$$\begin{aligned}\tilde{u}_1(x) &= -K^{(0)}_{+,12}|_{y=x}, \quad \tilde{u}_2(x) = -K^{(0)}_{+,21}|_{y=x},\\ \tilde{v}_2(x)|_{x_1=-\infty} &= \alpha_2(x_2), \quad \tilde{v}_1(x)|_{x_2=-\infty} = \alpha_1(x_1)\end{aligned} \tag{97}$$

hold for all $x \in \mathbb{R}^2$ and $y \in \mathrm{supp}\hat{K}_+(\Omega)$, where we take into account the singular series expansion

$$\hat{K}_+(\Omega) = \sum_{s=0}^{p(K_+)} K^{(s)}_+ \delta^{(s-1)}_{\sigma^{(1)}_x} \tag{98}$$

for some finite integer $p(K_+) \in \mathbb{Z}_+$ with respect to the Dirac function $\delta_{\sigma^{(1)}_x} : W^q_2(\mathbb{R}^2;\mathbb{C}) \to \mathbb{R}$, $q \in \mathbb{Z}_+$, and its derivatives, having the support (see [33], Chapter 3) coinciding with the closed cycle $\sigma^{(1)}_x \in \mathcal{K}(\mathbb{R}^2)$.

REMARK 3.1. Concerning the special case (88) discussed before in [15, 16], one gets easily that $p(K_+) = 1$ and $\sigma^{(1)}_x = \partial(\cap_{j=\overline{1,2}}\{y \in \mathbb{R}^2 : \langle y - x, \gamma_j\rangle = 0\}) \subset \mathrm{supp}\hat{K}_+(\Omega)$. It was shown also before that equations like (96) and (97) possess solutions if the Gelfand-Levitan-Marchenko equation (52) does.

Making use also of the exact forms of operators L_1 and $\mathrm{L}_2 \in \mathcal{L}(\mathcal{H})$, one obtains easily from (89) and (94) the corresponding set of differential equations for components of the kernel $\hat{\Phi}(\Omega) \in H_- \otimes H_-$:

$$\begin{aligned}\frac{\partial\Phi_{11}}{\partial x_1} + \frac{\partial\Phi_{11}}{\partial y_1} &= 0, \quad \frac{\partial\Phi_{12}}{\partial x_1} + \frac{\partial\Phi_{12}}{\partial y_1} = 0,\\ \frac{\partial\Phi_{21}}{\partial x_2} + \frac{\partial\Phi_{21}}{\partial y_2} &= 0, \quad \frac{\partial\Phi_{22}}{\partial x_2} + \frac{\partial\Phi_{22}}{\partial y_2} = 0,\end{aligned}$$

$$\begin{aligned}\frac{\partial\Phi_{11}}{\partial t} \pm (\frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial y_2^2})\Phi_{11} + (\alpha_2(y_2) - \alpha_2(x_2))\Phi_{11} &= 0,\\ \frac{\partial\Phi_{12}}{\partial t} \pm (\frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial y_2^2})\Phi_{12} + (\alpha_1(y_1) - \alpha_2(x_2))\Phi_{12} &= 0,\\ \frac{\partial\Phi_{21}}{\partial t} + (\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial y_1^2})\Phi_{21} + (\alpha_2(y_2) - \alpha_1(x_1))\Phi_{21} &= 0,\\ \frac{\partial\Phi_{22}}{\partial t} + (\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial y_1^2})\Phi_{22} + (\alpha_1(y_1) - \alpha_1(x_1))\Phi_{22} &= 0\end{aligned} \tag{99}$$

for all $(x, y) \in \mathbb{R}^2 \times \mathbb{R}^2$. The above equations (99) generalize those before found in [15, 16] and used for exact integrating the well known Devey-Stewartson differential equation [36, 34, 10] and finding so called soliton like solutions. Concerning our generalized case the kernel (98) is a solution to the following Gelfand-Levitan-Marchenko type equations:

$$K^{(0)}_+(x;y) + \Phi^{(0)}(x;y) + \int_{S^{(2)}_+(\sigma^{(1)}_x,\sigma^{(1)}_\infty)} K^{(0)}_+(x;\xi)\Phi^{(0)}(\xi;y)d\xi$$

$$+\int_{\sigma_x^{(1)}} K_+^{(1)}(x;\xi)\Phi^{(0)}(\xi;y)d\sigma_x^{(1)} = 0, \tag{100}$$

$$K_+^{(1)}(x;y) + \Phi^{(1)}(x;y) + \int_{S_+^{(2)}(\sigma_x^{(1)},\sigma_\infty^{(1)})} K_+^{(0)}(x;\xi)\Phi^{(1)}(\xi;y)d\xi$$

$$+\int_{\sigma_x^{(1)}} K_+^{(1)}(x;\xi)\Phi^{(1)}(\xi;y)d\sigma_x^{(1)} = 0,$$

where $y \in S_+^{(2)}(\sigma_x^{(1)},\sigma_\infty^{(1)})$ for all $x \in \mathbb{R}^2$ and, by definition,

$$\hat{\Phi}(\Omega) := \Phi^{(0)} + \Phi^{(1)}\delta_{\sigma_x^{(1)}} \tag{101}$$

is the corresponding to (98) kernel expansion. Since the kernel (101) is singular, the differential equations (99) must be treated naturally in the distributional sense [33].

Taking into account the exact forms of "dressed" differential operators $\mathrm{L}_j \in \mathcal{L}(\mathcal{H})$, $j = \overline{1,2}$, given by (85) and (90) one gets easily that the commutativity condition (93) gives rise to that of $\tilde{\mathrm{L}}_j \in \mathcal{L}(\mathcal{H})$, $j = \overline{1,2}$, being equivalent to the mentioned before Devey-Stewartson dynamical system

$$\begin{aligned} d\tilde{u}_1/dt &= -(\tilde{u}_{1,xx} + \tilde{u}_{1,yy}) + 2(\tilde{v}_1 - \tilde{v}_2), \\ d\tilde{u}_2/dt &= \tilde{u}_{2,xx} + \tilde{u}_{2,yy} + 2(\tilde{v}_2 - \tilde{v}_1), \\ \tilde{v}_{1,x} &= (\tilde{u}_1\tilde{u}_2)_y, \quad \tilde{v}_{2,x} = (\tilde{u}_1\tilde{u}_2)_x \end{aligned} \tag{102}$$

on a functional infinite-dimensional manifold $M_u \subset \mathcal{S}(\mathbb{R}^2;\mathbb{C})$. The exact soliton like solutions to (102) are given by expressions (97), where the kernel $K_+^{(1)}(\Omega)$ solves the second linear integral equation of (100). On the other hand, there exists the exact expression (31) which solves the set of "dressed" equations

$$\tilde{\mathrm{L}}_1\tilde{\psi}^{(0)}(\eta) = 0, \quad \tilde{\mathrm{L}}_2\tilde{\psi}^{(0)}(\eta) = 0. \tag{103}$$

Since the kernels $\Omega(\lambda,\mu) \in L_2^{(\rho)}(\Sigma;\mathbb{C}) \otimes L_2^{(\rho)}(\Sigma;\mathbb{C})$, for $\lambda,\mu \in \Sigma$, $(t;x) \in M_{\mathrm{T}} \cap S_+^{(2)}(\sigma_x^{(1)},\sigma_\infty^{(1)})$ are given by means of exact expressions (29), one can find via simple calculations the corresponding analytical expression for the functions $(\tilde{u}_1,\tilde{u}_2) \in M_u$, solving the dynamical system (102). This procedure is often called the Darboux type transformation and was recently extensively used in [22] as a particular case of the construction above for finding soliton-like solutions to the Devey-Stewartson (102) and related with it two-dimensional modified Korteweg-de Vries flows on M_u. Moreover, as it can be observed from the technique used for constructing the Delsarte-Darboux transmutation operators $\Omega_\pm \in \mathcal{L}(H)$, the set of solutions to (102) obtained by means of Darboux type transformations coincides completely with the corresponding set of solutions obtained by means of solving the related set of Gelfand-Levitan-Marchenko integral equations (99) and (100).

3.3 An affine generalized De Rham-Hodge differential complex and related generalized self-dual Yang-Mills flows. Consider the following set of affine differential expressions in $\mathcal{H} := C^1(\mathbb{R}^{m+1}; H),\ H := L_2(\mathbb{R}^m; \mathbb{C}^N)$:

$$\mathrm{L}_i(\lambda) := \mathbf{1}\frac{\partial}{\partial p_i} - \lambda \frac{\partial}{\partial x_i} + A_i(x;p|t), \tag{104}$$

where $x \in \mathbb{R}^m, (t,p) \in \mathbb{R}^{m+1}$, matrices $A_i \in C^1(\mathbb{R}^{m+1}; S(\mathbb{R}^m; End\mathbb{C}^N)),\ i = \overline{1,m}$, and a parameter $\lambda \in \mathbb{C}$. One can now easily construct an exact affine generalized De Rham-Hodge differential complex on $M_{\mathrm{T}} := \mathbb{R}^{m+1} \times \mathbb{R}^m$ as

$$\mathcal{H} \to \Lambda(M_{\mathrm{T}}; \mathcal{H}) \overset{d_{\mathcal{L}(\lambda)}}{\to} \Lambda^1(M_{\mathrm{T}}; \mathcal{H}) \to \overset{d_{\mathcal{L}(\lambda)}}{\cdots} \to \Lambda^{2m+1}(M_{\mathrm{T}}; \mathcal{H}) \overset{d_{\mathcal{L}(\lambda)}}{\to} 0, \tag{105}$$

where, by definition, the differentiation

$$d_{\mathcal{L}(\lambda)} := dt \wedge \mathrm{B}(\lambda) + \sum_{i=1}^{m} dp_i \wedge \mathrm{L}_i(\lambda) \tag{106}$$

and the affine matrix

$$\mathrm{B}(\lambda) := \partial/\partial t - \sum_{s=0}^{n(B)+q} B_s(x;p|t)\lambda^{n(B)-s} \tag{107}$$

with matrices $B_s \in C^1(\mathbb{R}^{m+1}; S(\mathbb{R}^m; End\mathbb{C}^N)),\ s = \overline{0, n(B)+q},\ n(B), q \in \mathbb{Z}_+$. The affine complex (105) will be exact for all $\lambda \in \mathbb{C}$ iff the following generalized self-dual Yang-Mills equations [42]

$$\partial A_i/\partial p_j - \partial A_j/\partial p_i - [A_i, A_j] = 0, \quad \partial A_i/\partial x_j - \partial A_j/\partial x_i = 0,$$

$$\partial B_0/\partial x_i = 0,\ \partial B_{n(B)+q}/\partial p_i = 0,\ \partial B_s/\partial x_i = \partial B_{s-1}/\partial p_i + [A_i, B_{s-1}] = 0,$$

$$\partial A_i/\partial t + \partial B_{n(B)}/\partial p_i - \partial B_{n(B)+1}/\partial x_i + [A_i, B_{n(B)}] = 0 \tag{108}$$

hold for all $i,j = \overline{1,m}$ and $s = \overline{0, n(B)} \vee \overline{n(B)+q, n(B)+2}$. Assume now that the conditions (108) are satisfied on M_{T}. Then, making the change $\mathbb{C} \ni \lambda \to \partial/\partial\tau : \mathcal{H} \to \mathcal{H}$, $\tau \in \mathbb{R}$, one finds the following set of pure differential expressions

$$\mathrm{L}_{i(\tau)} \ : \ = \mathbf{1}\frac{\partial}{\partial p_i} - \frac{\partial^2}{\partial\tau\partial x_i} + A_i(x;p|t), \tag{109}$$

$$\mathrm{B}_{(\tau)} \ : \ = \partial/\partial t - \sum_{s=0}^{n(B)+q} B_s(x;p|t)(\frac{\partial}{\partial\tau})^{n(B)-s},$$

where matrices $A_i,\ i = \overline{1,m}$, and $B_s,\ s = \overline{0, n(B)+q}$, do not depend on the variable $\tau \in \mathbb{R}$. By means of operator expressions (109) one can now naturally construct a new differential complex related with that of (105):

$$\mathcal{H}_{(\tau)} \to \Lambda(M_{\mathrm{T},\tau}; \mathcal{H}_{(\tau)}) \overset{d_{\mathcal{L}}}{\to} \Lambda^1(M_{\mathrm{T},\tau}; \mathcal{H}_{(\tau)}) \to \overset{d_{\mathcal{L}}}{\cdots} \to \Lambda^{2m+2}(M_{\mathrm{T},\tau}; \mathcal{H}_{(\tau)}) \overset{d_{\mathcal{L}}}{\to} 0, \tag{110}$$

where, by definition, $\mathcal{H}_{(\tau)} := C^1(\mathbb{R}^{m+1}; H_{(\tau)})$, $H_{(\tau)} := L_2(\mathbb{R}^m \times \mathbb{R}_\tau; \mathbb{C}^N)$ and

$$d_{\mathcal{L}} := dt \wedge \mathrm{B}_{(\tau)} + \sum_{i=1}^{m} dp_i \wedge \mathrm{L}_{i(\tau)}. \tag{111}$$

Owing to the condition (108) the following lemma holds.

LEMMA 3.2 The differential complex (110) is exact.

Therefore, one can build the standard generalized De Rham-Hodge type Hilbert space decomposition

$$\mathcal{H}_\Lambda(M_{\mathrm{T},\tau}) := \oplus_{k=0}^{k=2m+2} \mathcal{H}_\Lambda^k(M_{\mathrm{T},\tau}) \tag{112}$$

as well the corresponding Hilbert-Schmidt rigging

$$\mathcal{H}_{\Lambda,+}(M_{\mathrm{T},\tau}) \subset \mathcal{H}_\Lambda(M_{\mathrm{T},\tau}) \subset \mathcal{H}_{\Lambda,-}(M_{\mathrm{T},\tau}). \tag{113}$$

Making use now of the results obtained in subsection 1.5, one can define the Delsarte closed subspaces $\mathcal{H}_{0(\tau)}$ and $\tilde{\mathcal{H}}_{0(\tau)} \subset \mathcal{H}_{(\tau)-}$, related with the exact complex (110):

$$\begin{aligned} \mathcal{H}_{0(\tau)} \quad &: \; = \{\psi_{(\tau)}^{(0)}(\xi) \in \mathcal{H}_{\Lambda,-}^0(M_{\mathrm{T},\tau}) : \mathrm{L}_{j(\tau)}\psi_{(\tau)}^{(0)}(\xi) = 0, \\ \mathrm{B}_{(\tau)}\psi_{(\tau)}^{(0)}(\xi) \quad &= \quad 0,\ \psi_{(\tau)}^{(0)}(\xi)|_\Gamma = 0,\ \psi_{(\tau)}^{(0)}(\xi)|_{t=0} = e^{\lambda\tau}\psi_\lambda^{(0)}(\eta) \in \mathcal{H}_{\Lambda,-}^0(M_{\mathbb{R}^m,\tau}), \\ \mathrm{L}_j(\lambda)\psi_\lambda^{(0)}(\eta) \quad &= \quad 0,\ \xi = (\lambda;\eta) \in \Sigma : = \mathbb{C} \times \Sigma_{\mathbb{C}}^{(m)}\}, \end{aligned} \tag{114}$$

$$\begin{aligned} \tilde{\mathcal{H}}_{0(\tau)} \quad &: \; = \{\tilde{\psi}_{(\tau)}^{(0)}(\xi) \in \mathcal{H}_{\Lambda,-}^0(M_{\mathrm{T},\tau}) : \tilde{\mathrm{L}}_{j(\tau)}^{(0)}\tilde{\psi}_{(\tau)}^{(0)}(\xi) = 0, \\ \tilde{\mathrm{B}}_{(\tau)}\tilde{\psi}_{(\tau)}^{(0)}(\xi) \quad &= \quad 0,\ \tilde{\psi}_{(\tau)}^{(0)}(\xi)|_{\tilde{\Gamma}} = 0,\ \tilde{\psi}_{(\tau)}^{(0)}(\xi)|_{t=0} = e^{\lambda\tau}\tilde{\psi}_\lambda^{(0)}(\eta) \in \mathcal{H}_{\Lambda,-}^0(M_{\mathbb{R}^m,\tau}), \\ \tilde{\mathrm{L}}_j(\lambda)\tilde{\psi}_\lambda^{(0)}(\eta) \quad &= \quad 0,\ \xi = (\lambda;\eta) \in \Sigma : = \mathbb{C} \times \Sigma_{\mathbb{C}}^{(m)}\}, \end{aligned}$$

where Γ and $\tilde{\Gamma} \subset M_{\mathrm{T},\tau}$ are some smooth hyper-surfaces. The similar expressions correspond to the adjoint closed subspaces $\mathcal{H}_{0(\tau)}^*$ and $\tilde{\mathcal{H}}_{0(\tau)}^* \subset \mathcal{H}_{\tau,-}^*$:

$$\begin{aligned} \tilde{\mathcal{H}}_{0(\tau)} \quad &: \; = \{\varphi_{(\tau)}^{(0)}(\xi) \in \mathcal{H}_{\Lambda,-}^0(M_{\mathrm{T},\tau}) : \mathrm{L}_{j(\tau)}^*\varphi_{(\tau)}^{(0)}(\xi) = 0, \\ \mathrm{B}_{(\tau)}\varphi_{(\tau)}^{(0)}(\xi) \quad &= \quad 0,\ \varphi_{(\tau)}^{(0)}(\xi)|_\Gamma = 0,\ \varphi_{(\tau)}^{(0)}(\xi)|_{t=0} = e^{-\bar{\lambda}\tau}\varphi_\lambda^{(0)}(\eta) \in \mathcal{H}_{\Lambda,-}^0(M_{\mathbb{R}^m,\tau}), \\ \mathrm{L}_j^*(\lambda)\varphi_\lambda^{(0)}(\eta) \quad &= \quad 0,\ \xi = (\lambda;\eta) \in \Sigma : = \mathbb{C} \times \Sigma_{\mathbb{C}}^{(m)}\}, \end{aligned} \tag{115}$$

$$\begin{aligned} \tilde{\mathcal{H}}_{0(\tau)} \quad &: \; = \{\tilde{\varphi}_{(\tau)}^{(0)}(\xi) \in \mathcal{H}_{\Lambda,-}^0(M_{\mathrm{T},\tau}) : \tilde{\mathrm{L}}_{j(\tau)}^*\tilde{\varphi}_{(\tau)}^{(0)}(\xi) = 0, \\ \tilde{\mathrm{B}}_{(\tau)}^*\tilde{\varphi}_{(\tau)}^{(0)}(\xi) \quad &= \quad 0,\ \tilde{\varphi}_{(\tau)}^{(0)}(\xi)|_{\tilde{\Gamma}} = 0,\ \tilde{\varphi}_{(\tau)}^{(0)}(\xi)|_{t=0} = e^{-\bar{\lambda}\tau}\tilde{\varphi}_\lambda^{(0)}(\eta) \in \mathcal{H}_{\Lambda,-}^0(M_{\mathbb{R}^m,\tau}), \\ \tilde{\mathrm{L}}_j^*(\lambda)\tilde{\varphi}_\lambda^{(0)}(\eta) \quad &= \quad 0,\ \xi = (\lambda;\eta) \in \Sigma : = \mathbb{C} \times \Sigma_{\mathbb{C}}^{(m)}\}. \end{aligned}$$

Based on the closed subspaces (115) and (114), one can suitably build the Darboux type kernel $\tilde{\Omega}_{(t,x;\tau)}(\eta,\xi) \in L_2^{(\rho)}(\Sigma_{\mathbb{C}}^{(m)};\mathbb{C}) \otimes L_2^{(\rho)}(\Sigma_{\mathbb{C}}^{(m)};\mathbb{C})$, $\eta,\xi \in \Sigma_{\mathbb{C}}^{(m)}$, and further, the corresponding Delsarte transmutation mappings $\Omega_{\pm} \in \mathcal{L}(H_{(\tau)})$. Namely, assume that the following conditions

$$\psi^{(0)}_{(\tau)}(\xi) := \tilde{\psi}^{(0)}_{(\tau)}(\xi) \cdot \tilde{\Omega}^{-1}_{(t,p;x;\tau)} \tilde{\Omega}_{(t_0,p_0,x_0;\tau)} \tag{116}$$

for any $\xi \in \mathbb{C}\times\Sigma^{(m)}_{\mathbb{C}}$ hold, where

$$\tilde{\Omega}_{(t,x;\tau)}(\mu,\xi) := \int_{\sigma_{(t;x;\tau)}} \tilde{\Omega}^{(2m+1)}_{(\tau)}[e^{-\bar{\lambda}\tau}\tilde{\varphi}^{(0)}(\mu), e^{\lambda\tau}\tilde{\psi}^{(0)}(\eta)dx \wedge dp \wedge dt],$$

$$\begin{aligned} & \tilde{Z}^{(2m+1)}_{(\tau)}[e^{-\bar{\lambda}\tau}\tilde{\varphi}^{(0)}(\mu), \sum_{i=1}^{m} e^{\lambda\tau}\tilde{\psi}^{(0)}(\xi_{(i)}) \wedge d\tau \wedge dx \wedge^{m}_{j\neq i} dp_j] \\ : \quad & = d\tilde{\Omega}^{(2m)}_{(\tau)}[e^{-\bar{\lambda}\tau}\tilde{\varphi}^{(0)}(\mu), \sum_{i=1}^{m} e^{\lambda\tau}\tilde{\psi}^{(0)}(\xi_{(i)}) \wedge d\tau \wedge dx \wedge^{m}_{j\neq i} dp_j], \end{aligned} \tag{117}$$

and, similarly to (24), there holds the relationship

$$\begin{aligned} & \left\langle d^{*}_{\tilde{\mathcal{L}}}\tilde{\varphi}^{(0)}(\mu)e^{-\bar{\lambda}\tau}, *\sum_{i=1}^{m} e^{\lambda\tau}\tilde{\psi}^{(0)}(\xi_{(i)})dt \wedge d\tau \wedge dx \wedge^{m}_{j\neq i} dp_j \right\rangle \\ = \quad & \left\langle (*)^{-1}\tilde{\varphi}^{(0)}(\mu)e^{-\bar{\lambda}\tau}, d_{\tilde{\mathcal{L}}}(\sum_{i=1}^{m} e^{\lambda\tau}\tilde{\psi}^{(0)}(\xi_{(i)})dt \wedge d\tau \wedge dx \wedge^{m}_{j\neq i} dp_j) \right\rangle \\ & + d\tilde{Z}^{(2m+1)}_{(\tau)}[\tilde{\varphi}^{(0)}(\mu)e^{-\bar{\lambda}\tau}, \sum_{i=1}^{m} e^{\lambda\tau}\tilde{\psi}^{(0)}(\xi_{(i)})dt \wedge d\tau \wedge dx \wedge^{m}_{j\neq i} dp_j], \end{aligned} \tag{118}$$

defining the exact $(2m+1)$-form $\tilde{Z}^{(2m+1)}_{(\tau)} \in \Lambda^{2m+1}(M_{\mathrm{T},\tau};\mathbb{C})$. Compute now the Delsarte transformed differential expressions

$$\mathrm{L}_{j(\tau)} := \hat{\Omega}^{-1}_{(\tau)\pm}\tilde{\mathrm{L}}_{j(\tau)}\hat{\Omega}_{(\tau)\pm}, \quad \mathrm{B}_{(\tau)} := \hat{\Omega}^{-1}_{(\tau)\pm}\tilde{\mathrm{B}}_{(\tau)}\hat{\Omega}_{(\tau)\pm} \tag{119}$$

for any $j = \overline{1,m}$, where, by definition,

$$\begin{aligned} \tilde{\mathrm{L}}_{j(\tau)} \quad : \quad & = \mathbf{1}\frac{\partial}{\partial p_j} - \frac{\partial^2}{\partial\tau\partial x_j} + \bar{A}_j, \\ \mathrm{B}_{(\tau)} \quad : \quad & = \partial/\partial t - \sum_{s=0}^{n(B)+q} \bar{B}_s(\frac{\partial}{\partial\tau})^{n(B)-s} \end{aligned} \tag{120}$$

with all matrices $\bar{A}_j \in End\mathbb{C}^m$, $j = \overline{1,m}$, and $\bar{B}_s \in End\mathbb{C}^m$, $s = \overline{0, n(B)+q}$, being constant. This means, in particular, the commuting relationships

$$[\tilde{\mathrm{L}}_{j(\tau)}, \tilde{\mathrm{L}}_{i(\tau)}] = 0, \ [\tilde{\mathrm{L}}_{j(\tau)}, \tilde{\mathrm{B}}_{(\tau)}] = 0 \tag{121}$$

hold for all $i,j=\overline{1,m}$. Owing to the expressions (119) the induced commuting relationships

$$[\mathrm{L}_{j(\tau)},\mathrm{L}_{i(\tau)}]=0,\ [\mathrm{L}_{j(\tau)},\mathrm{B}_{(\tau)}]=0 \tag{122}$$

evidently hold, coinciding exactly with relationships (108). Moreover, reducing our differential expressions (119) upon functional subspaces $\mathcal{H}_{(\lambda)}:=e^{\lambda\tau}\mathcal{H}$, $\lambda\in\mathbb{C}$, one gets easily the set of affine differential expressions (104) and (107). Write down now the respectively reduced Delsarte transmutation operators

$$\begin{aligned}\hat{\Omega}_{\pm} &= \mathbf{1}-\int_{\Sigma_{\mathbb{C}}^{(m)}} d\rho_{\Sigma_{\mathbb{C}}^{(m)}}(\nu)\int_{\Sigma_{\mathbb{C}}^{(m)}} d\rho_{\Sigma_{\mathbb{C}}^{(m)}}(\eta)\psi^{(0)}(\lambda;\nu)\tilde{\Omega}^{-1}_{(t_0,p_0;x_0)}(\lambda;\nu,\eta) \\ &\times\int_{S_{\pm}^{(2m+1)}(\sigma^{(2m)}_{(t,p;x)},\sigma^{(2m)}_{(tt_0,p_0;x_0)})}\tilde{Z}^{(2m+1)}[\tilde{\varphi}^{(0)}(\lambda;\nu),(\cdot)\sum_{i=1}^{m}dt\wedge dx\wedge^{m}_{j\neq i}dp_j],\end{aligned} \tag{123}$$

where $\sigma^{(2m)}_{(t,p;x)}$ and $\sigma^{(2m)}_{(tt_0,p_0;x_0)}\in\mathcal{K}(M_{\mathrm{T}})$ are some $2m$-dimensional closed singular simplexes, and by definition,

$$\begin{aligned}&\tilde{Z}^{(2m+1)}[\tilde{\varphi}^{(0)}(\lambda;\nu),\sum_{i=1}^{m}\tilde{\psi}^{(0)}(\lambda;\eta_{(i)})dt\wedge dx\wedge^{m}_{j\neq i}dp_j] \\ :&=\tilde{Z}^{(2m+1)}_{(\tau)}[e^{-\bar{\lambda}\tau}\tilde{\varphi}^{(0)}(\lambda;\nu),\sum_{i=1}^{m}e^{\lambda\tau}\tilde{\psi}^{(0)}(\lambda;\eta_{(i)})d\tau\wedge dt\wedge dx\wedge^{m}_{j\neq i}dp_j]|_{d\tau=0},\end{aligned}$$

$$d\tilde{\Omega}_{(t,p;x)}(\lambda;\nu,\eta):=\tilde{Z}^{(2m+1)}[\tilde{\varphi}^{(0)}(\lambda;\nu),\sum_{i=1}^{m}\tilde{\psi}^{(0)}(\lambda;\eta_{(i)})dt\wedge dx\wedge^{m}_{j\neq i}dp_j], \tag{124}$$

since the $(2m+1)$-form (124) is, owing to (118), also exact for any $(\lambda;\nu,\eta)\in\mathbb{C}\times\left(\Sigma_{\mathbb{C}}^{(m)}\times\Sigma_{\mathbb{C}}^{(m)}\right)$. Thus, the operator expression (123) if applied to the operators (120) reduced upon the functional subspace $\mathcal{H}_{(\lambda)}\simeq\mathcal{H}$, $\lambda\in\mathbb{C}$, gives rise to the differential expressions

$$\mathrm{L}_j(\lambda):=\hat{\Omega}_{\pm}^{-1}\tilde{\mathrm{L}}_j(\lambda)\hat{\Omega}_{\pm}\quad \mathrm{B}(\lambda):=\hat{\Omega}_{\pm}^{-1}\tilde{\mathrm{B}}(\lambda)\hat{\Omega}_{\pm}, \tag{125}$$

where $\mathrm{L}_j(\lambda)\mathcal{H}_{(\lambda)}=\mathrm{L}_{j(\tau)}\mathcal{H}_{(\lambda)}$, $\mathrm{B}(\lambda)\mathcal{H}_{(\lambda)}=\mathrm{B}_{(\tau)}(\lambda)\mathcal{H}_{(\lambda)}$, $j=\overline{1,m}$, coinciding with affine differential expressions (104) and (107). Concerning application of these results to finding exact soliton like solutions to self-dual Yang-Mills equations (108), it is enough to mention that the relationship (116) reduced upon the subspace $\mathcal{H}_{(\lambda)}\simeq\mathcal{H}$, $\lambda\in\mathbb{C}$, gives rise the following mapping:

$$\psi^{(0)}(\lambda;\eta):=\tilde{\psi}^{(0)}(\lambda;\eta)\cdot\tilde{\Omega}^{-1}_{(t,p;x)}\tilde{\Omega}_{(t_0,p_0;x_0)}, \tag{126}$$

where kernels $\tilde{\Omega}_{(t,p;x;\tau)}(\lambda;\eta,\xi)\in L_2^{(\rho)}(\Sigma_{\mathbb{C}}^{(m)};\mathbb{C})\otimes L_2^{(\rho)}(\Sigma_{\mathbb{C}}^{(m)};\mathbb{C})$, $\eta,\xi\in\Sigma_{\mathbb{C}}^{(m)}$, for all $(t,p;x)\in M_{\mathrm{T}}$ and $\lambda\in\mathbb{C}$. Since the element $\psi^{(0)}(\lambda;\eta)\in\mathcal{H}_{-}$ for any $(\lambda;\xi)\in\mathbb{C}\times\Sigma_{\mathbb{C}}^{(m)}$ satisfies the set of differential equations

$$L_i(\lambda)\psi^{(0)}(\lambda;\eta)=0,\quad B(\lambda)\psi^{(0)}(\lambda;\eta)=0, \tag{127}$$

for all $i=\overline{1,m}$, from (126) and (127) one finds easily exact expressions for the corresponding matrices A_j and $B_s \in C^1(\mathbb{R}\times\mathbb{R}^{m+1}; S(\mathbb{R}^m; End\mathbb{C}^N))$, $j=\overline{1,m}$, $s=\overline{0,n(B)+q}$, satisfying the self-dual Yang-Mills equations (108). Thereby, the following theorem is stated.

THEOREM 3.3. The integral expressions (123) in $\mathcal{H}$ are the Delsarte transmutation operators corresponding to the affine differential expressions (104), (108) and constant operators

$$\tilde{\mathrm{L}}_i(\lambda) := \mathbf{1}\frac{\partial}{\partial p_i} - \lambda\frac{\partial}{\partial x_i} + \bar{A}, \quad \tilde{\mathrm{B}}(\lambda) := \partial/\partial t - \sum_{s=0}^{n(B)+q} \bar{B}_s \lambda^{n(B)-s} \tag{128}$$

for any $\lambda\in\mathbb{C}$. The mapping (126) realizes the isomorphisms between the closed subspaces

$$\begin{aligned}\mathcal{H}_0 \quad : \quad &= \{\psi^{(0)}(\lambda;\eta)\in\mathcal{H}_- : d_{\tilde{\mathcal{L}}(\lambda)}\psi^{(0)}(\lambda;\eta)=0,\ \psi^{(0)}(\lambda;\eta)|_{t=0} \\ &= \psi_\lambda^{(0)}(\eta)\in H_-,\ \psi^{(0)}(\lambda;\eta)|_\Gamma = 0, (\lambda;\eta)\in\mathbb{C}\times\Sigma_{\mathbb{C}}^{(m)}\}\end{aligned} \tag{129}$$

and

$$\begin{aligned}\tilde{\mathcal{H}}_0 \quad : \quad &= \{\tilde{\psi}^{(0)}(\lambda;\eta)\in\mathcal{H}_- : d^{(0)}_{\tilde{\mathcal{L}}(\lambda)}\tilde{\psi}(\lambda;\eta)=0,\ \tilde{\psi}^{(0)}(\lambda;\eta)|_{t=0} \\ &= \tilde{\psi}_\lambda^{(0)}(\eta)\in H_-,\ \tilde{\psi}^{(0)}(\lambda;\eta)|_{\tilde{\Gamma}} = 0, (\lambda;\eta)\in\mathbb{C}\times\Sigma_{\mathbb{C}}^{(m)}\}\end{aligned} \tag{130}$$

for any parameter $\lambda\in\mathbb{C}$. Moreover, the expressions (126) generate the standard Darboux type transformations for the set of operators (128) and (104), (107) via the corresponding set of linear equations (127), thereby producing exact soliton-like solutions to the self-dual Yang-Mills equations (108).

As a simple partial consequence from Theorem 3.2 one retrieves all of results obtained before in [42], where the Delsarte-Darboux mapping (126) was chosen completely a priori without any proof and motivation in the form of some affine gauge transformation.

The results similar to the above can be with a minor change applied also to the affine differential generalized De Rham-Hodge complex (105) with the external differentiation (106), where

$$\begin{aligned}\mathrm{L}_i(\lambda) \quad : \quad &= \mathbf{1}\frac{\partial}{\partial p_i} - (\sum_{k=0}^{n_i(L)} a_{ik}\lambda^{k+1})\frac{\partial}{\partial x_i} + \sum_{k=0}^{n_i(L)} A_{ik}\lambda^k, \\ \tilde{\mathrm{B}}(\lambda) \quad : \quad &= \partial/\partial t - \sum_{s=0}^{n(B)+q} \bar{B}_s\lambda^{n(B)-s},\end{aligned} \tag{131}$$

or

$$\begin{aligned}\mathrm{L}_i(\lambda) \quad : \quad &= \mathbf{1}\frac{\partial}{\partial p_i} - (\sum_{k=0}^{n_i(L)} a_{ik}^{(j)}\lambda^{k+1})\frac{\partial}{\partial x_j} + \sum_{k=0}^{n_i(L)} A_{ik}\lambda^k, \\ \tilde{\mathrm{B}}(\lambda) \quad : \quad &= \partial/\partial t - \sum_{s=0}^{n(B)+q} \bar{B}_s\lambda^{n(B)-s},\end{aligned} \tag{132}$$

for $i = \overline{1, m}$, $\lambda \in \mathbb{C}$. The case (131) was analyzed recently in [41] by means of the suitable affine of gauge type transformation which was used before in [42]. To our regret, the obtained there results are too complicated and unwieldy, thereby one needs to use more mathematically motivated, clear and less cumbersome techniques for finding Delsarte-Darboux transformations and related with them soliton-like exact solutions.

4 Acknowledgments

One of the authors (A.P.) cordially thanks Prof. I.V. Skrypnik (†, 2005; IM, Kyiv and IAM, Donetsk) for fruitful discussions of some aspects of the generalized De Rham-Hodge theory and its applications presented in the article during his Kyiv-city seminar "Nonlinear Analysis" (April 14, 2004). The authors are grateful to Profs. L.P. Nizhnik (IM of NAS, Kyiv), Holod P.I. (UKMA, Kyiv), T. Winiarska (IM, Politechnical University, Krakow), A. Pelczar and J. Ombach (Jagiellonian University, Krakow), J. Janas (Institute of Mathematics of PAN, Krakow), Z. Peradzynski (Warsaw University, Warsaw) and D.L. Blackmore (NJ Institute of Technology, Newark, NJ, USA) for valuable comments on diverse problems related with results presented in the review. The last but not least thanks is addressed to our friends Profs V. V. Gafiychuk (IAPMM, Lviv) and Ya. V. Mykytiuk (I. Ya. Franko National University, Lviv) for the permanent support and help in editing this review.

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